GENERALIZED LAGUERRE FUNCTIONS TO CALCULATE THE INVERSE LAPLACE TRANSFORM

Vilém Kárský
Doctoral Programme (1), FEEC BUT
E-mail: xkarsk01@stud.feec.vutbr.cz

Supervised by: Pavel Jura
E-mail: jura@feec.vutbr.cz

Abstract: This paper concentrates on using generalized Laguerre functions to calculate the inverse Laplace transform. The actual application of the method is demonstrated via transforming two transfer functions, one ranging within the integer-order category and the other being of the fractional-order type.

Keywords: Inverse Laplace transform, generalized Laguerre functions, fractional order systems

1 INTRODUCTION

The popularity of employing fractional order transfer functions for system description has been rising in recent years. This trend arises from the fact that such description is more accurate for some systems. However, using the technique is also associated with certain problems, prominently including the significantly increased complexity of each computational cycle. The drawback then relates mainly to the computation of the inverse Laplace transform.

2 MATHEMATICAL BACKGROUND

2.1 Generalized Laguerre Function

Generalized Laguerre polynomials are underlying generalized Laguerre functions. The polynomials are defined as

$$L_n^{\alpha}(x) = \frac{\mathrm{e}^x x^{-\alpha}}{n!} \frac{\mathrm{d^n}}{\mathrm{d}x^n} \left(x^{n+\alpha}\mathrm{e}^{-x}\right) [1]. \tag{1}$$

According to [2, 3], useful formulae are available for generalized Laguerre polynomials with $\alpha = 1$, as shown below:

$$L_{n-1}^{1}(x) = -\frac{\mathrm{d}}{\mathrm{d}x} L_n(x), \tag{2}$$

$$L_n(x) = L_n^1(x) - L_{n-1}^1(x). \tag{3}$$

The Laplace transform of non-generalized Laguerre polynomials is

$$\mathscr{L}\left\{L_n(x)\right\} = \frac{1}{s}\left(\frac{s-1}{s}\right)^n. \tag{4}$$

It then follows from formulae 2 to 4 that the Laplace transform of generalized Laguerre polynomials with $\alpha = 1$ reads

$$\mathscr{L}\left\{L_n^1(x)\right\} = (-1)\left(\frac{s-1}{s}\right)^{n+1}. \tag{5}$$

Generalized Laguerre polynomials are orthogonal in $\langle 0,\infty)$, with weight function $w(x)=x^{\alpha}\mathrm{e}^{-x}$. Using substitution $x=2\lambda t$, we can write

$$\int_0^{\infty}\frac{n!(2\lambda t)^{\alpha}}{\Gamma(n+\alpha+1)}\sqrt{2\lambda t}\mathrm{e}^{-\lambda t}L_n^{\alpha}(2\lambda t)\cdot\sqrt{2\lambda t}\mathrm{e}^{-\lambda t}L_m^{\alpha}(2\lambda t)\mathrm{d}t=\begin{cases}0 \; for\; m\neq n\\ 1\; for\; m=n\end{cases}. \tag{6}$$

Applying equation 6 enables us to define generalized Laguerre functions as

$$l_n^{\alpha}(2\lambda t)=\sqrt{2\lambda t}\mathrm{e}^{-\lambda t}L_n^{\alpha}(2\lambda t). \tag{7}$$

2.2 Mittag-Leffler Function

Mittag-Leffler functions embody a generalization of the exponential function. As proposed in [4], Mittag-Leffler functions are defined by formula

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \tag{8}$$

where $\Gamma(k)$ is the Gamma function.

2.3 Laplace transform of fractional order derivative

As outlined in [3], the Laplace transform of the Caputo fractional order derivative is

$$\mathscr{L}\left\{D_0^{\alpha}f(t)\right\}(s)=s^{\alpha}F(s)-\sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0). \tag{9}$$

3 INVERSE LAPLACE TRANSFORM USING GENERALIZED LAGUERRE FUNCTIONS

In integer-order transfer functions, several well-known methods can be utilized to solve the inverse Laplace transform. When calculating the inverse Laplace transform of a fractional order transfer function, we can employ the formulae from [4]; the solution, however, takes the form of an infinite series. Another (and considerably more advantageous) method is characterized in [3]; this procedure exploits generalized Laguerre functions to calculate the inverse Laplace transform. Let us assume a system expressed as

$$F(s)=\frac{\sum_{m=0}^{M}b_m s^{qm}}{\sum_{n=0}^{N}a_n s^{qn}}=\sum_{i=0}^{I}\frac{K_i}{(s^q-w_i)^{r_i}}. \tag{10}$$

We can then write $g(t)$ in the form

$$g(t)=\sum_{i=0}^{\infty}c_i^1 l_i^1=\sqrt{2\lambda}\mathrm{e}^{-\lambda t}\sum_{i=0}^{\infty}c_i^1 L_i^1(2\lambda t). \tag{11}$$

According to [3], the coefficients c_i^1 are computable using the formula

$$c_i^1=\frac{-\sqrt{2\lambda}}{(i+1)!}\left[\frac{\mathrm{d}^{i+1}}{\mathrm{d}z^{i+1}}F(z)\right]_{z=0}, \tag{12}$$

where $F(z)$ is our transfer function after being subjected to the bilinear transform in the form

$$s=\lambda\frac{1+z}{1-z}. \tag{13}$$

4 APPLICATIONS

This section introduces two examples of solving the inverse Laplace transform via generalized Laguerre functions. The given procedures transform the transfer functions of an integer-order and a fractional-order system, respectively.

