Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 159, pp. 1–13.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

SOLVABILITY OF A CLOSE TO SYMMETRIC SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. The problem of solvability of a close to symmetric product-type system of difference equations of second order is investigated. Some recent results in the literature are extended.

1. Introduction

Various types of nonlinear difference equations and systems have been considerably studied recently (see, e.g., [1]-[4], [6, 7, 9, 10], [12]-[31]). Among other topics, there has been some renewed interest in the equations and systems which can be solved (see, e.g., [1]-[4], [16, 21], [23]-[29], [31]). Many of these papers essentially used a transformation method by Stević (see, e.g., [1, 2, 16, 21, 23, 24, 25, 31] where can be also found original sources and many other references). Some known classes of difference equations and systems, including solvable ones, can be found, for example, in [5, 8, 11, 18]. After the publication of some papers on concrete systems of difference equations by Papaschinopoulos and Schinas almost two decades ago (see, e.g., [12, 13, 14]), some interest in the area has also started (see, e.g., [4, 15, 17, 19, 20, 24, 25, 26, 27, 28, 29, 30, 31]).

An investigation of the long-term behavior of solutions to some classes of difference equations which are modifications/perturbations of product-type ones has been also started by Stević (see, e.g., [22] and the references therein). The corresponding investigation of related systems of difference equations has been started somewhat later. For example, in [30] the boundedness character of positive solutions of the following system

$$z_{n+1} = \max\left\{f, w_n^p/z_{n-1}^q\right\}, \quad w_{n+1} = \max\left\{f, z_n^p/w_{n-1}^q\right\}, \quad n \in \mathbb{N}_0, \tag{1.1}$$

with positive parameters f, p and q, was investigated. There are also some solvable max-type systems of difference equations [24]. The corresponding product-type system to (1.1) (system (1.2) below with $\hat{a} = \hat{c} = p$ and $\hat{b} = \hat{d} = q$) with positive initial values is solvable. However, the case of complex initial values seems has not been studied in detail. These observations motivated us to study product-type systems with such initial values. One of the first papers on the problem is [29],

2010 *Mathematics Subject Classification.* 39A10, 39A20.
Key words and phrases. Symmetric system of difference equations; product-type system.

Submitted April 10, 2016. Published June 27, 2016.

where we have studied the product-type system

$$z_{n+1}=\frac{w_n^{\hat{a}}}{z_{n-1}^{\hat{b}}},\quad w_{n+1}=\frac{z_n^{\hat{c}}}{w_{n-1}^{\hat{d}}},\quad n\in\mathbb{N}_0,\tag{1.2}$$

where $\hat{a},\hat{b},\hat{c},\hat{d}$ are integers (the condition is posed to avoid multi-valued sequences).

Motivated by the close to symmetric systems in [30] and [31], in [26] S. Stević has noticed that some complex parameters/coefficients can be included into a product-type system of difference equations so that the solvability of such obtained system is preserved. For some other results in the topic, see also [28].

Our aim is to investigate the solvability of the following close to symmetric system of difference equations

$$z_{n+1}=\alpha w_n^a z_{n-1}^b,\quad w_{n+1}=\beta z_n^c w_{n-1}^d,\quad n\in\mathbb{N}_0,\tag{1.3}$$

where $a,b,c,d\in\mathbb{Z}$, $\alpha,\beta\in\mathbb{C}$ and $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}$. Actually, since the cases $\alpha=0$ and $\beta=0$ are simple, we will study only the case when $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ in detail.

We want to point out that system (1.3) is not only an interesting and important extension of system (1.2), but also our approach in the paper will be different from the one in [29], but in the spirit of [26].

Note that the domain of undefinable solutions ([25]) to system (1.3) is a subset of

$$\mathcal{U}=\{(z_{-1},z_0,w_{-1},w_0)\in\mathbb{C}^4: z_{-1}=0\text{ or }z_0=0\text{ or }w_{-1}=0\text{ or }w_0=0\}.$$

This domain is equal to $\mathcal{U}$ if $\min\{a,b,c,d\}<0$, but it can be also an empty set if $\min\{a,b,c,d\}>0$. To avoid some quite simple and not so interesting discussions we will also assume that all the initial values belong to $\mathbb{C}\setminus\{0\}$. Throughout the paper we will frequently use the convention $\sum_{j=l}^m a_j=0$, for $m<l$.

2. Main results

The problem of solvability of system (1.3) will be treated in this section. Three cases will be separately studied, namely, $a=0$, $c=0$ and $ac\neq 0$.

Theorem 2.1. *Assume that $b,c,d\in\mathbb{Z}$, $a=0$, $\alpha,\beta\in\mathbb{C}\setminus\{0\}$ and initial values $z_{-1},z_0,w_{-1},w_0\in\mathbb{C}\setminus\{0\}$. Then system* (1.3) *is solvable in closed form.*

Proof. Since $a=0$ we have

$$z_{n+1}=\alpha z_{n-1}^b,\quad w_{n+1}=\beta z_n^c w_{n-1}^d,\quad n\in\mathbb{N}_0.\tag{2.1}$$

The first equation in (2.1) easily yields

$$z_{2n}=\alpha^{\sum_{j=0}^{n-1}b^j}z_0^{b^n},\quad n\in\mathbb{N},\tag{2.2}$$

$$z_{2n-1}=\alpha^{\sum_{j=0}^{n-1}b^j}z_{-1}^{b^n},\quad n\in\mathbb{N}.\tag{2.3}$$

From (2.2) and (2.3) we have

$$z_{2n}=\alpha^{\frac{1-b^n}{1-b}}z_0^{b^n},\quad n\in\mathbb{N},\tag{2.4}$$

$$z_{2n-1}=\alpha^{\frac{1-b^n}{1-b}}z_{-1}^{b^n},\quad n\in\mathbb{N},\tag{2.5}$$

when $b\neq 1$, while

$$z_{2n}=\alpha^n z_0,\quad n\in\mathbb{N},\tag{2.6}$$

$$z_{2n-1} = \alpha^n z_{-1}, \quad n \in \mathbb{N}, \tag{2.7}$$

when $b = 1$ (note that (2.2) and (2.3) obviously hold for $n = 0$, when $b \neq 0$).

By using (2.2) in the second equation in (2.1) with $n \to 2n$, we obtain

$$\begin{aligned} w_{2n+1} &= \beta z_{2n}^c w_{2n-1}^d = \beta(\alpha^{\sum_{j=0}^{n-1} b^j} z_0^{b^n})^c w_{2n-1}^d \\ &= \beta\alpha^{c\sum_{j=0}^{n-1} b^j} z_0^{cb^n} w_{2n-1}^d, \end{aligned} \tag{2.8}$$

for $n \in \mathbb{N}$.

Suppose that for some $k \in \mathbb{N}$ we have proved

$$w_{2n+1} = \beta^{\sum_{i=0}^{k-1} d^i} \alpha^{c\sum_{i=0}^{k-1} d^i \sum_{j=0}^{n-i-1} b^j} z_0^{c\sum_{i=0}^{k-1} d^i b^{n-i}} w_{2(n-k)+1}^{d^k}, \tag{2.9}$$

for $n \geq k$. By using (2.8) with $n \to n-k$ into (2.9) it follows that

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{i=0}^{k-1} d^i} \alpha^{c\sum_{i=0}^{k-1} d^i \sum_{j=0}^{n-i-1} b^j} z_0^{c\sum_{i=0}^{k-1} d^i b^{n-i}} \\ &\quad \times (\beta\alpha^{c\sum_{j=0}^{n-k-1} b^j} z_0^{cb^{n-k}} w_{2(n-k-1)+1}^d)^{d^k} \\ &= \beta^{\sum_{i=0}^{k} d^i} \alpha^{c\sum_{i=0}^{k} d^i \sum_{j=0}^{n-i-1} b^j} z_0^{c\sum_{i=0}^{k} d^i b^{n-i}} w_{2(n-k-1)+1}^{d^{k+1}}, \end{aligned} \tag{2.10}$$

for $n \geq k+1$.

Formulas (2.8), (2.10) along with the induction shows that (2.9) holds for all natural numbers k and n such that $1 \leq k \leq n$. For $k = n$, (2.9) becomes

$$w_{2n+1} = \beta^{\sum_{i=0}^{n-1} d^i} \alpha^{c\sum_{i=0}^{n-1} d^i \sum_{j=0}^{n-i-1} b^j} z_0^{c\sum_{i=0}^{n-1} d^i b^{n-i}} w_1^{d^n}, \quad n \in \mathbb{N}. \tag{2.11}$$

Using the relation $w_1 = \beta z_0^c w_{-1}^d$ into (2.11), we obtain

$$\begin{aligned} w_{2n+1} &= \beta^{\sum_{i=0}^{n-1} d^i} \alpha^{c\sum_{i=0}^{n-1} d^i \sum_{j=0}^{n-i-1} b^j} z_0^{c\sum_{i=0}^{n-1} d^i b^{n-i}} (\beta z_0^c w_{-1}^d)^{d^n} \\ &= \beta^{\sum_{i=0}^{n} d^i} \alpha^{c\sum_{i=0}^{n-1} d^i \sum_{j=0}^{n-i-1} b^j} z_0^{c\sum_{i=0}^{n} d^i b^{n-i}} w_{-1}^{d^{n+1}}, \quad n \in \mathbb{N}_0. \end{aligned} \tag{2.12}$$

By using (2.3) into the second equation in (2.1) with $n \to 2n-1$, we obtain

$$\begin{aligned} w_{2n} &= \beta z_{2n-1}^c w_{2n-2}^d = \beta(\alpha^{\sum_{j=0}^{n-1} b^j} z_{-1}^{b^n})^c w_{2n-2}^d \\ &= \beta\alpha^{c\sum_{j=0}^{n-1} b^j} z_{-1}^{cb^n} w_{2n-2}^d, \end{aligned} \tag{2.13}$$

for $n \in \mathbb{N}$.

Assume that for some $k \in \mathbb{N}$ we have proved that

$$w_{2n} = \beta^{\sum_{i=0}^{k-1} d^i} \alpha^{c\sum_{i=0}^{k-1} d^i \sum_{j=0}^{n-i-1} b^j} z_{-1}^{c\sum_{i=0}^{k-1} d^i b^{n-i}} w_{2(n-k)}^{d^k}, \tag{2.14}$$

for $n \geq k$.