4.1 INTEGER ORDER SYSTEM

The first example consists in an integer-order system with the transfer function

$$F(s) = \frac{1}{s^2+s+1} = \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{s+\frac{1+\mathrm{j}\sqrt{3}}{2}} - \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{s+\frac{1-\mathrm{j}\sqrt{3}}{2}}. \tag{14}$$

Using well-known formulae, we can then transform both partial fractions independently to obtain

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+s+1}\right\} = \frac{2}{\sqrt{2}}\mathrm{e}^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{3}}{2}t\right). \tag{15}$$

The process is illustrated by the blue line in Fig. 1.

We can nevertheless calculate the inverse Laplace transform by means of generalized Laguerre functions; here, those of the eight order with $\lambda = 0.4175$ are used. First, the bilinear transform has to be employed to yield

$$F(z) = \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{\lambda\frac{1+z}{1-z}+\frac{1+\mathrm{j}\sqrt{3}}{2}} - \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{\lambda\frac{1+z}{1-z}+\frac{1-\mathrm{j}\sqrt{3}}{2}}. \tag{16}$$

Subsequently, we can evaluate the coefficients c_0^1 to c_7^1; the corresponding values are summarized in Table 1. The final approximation is represented by the green line in Figure 1.

Table 1: The coefficients' spectrum: the integer-order transfer function

i	0	1	2	3	4	5	6	7
c_i^1	0.8986	-0.1216	-0.3350	-0.0386	0.1048	0.0382	-0.0260	-0.0103

4.2 FRACTIONAL ORDER SYSTEM

The second example is embodied in the fractional-order transfer function

$$F(s) = \frac{1}{s^{1.4}+s^{0.7}+1} = \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{s^{0.7}+\frac{1+\mathrm{j}\sqrt{3}}{2}} - \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{s^{0.7}+\frac{1-\mathrm{j}\sqrt{3}}{2}}. \tag{17}$$

First of all, we calculate the inverse Laplace transform, using the equation

$$\mathscr{L}\left\{t^{\alpha k+\beta-1}E_{\alpha,\beta}^{(k)}(\pm a t^{\alpha})\right\} = \frac{k!s^{\alpha-\beta}}{(s^{\alpha}\mp a)^{k+1}} \quad [4]; \tag{18}$$

thus, we get

$$\mathscr{L}^{-1}\left\{\frac{1}{s^{1.4}+s^{0.7}+1}\right\} = \mathrm{j}\frac{t^{-0.3}}{\sqrt{3}}\left[E_{0.7,\,0.7}\left(-\frac{1+\mathrm{j}\sqrt{3}}{2}t^{0.7}\right) - E_{0.7,\,0.7}\left(-\frac{1-\mathrm{j}\sqrt{3}}{2}t^{0.7}\right)\right]. \tag{19}$$

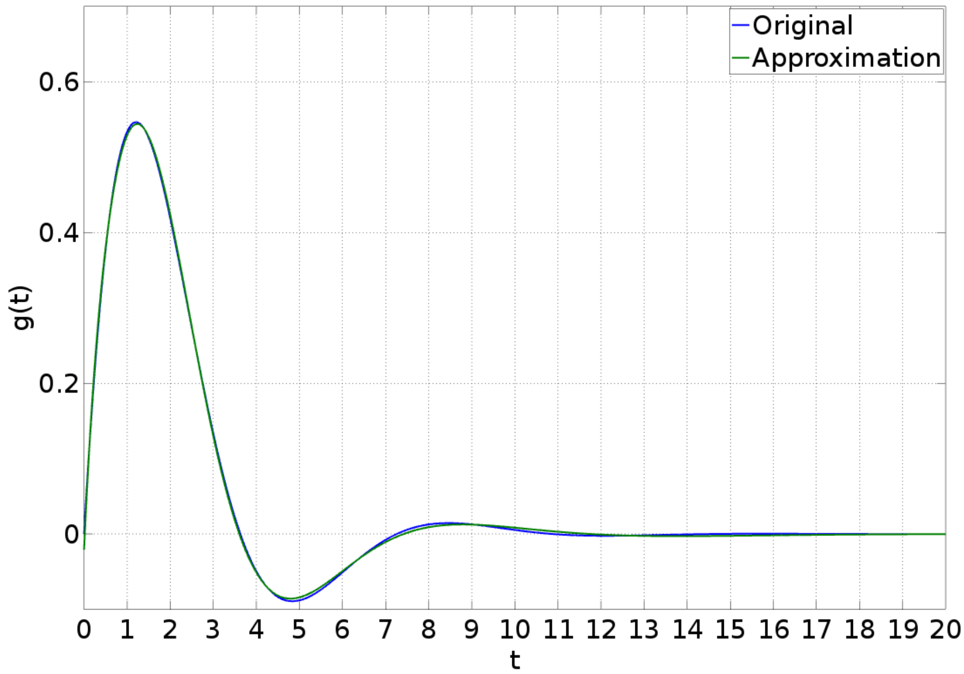

Figure 1: The approximation of the integer-order system

The process is plotted via the blue line in Figure 2; for the given purpose, the initial 50 terms were utilized.