By using (2.13) with $n \to n-k$ into (2.14) we obtain

$$\begin{aligned} w_{2n} &= \beta^{\sum_{i=0}^{k-1} d^i} \alpha^{c\sum_{i=0}^{k-1} d^i \sum_{j=0}^{n-i-1} b^j} z_{-1}^{c\sum_{i=0}^{k-1} d^i b^{n-i}} \\ &\quad \times (\beta\alpha^{c\sum_{j=0}^{n-k-1} b^j} z_{-1}^{cb^{n-k}} w_{2(n-k-1)}^d)^{d^k} \\ &= \beta^{\sum_{i=0}^{k} d^i} \alpha^{c\sum_{i=0}^{k} d^i \sum_{j=0}^{n-i-1} b^j} z_{-1}^{c\sum_{i=0}^{k} d^i b^{n-i}} w_{2(n-k-1)}^{d^{k+1}}, \end{aligned} \tag{2.15}$$

for $n \geq k+1$.

From (2.13), (2.15) and the induction we have that (2.14) holds for all natural numbers k and n such that $1 \leq k \leq n$. For $k = n$, (2.14) becomes

$$w_{2n} = \beta^{\sum_{i=0}^{n-1} d^i} \alpha^{c\sum_{i=0}^{n-1} d^i \sum_{j=0}^{n-i-1} b^j} z_{-1}^{c\sum_{i=0}^{n-1} d^i b^{n-i}} w_0^{d^n}, \tag{2.16}$$

for $n \in \mathbb{N}$.

Case $b \neq d$. From (2.12) and (2.16) we have that

$$w_{2n+1} = \beta^{\sum_{i=0}^{n} d^i} \alpha^{c\sum_{i=0}^{n-1} d^i \sum_{j=0}^{n-i-1} b^j} z_0^{c\frac{d^{n+1}-b^{n+1}}{d-b}} w_{-1}^{d^{n+1}}, \tag{2.17}$$

for $n \in \mathbb{N}_0$, and

$$w_{2n} = \beta^{\sum_{i=0}^{n-1} d^i} \alpha^{c\sum_{i=0}^{n-1} d^i \sum_{j=0}^{n-i-1} b^j} z_{-1}^{bc\frac{d^n-b^n}{d-b}} w_0^{d^n}, \quad n \in \mathbb{N}. \tag{2.18}$$

Subcase $b \neq 1 \neq d$. From (2.17) we have

$$\begin{aligned} w_{2n+1} &= \beta^{\frac{1-d^{n+1}}{1-d}} \alpha^{c\sum_{i=0}^{n-1} d^i \frac{1-b^{n-i}}{1-b}} z_0^{c\frac{d^{n+1}-b^{n+1}}{d-b}} w_{-1}^{d^{n+1}} \\ &= \beta^{\frac{1-d^{n+1}}{1-d}} \alpha^{\frac{c}{1-b}\left(\frac{1-d^n}{1-d} - b\frac{b^n-d^n}{b-d}\right)} z_0^{c\frac{d^{n+1}-b^{n+1}}{d-b}} w_{-1}^{d^{n+1}} \\ &= \beta^{\frac{1-d^{n+1}}{1-d}} \alpha^{\frac{c(b-d+d^{n+1}-b^{n+1}+db^{n+1}-bd^{n+1})}{(1-b)(1-d)(b-d)}} z_0^{c\frac{d^{n+1}-b^{n+1}}{d-b}} w_{-1}^{d^{n+1}}, \end{aligned} \tag{2.19}$$

for $n \in \mathbb{N}_0$. From (2.18) and by employing a formula used in getting (2.19), we have

$$\begin{aligned} w_{2n} &= \beta^{\sum_{i=0}^{n-1} d^i} \alpha^{c\sum_{i=0}^{n-1} d^i \frac{1-b^{n-i}}{1-b}} z_{-1}^{bc\frac{d^n-b^n}{d-b}} w_0^{d^n} \\ &= \beta^{\frac{1-d^n}{1-d}} \alpha^{\frac{c(b-d+d^{n+1}-b^{n+1}+db^{n+1}-bd^{n+1})}{(1-b)(1-d)(b-d)}} z_{-1}^{bc\frac{d^n-b^n}{d-b}} w_0^{d^n}, \end{aligned} \tag{2.20}$$

for $n \in \mathbb{N}$.

Subcase $b \neq 1 = d$. From (2.17) we have

$$\begin{aligned} w_{2n+1} &= \beta^{n+1} \alpha^{c\sum_{i=0}^{n-1} \frac{1-b^{n-i}}{1-b}} z_0^{c\frac{1-b^{n+1}}{1-b}} w_{-1} \\ &= \beta^{n+1} \alpha^{\frac{c}{1-b}\left(n - b\frac{1-b^n}{1-b}\right)} z_0^{c\frac{1-b^{n+1}}{1-b}} w_{-1} \\ &= \beta^{n+1} \alpha^{\frac{c(n-(n+1)b+b^{n+1})}{(1-b)^2}} z_0^{c\frac{1-b^{n+1}}{1-b}} w_{-1}, \end{aligned} \tag{2.21}$$

for $n \in \mathbb{N}_0$. From (2.18) and by employing a formula used in getting (2.21), we have

$$\begin{aligned} w_{2n} &= \beta^{n} \alpha^{c\sum_{i=0}^{n-1} \frac{1-b^{n-i}}{1-b}} z_{-1}^{bc\frac{1-b^{n}}{1-b}} w_{0} \\ &= \beta^{n} \alpha^{\frac{c(n-(n+1)b+b^{n+1})}{(1-b)^2}} z_{-1}^{bc\frac{1-b^{n}}{1-b}} w_{0}, \end{aligned} \tag{2.22}$$

for $n \in \mathbb{N}$.

Subcase $b = 1 \neq d$. From (2.17) we have

$$\begin{aligned} w_{2n+1} &= \beta^{\frac{1-d^{n+1}}{1-d}} \alpha^{c\sum_{i=0}^{n-1} d^i (n-i)} z_0^{c\frac{d^{n+1}-1}{d-1}} w_{-1}^{d^{n+1}} \\ &= \beta^{\frac{1-d^{n+1}}{1-d}} \alpha^{c\left(n\frac{1-d^n}{1-d} - d\frac{1-nd^{n-1}+(n-1)d^n}{(1-d)^2}\right)} z_0^{c\frac{d^{n+1}-1}{d-1}} w_{-1}^{d^{n+1}} \\ &= \beta^{\frac{1-d^{n+1}}{1-d}} \alpha^{\frac{c(n-(n+1)d+d^{n+1})}{(1-d)^2}} z_0^{c\frac{d^{n+1}-1}{d-1}} w_{-1}^{d^{n+1}}, \end{aligned} \tag{2.23}$$

for $n \in \mathbb{N}_0$. From (2.18) and by employing a formula used in getting (2.23), we have

$$\begin{aligned} w_{2n} &= \beta^{\frac{1-d^n}{1-d}} \alpha^{c\sum_{i=0}^{n-1} d^i (n-i)} z_{-1}^{c\frac{1-d^n}{1-d}} w_0^{d^n} \\ &= \beta^{\frac{1-d^n}{1-d}} \alpha^{\frac{c(n-(n+1)d+d^{n+1})}{(1-d)^2}} z_{-1}^{c\frac{d^n-1}{d-1}} w_0^{d^n}, \end{aligned} \tag{2.24}$$

for $n \in \mathbb{N}$.

Case $b = d \neq 1$. From (2.12) we have

$$\begin{aligned} w_{2n+1} &= \beta^{\frac{1-b^{n+1}}{1-b}} \alpha^{c\sum_{i=0}^{n-1} b^i \frac{1-b^{n-i}}{1-b}} z_0^{cb^n(n+1)} w_{-1}^{b^{n+1}} \\ &= \beta^{\frac{1-b^{n+1}}{1-b}} \alpha^{\frac{c}{1-b}\left(\frac{1-b^n}{1-b} - nb^n\right)} z_0^{cb^n(n+1)} w_{-1}^{b^{n+1}} \\ &= \beta^{\frac{1-b^{n+1}}{1-b}} \alpha^{\frac{c(1-(n+1)b^n + nb^{n+1})}{(1-b)^2}} z_0^{cb^n(n+1)} w_{-1}^{b^{n+1}}, \end{aligned} \tag{2.25}$$

for $n \in \mathbb{N}_0$.

From (2.16) and by employing a formula used in getting (2.25), we have

$$\begin{aligned} w_{2n} &= \beta^{\frac{1-b^n}{1-b}} \alpha^{c\sum_{i=0}^{n-1} b^i \frac{1-b^{n-i}}{1-b}} z_{-1}^{cnb^n} w_0^{b^n} \\ &= \beta^{\frac{1-b^n}{1-b}} \alpha^{\frac{c(1-(n+1)b^n + nb^{n+1})}{(1-b)^2}} z_{-1}^{cnb^n} w_0^{b^n}, \end{aligned} \tag{2.26}$$

for $n \in \mathbb{N}_0$.

Case $b = d = 1$. From (2.12) we have

$$\begin{aligned} w_{2n+1} &= \beta^{n+1} \alpha^{c\sum_{i=0}^{n-1}(n-i)} z_0^{c(n+1)} w_{-1} \\ &= \beta^{n+1} \alpha^{\frac{cn(n+1)}{2}} z_0^{c(n+1)} w_{-1}, \end{aligned} \tag{2.27}$$

for $n \in \mathbb{N}_0$. From (2.16) we have

$$\begin{aligned} w_{2n} &= \beta^n \alpha^{c\sum_{i=0}^{n-1}(n-i)} z_{-1}^{cn} w_0 \\ &= \beta^n \alpha^{\frac{cn(n+1)}{2}} z_{-1}^{cn} w_0, \end{aligned} \tag{2.28}$$

for $n \in \mathbb{N}_0$. This completes the proof. □

Corollary 2.2. *Consider system* (1.3). *Assume that $b, c, d \in \mathbb{Z}$, $a = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

- (a) *If $b \neq 1 \neq d \neq b$, then the general solution of system* (1.3) *is given by* (2.4), (2.5), (2.19) *and* (2.20).
- (b) *If $b \neq 1 = d$, then the general solution of system* (1.3) *is given by* (2.4), (2.5), (2.21) *and* (2.22).
- (c) *If $b = 1 \neq d$, then the general solution of system* (1.3) *is given by* (2.6), (2.7), (2.23) *and* (2.24).
- (d) *If $b = d \neq 1$, then the general solution of system* (1.3) *is given by* (2.4), (2.5), (2.25) *and* (2.26).
- (e) *If $b = d = 1$, then the general solution of system* (1.3) *is given by* (2.6), (2.7), (2.27) *and* (2.28).