At this stage, we transform the system, using the first eight generalized Laguerre functions with the time-scale
$\lambda = 0.5805$. Naturally, the bilinear transform needs to be applied to yield

$$F(z) = \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{\left(\lambda\frac{1+z}{1-z}\right)^{0.7} + \frac{1+\mathrm{j}\sqrt{3}}{2}} - \frac{\mathrm{j}\frac{\sqrt{3}}{3}}{\left(\lambda\frac{1+z}{1-z}\right)^{0.7} + \frac{1-\mathrm{j}\sqrt{3}}{2}}. \tag{20}$$

It is then possible to evaluate the coefficients c_0^1 to c_7^1; the corresponding values are indicated within Table 2. The final approximation is represented by the green line in Figure 2.

Table 2: The coefficients' spectrum: the fractional-order transfer function

i	0	1	2	3	4	5	6	7
c_i^1	0.6826	-0.2148	0.0246	-0.0288	0.0110	-0.0139	0.0073	-0.0070

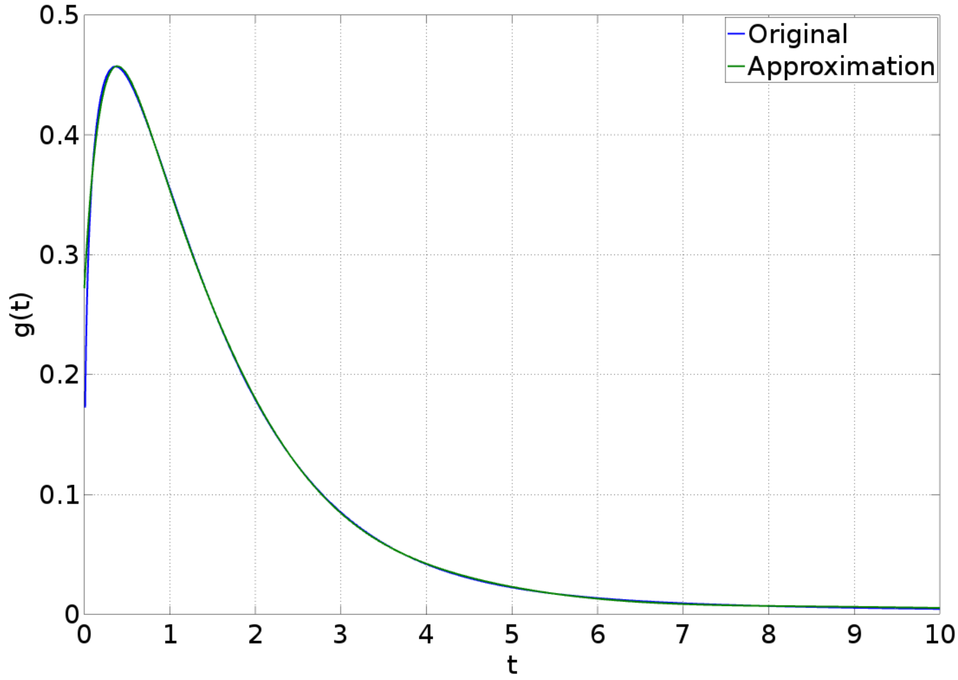

Figure 2: The approximation of the fractional-order system

5 CONCLUSION

As proposed through the demonstration above, the generalized Laguerre function is a suitable tool to compute the inverse Laplace transform, mainly for fractional order transfer functions. Generalized Laguerre functions converge considerably faster than Mittag-Leffler functions; however, there still remains the question of how to choose an applicable time-scale (parameter λ), which embodies a factor having remarkable effect on the quality of approximation.

ACKNOWLEDGEMENT

The research was funded from grant No. FEKT-S-17-4234 - “Industry 4.0 in automation and cybernetics” and further supported through the Internal science fund of Brno University of Technology.

REFERENCES

[1] Associated Laguerre Polynomial. *Wolfram Math World* [online]. 2017 [cit. 2018-03-11]. Available from: <http://mathworld.wolfram.com/AssociatedLaguerrePolynomial.html>

[2] WOLFRAM RESEARCH. *The Wolfram Functions Site: Polynomials* [online]. 2017 [cit. 2017-05-14]. Available from: <http://functions.wolfram.com/Polynomials/>

[3] MAIONE, G. Inverting fractional order transfer functions through Laguerre approximation. *Systems and Control Letters*, 52(5), 387–393, 2004.

[4] PODLUBNÝ, Igor. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. San Diego, Calif.: Academic Press, c1999. Mathematics in science and engineering, v. 198. ISBN 0-12-558840-2.