Theorem 2.3. *Assume that $a, b, d \in \mathbb{Z}$, $c = 0$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (1.3) *is solvable in closed form.*

Proof. This theorem follows from the proof of Theorem 2.1, since essentially the same system is obtained in this case. Namely, it is enough to change letter a to c, letter b to d, letter z to w, and letter α to β, in the system

$$z_{n+1} = \alpha w_n^a z_{n-1}^b, \quad w_{n+1} = \beta w_{n-1}^d, \quad n \in \mathbb{N}_0,$$

and it will become system (2.1). □

Theorem 2.4. *Assume that $a, b, c, d \in \mathbb{Z}$, $a \neq 0 \neq c$, $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and z_{-1}, z_0, w_{-1},$w_0 \in \mathbb{C} \setminus \{0\}$. Then system* (1.3) *is solvable in closed form.*

Proof. Using the conditions $\alpha, \beta \in \mathbb{C} \setminus \{0\}$ and $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$ into the equations in (1.3) it is easy to see by using the induction that $z_n \neq 0 \neq w_n$ for every $n \geq -1$. Hence, from the first equation in (1.3) for any such solution we have

$$w_n^a = \frac{z_{n+1}}{\alpha z_{n-1}^b}, \quad n \in \mathbb{N}_0. \tag{2.29}$$

By taking the second equation in (1.3) to the a-th power is obtained

$$w_{n+1}^a = \beta^a z_n^{ac} w_{n-1}^{ad}, \quad n \in \mathbb{N}_0. \tag{2.30}$$

If we use (2.29) into (2.30) we easily get

$$z_{n+2} = \alpha^{1-d} \beta^a z_n^{ac+b+d} z_{n-2}^{-bd}, \quad n \in \mathbb{N}. \tag{2.31}$$

Let $\gamma := \alpha^{1-d}\beta^a$,

$$x_1 = 1, \quad a_1 = ac + b + d, \quad b_1 = -bd. \tag{2.32}$$

From (2.31) we have that

$$z_{2(n+1)+i} = \gamma^{x_1} z_{2n+i}^{a_1} z_{2(n-1)+i}^{b_1}, \quad n \in \mathbb{N}, \tag{2.33}$$

for $i = -1, 0$.

Using (2.33) with $n \to n-1$ into itself, we obtain

$$\begin{aligned} z_{2(n+1)+i} &= \gamma^{x_1} (\gamma z_{2(n-1)+i}^{a_1} z_{2(n-2)+i}^{b_1})^{a_1} z_{2(n-1)+i}^{b_1} \\ &= \gamma^{x_1+a_1} z_{2(n-1)+i}^{a_1 a_1 + b_1} z_{2(n-2)+i}^{b_1 a_1} \\ &= \gamma^{x_2} z_{2(n-1)+i}^{a_2} z_{2(n-2)+i}^{b_2}, \end{aligned} \tag{2.34}$$

for $n \geq 2$ and $i = -1, 0$, where

$$x_2 := x_1 + a_1, \quad a_2 := a_1 a_1 + b_1, \quad b_2 := b_1 a_1. \tag{2.35}$$

Assume that for a $k \geq 2$ have been proved the following equality

$$z_{2(n+1)+i} = \gamma^{x_k} z_{2(n-k+1)+i}^{a_k} z_{2(n-k)+i}^{b_k}, \tag{2.36}$$

for $n \geq k$ and $i = -1, 0$, where

$$x_k := x_{k-1} + a_{k-1}, \quad a_k := a_1 a_{k-1} + b_{k-1}, \quad b_k := b_1 a_{k-1}. \tag{2.37}$$

Then, by using (2.33) with $n \to n-k$ into (2.36), we have

$$\begin{aligned} z_{2(n+1)+i} &= \gamma^{x_k} z_{2(n-k+1)+i}^{a_k} z_{2(n-k)+i}^{b_k} \\ &= \gamma^{x_k} (\gamma z_{2(n-k)+i}^{a_1} z_{2(n-k-1)+i}^{b_1})^{a_k} z_{2(n-k)+i}^{b_k} \\ &= \gamma^{x_k+a_k} z_{2(n-k)+i}^{a_1 a_k + b_k} z_{2(n-k-1)+i}^{b_1 a_k} \\ &= \gamma^{x_{k+1}} z_{2(n-k)+i}^{a_{k+1}} z_{2(n-k-1)+i}^{b_{k+1}}, \end{aligned} \tag{2.38}$$

for $n \geq k+1$ and $i = -1, 0$, where

$$x_{k+1} := x_k + a_k, \quad a_{k+1} := a_1 a_k + b_k, \quad b_{k+1} := b_1 a_k. \tag{2.39}$$

From (2.34), (2.35), (2.38), (2.39) and the induction we obtain that (2.36) and (2.37) hold for all natural numbers k and n such that $2 \leq k \leq n$. Note that (2.36) also holds for $1 \leq k \leq n$.

For $k = n$, (2.36) becomes

$$z_{2(n+1)+i} = \gamma^{x_n} z_{2+i}^{a_n} z_i^{b_n},$$

for $n \in \mathbb{N}$ and $i = -1, 0$, from which along with

$$z_1 = \alpha w_0^a z_{-1}^b, \quad z_2 = \alpha w_1^a z_0^b = \alpha\beta^a z_0^{ac+b} w_{-1}^{ad},$$

it follows that

$$\begin{aligned} z_{2n} &= \gamma^{x_{n-1}} z_2^{a_{n-1}} z_0^{b_{n-1}} \\ &= (\alpha^{1-d}\beta^a)^{x_{n-1}} (\alpha\beta^a z_0^{ac+b} w_{-1}^{ad})^{a_{n-1}} z_0^{b_{n-1}} \\ &= \alpha^{(1-d)x_{n-1}+a_{n-1}} \beta^{a x_{n-1}+a a_{n-1}} z_0^{(ac+b)a_{n-1}+b_{n-1}} w_{-1}^{ad a_{n-1}}, \end{aligned} \tag{2.40}$$

$$\begin{aligned} z_{2n-1} &= \gamma^{x_{n-1}} z_1^{a_{n-1}} z_{-1}^{b_{n-1}} \\ &= (\alpha^{1-d}\beta^a)^{x_{n-1}} (\alpha w_0^a z_{-1}^b)^{a_{n-1}} z_{-1}^{b_{n-1}} \\ &= \alpha^{(1-d)x_{n-1}+a_{n-1}} \beta^{a x_{n-1}} w_0^{a a_{n-1}} z_{-1}^{b a_{n-1}+b_{n-1}}, \end{aligned} \tag{2.41}$$

for $n \geq 2$. From (2.37) and since $x_1 = 1$, we have that

$$a_k = a_1 a_{k-1} + b_1 a_{k-2}, \quad k \geq 3, \tag{2.42}$$

$$x_k = 1 + \sum_{j=1}^{k-1} a_j, \quad k \in \mathbb{N}. \tag{2.43}$$

In what follows we consider three cases separately, that is, $b = 0$, $d = 0$ and $bd \neq 0$.

Case $b = 0$. In this case (2.42) is

$$a_k = a_1 a_{k-1} = (ac + d) a_{k-1}, \quad k \in \mathbb{N},$$

from which it follows that

$$a_k = a_1 (ac + d)^{k-1} = (ac + d)^k, \quad k \in \mathbb{N}, \tag{2.44}$$

and which along with $b_1 = 0$ and $b_k = b_1 a_{k-1}$, $k \geq 2$, implies that

$$b_k = 0, \quad k \in \mathbb{N}. \tag{2.45}$$

From (2.43) and (2.44) we have

$$x_k = 1 + \sum_{j=1}^{k-1} (ac + d)^j, \quad k \in \mathbb{N},$$

from which it follows that

$$x_k = \frac{1 - (ac + d)^k}{1 - ac - d}, \quad k \in \mathbb{N}, \tag{2.46}$$

if $ac + d \neq 1$, while

$$x_k = k, \quad k \in \mathbb{N}, \tag{2.47}$$

if $ac + d = 1$.

From (2.40), (2.41), (2.44), (2.45) and (2.46), we have that

$$z_{2n} = \alpha^{\frac{1-d-ac(ac+d)^{n-1}}{1-ac-d}} \beta^{a\frac{1-(ac+d)^n}{1-ac-d}} z_0^{ac(ac+d)^{n-1}} w_{-1}^{ad(ac+d)^{n-1}}, \tag{2.48}$$

$$z_{2n-1} = \alpha^{\frac{1-d-ac(ac+d)^{n-1}}{1-ac-d}} \beta^{a\frac{1-(ac+d)^{n-1}}{1-ac-d}} w_0^{a(ac+d)^{n-1}}, \tag{2.49}$$

for $n \geq 2$, if $ac + d \neq 1$, while from (2.40), (2.41), (2.44), (2.45) and (2.47), we have

$$z_{2n} = \alpha^{(1-d)n+d} \beta^{an} z_0^{ac} w_{-1}^{ad}, \tag{2.50}$$

$$z_{2n-1} = \alpha^{(1-d)n+d} \beta^{a(n-1)} w_0^a, \tag{2.51}$$

for $n \ge 2$, if $ac + d = 1$.

Case $d = 0$**.** In this case (2.42) is

$$a_k = a_1 a_{k-1} = (ac + b)a_{k-1}, \quad k \in \mathbb{N},$$

from which it follows that

$$a_k = a_1(ac + b)^{k-1} = (ac + b)^k, \quad k \in \mathbb{N}, \tag{2.52}$$

and which along with $b_1 = 0$ and $b_k = b_1 a_{k-1}$, $k \ge 2$, implies that (2.45) holds.

From (2.43) and (2.52) we have

$$x_k = 1 + \sum_{j=1}^{k-1}(ac + b)^j, \quad k \in \mathbb{N},$$

from which it follows that

$$x_k = \frac{1 - (ac + b)^k}{1 - ac - b}, \quad k \in \mathbb{N}, \tag{2.53}$$

if $ac + b \ne 1$, while

$$x_k = k, \quad k \in \mathbb{N}, \tag{2.54}$$

if $ac + b = 1$. From (2.40), (2.41), (2.45), (2.52) and (2.53), we have

$$z_{2n} = \alpha^{\frac{1-(ac+b)^n}{1-ac-b}} \beta^{a\frac{1-(ac+b)^n}{1-ac-b}} z_0^{(ac+b)^n}, \tag{2.55}$$

$$z_{2n-1} = \alpha^{\frac{1-(ac+b)^n}{1-ac-b}} \beta^{a\frac{1-(ac+b)^{n-1}}{1-ac-b}} w_0^{a(ac+b)^{n-1}} z_{-1}^{b(ac+b)^{n-1}}, \tag{2.56}$$

for $n \ge 2$, if $ac + b \ne 1$, while from (2.40), (2.41), (2.45), (2.52) and (2.54), we have

$$z_{2n} = \alpha^n \beta^{an} z_0, \tag{2.57}$$

$$z_{2n-1} = \alpha^n \beta^{a(n-1)} w_0^a z_{-1}^b, \tag{2.58}$$

for $n \in \mathbb{N}$, if $ac + b = 1$.

Case $b \ne 0 \ne d$**.** Let $\lambda_{1,2}$ be the roots of the characteristic polynomial

$$P(\lambda) = \lambda^2 - (ac + b + d)\lambda + bd, \tag{2.59}$$

associate to difference equation (2.42).

Recall that then the general solution to equation (2.42) is

$$a_n = c_1\lambda_1^n + c_2\lambda_2^n, \quad n \in \mathbb{N},$$

if $(ac + b + d)^2 \ne 4bd$, where c_1 and c_2 are arbitrary constants, while in the case $(ac + b + d)^2 = 4bd$, it has the following form

$$u_n = (d_1 n + d_2)\lambda_1^n, \quad n \in \mathbb{N},$$

where d_1 and d_2 are arbitrary constants.

By some calculation and using the values for a_1 and a_2, if $(ac + b + d)^2 \ne 4bd$, we obtain

$$a_k = \frac{\lambda_1^{k+1} - \lambda_2^{k+1}}{\lambda_1 - \lambda_2}, \quad k \in \mathbb{N}, \tag{2.60}$$

while if $(ac + b + d)^2 = 4bd$, we obtain

$$a_k = (k + 1)\lambda_1^k, \quad k \in \mathbb{N}. \tag{2.61}$$

By using (2.60) into the third equation in (2.37) we obtain

$$b_k = -bda_{k-1} = -bd\frac{\lambda_1^k - \lambda_2^k}{\lambda_1 - \lambda_2}, \quad k \ge 2, \tag{2.62}$$

if $(ac+b+d)^2 \neq 4bd$, while if $(ac+b+d)^2 = 4bd$, by using (2.61) into the third equation in (2.37) we obtain

$$b_k = -bda_{k-1} = -bdk\lambda_1^{k-1}, \quad k \geq 2. \tag{2.63}$$

On the other hand, by using (2.60) into (2.43) we obtain

$$x_k = 1 + \sum_{j=1}^{k-1} \frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} = \frac{(\lambda_2 - 1)\lambda_1^{k+1} - (\lambda_1 - 1)\lambda_2^{k+1} + \lambda_1 - \lambda_2}{(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2)}, \tag{2.64}$$

for $k \in \mathbb{N}$, if $(ac+b+d)^2 \neq 4bd$, while if $(ac+b+d)^2 = 4bd$, by using (2.61) into (2.43) we obtain

$$x_k = 1 + \sum_{j=1}^{k-1} (j+1)\lambda_1^j = \frac{1 - (k+1)\lambda_1^k + k\lambda_1^{k+1}}{(1-\lambda_1)^2}, \quad k \in \mathbb{N}. \tag{2.65}$$

From (2.40), (2.41), (2.60), (2.62) and (2.64), we obtain formulas for z_n in the case $(ac+b+d)^2 \neq 4bd$, while from (2.40), (2.41), (2.61), (2.63) and (2.65), we obtain formulas for z_n in the case $(ac+b+d)^2 = 4bd$.

From the second equation in (1.3) we have

$$z_n^c = \frac{w_{n+1}}{\beta w_{n-1}^d}, \quad n \in \mathbb{N}_0. \tag{2.66}$$

By taking the first equation in (1.3) to the c-th power is obtained

$$z_{n+1}^c = \alpha^c w_n^{ac} z_{n-1}^{bc}, \quad n \in \mathbb{N}_0. \tag{2.67}$$

Using (2.66) into (2.67) we easily obtain

$$w_{n+2} = \alpha^c \beta^{1-b} w_n^{ac+b+d} w_{n-2}^{-bd}, \quad n \in \mathbb{N}, \tag{2.68}$$

which differs from (2.31) only for the coefficient $\alpha^c\beta^{1-b}$.

Let $\delta := \alpha^c\beta^{1-b}$,

$$y_1 = 1, \quad a_1 = ac+b+d, \quad b_1 = -bd. \tag{2.69}$$

By the above described procedure for z_{2n+i}, $n \in \mathbb{N}_0$, $i = -1, 0$, it can be shown that for any $k \in \mathbb{N}$ such that $1 \leq k \leq n$, hold

$$w_{2(n+1)+i} = \delta^{y_k} w_{2(n-k+1)+i}^{a_k} w_{2(n-k)+i}^{b_k}, \tag{2.70}$$

for $n \geq k$ and $i = -1, 0$, where

$$y_k := y_{k-1} + a_{k-1}, \quad a_k := a_1 a_{k-1} + b_{k-1}, \quad b_k = b_1 a_{k-1}. \tag{2.71}$$

For $k = n$, (2.70) becomes

$$w_{2(n+1)+i} = \delta^{y_n} w_{2+i}^{a_n} w_i^{b_n},$$

for $n \in \mathbb{N}$ and $i = -1, 0$, from which along with

$$w_1 = \beta z_0^c w_{-1}^d, \quad w_2 = \beta z_1^c w_0^d = \alpha^c \beta w_0^{ac+d} z_{-1}^{bc},$$

it follows that

$$\begin{aligned} w_{2n} &= \delta^{y_{n-1}} w_2^{a_{n-1}} w_0^{b_{n-1}} = (\alpha^c\beta^{1-b})^{y_{n-1}} (\alpha^c \beta w_0^{ac+d} z_{-1}^{bc})^{a_{n-1}} w_0^{b_{n-1}} \\ &= \alpha^{cy_{n-1}+ca_{n-1}} \beta^{(1-b)y_{n-1}+a_{n-1}} z_{-1}^{bca_{n-1}} w_0^{(ac+d)a_{n-1}+b_{n-1}}, \end{aligned} \tag{2.72}$$

$$\begin{aligned} w_{2n-1} &= \delta^{y_{n-1}} w_1^{a_{n-1}} w_{-1}^{b_{n-1}} = (\alpha^c\beta^{1-b})^{y_{n-1}} (\beta z_0^c w_{-1}^d)^{a_{n-1}} w_{-1}^{b_{n-1}} \\ &= \alpha^{cy_{n-1}} \beta^{(1-b)y_{n-1}+a_{n-1}} w_{-1}^{da_{n-1}+b_{n-1}} z_0^{ca_{n-1}}, \end{aligned} \tag{2.73}$$

for $n \geq 2$. From (2.71) and the fact that $y_1 = 1$, we see that the sequence $(a_k)_{k\in\mathbb{N}}$ satisfies recurrent relation (2.42), while

$$y_k = 1 + \sum_{j=1}^{k-1} a_j, \quad k \in \mathbb{N}, \tag{2.74}$$

which means that

$$y_k = x_k, \quad k \in \mathbb{N}. \tag{2.75}$$

Now we consider three cases separately, that is, $b = 0$, $d = 0$ and $bd \neq 0$.

Case $b = 0$. From the above consideration it is clear that for a_k holds formula (2.44), for b_k formula (2.45), while for y_k, we have

$$y_k = \frac{1-(ac+d)^k}{1-ac-d}, \quad k \in \mathbb{N}, \tag{2.76}$$

if $ac + d \neq 1$, while

$$y_k = k, \quad k \in \mathbb{N}, \tag{2.77}$$

if $ac + d = 1$.

From (2.44), (2.45), (2.72), (2.73) and (2.76), we have that

$$w_{2n} = \alpha^{c\frac{1-(ac+d)^n}{1-ac-d}} \beta^{\frac{1-(ac+d)^n}{1-ac-d}} w_0^{(ac+d)^n}, \tag{2.78}$$

$$w_{2n-1} = \alpha^{c\frac{1-(ac+d)^{n-1}}{1-ac-d}} \beta^{\frac{1-(ac+d)^n}{1-ac-d}} w_{-1}^{d(ac+d)^{n-1}} z_0^{c(ac+d)^{n-1}}, \tag{2.79}$$

for $n \geq 2$, if $ac+d \neq 1$, while from (2.44), (2.45), (2.72), (2.73) and (2.77), we have that

$$w_{2n} = \alpha^{cn}\beta^n w_0, \tag{2.80}$$

$$w_{2n-1} = \alpha^{c(n-1)}\beta^n w_{-1}^d z_0^c, \tag{2.81}$$

for $n \in \mathbb{N}$, if $ac + d = 1$.

Case $d = 0$. From the above consideration it is clear that for a_k holds formula (2.52), for b_k formula (2.45), while for y_k, we have

$$y_k = \frac{1-(ac+b)^k}{1-ac-b}, \quad k \in \mathbb{N}, \tag{2.82}$$

if $ac + b \neq 1$, while

$$y_k = k, \quad k \in \mathbb{N}, \tag{2.83}$$

if $ac + b = 1$.

From (2.45), (2.52), (2.72), (2.73) and (2.82), we have that

$$w_{2n} = \alpha^{c\frac{1-(ac+b)^n}{1-ac-b}} \beta^{\frac{1-b-ac(ac+b)^{n-1}}{1-ac-b}} z_{-1}^{bc(ac+b)^{n-1}} w_0^{ac(ac+b)^{n-1}}, \tag{2.84}$$

$$w_{2n-1} = \alpha^{c\frac{1-(ac+b)^{n-1}}{1-ac-b}} \beta^{\frac{1-b-ac(ac+b)^{n-1}}{1-ac-b}} z_0^{c(ac+b)^{n-1}}, \tag{2.85}$$

for $n \geq 2$, if $ac+b \neq 1$, while from (2.45), (2.52), (2.72), (2.73) and (2.83), we have that

$$w_{2n} = \alpha^{cn}\beta^{(1-b)n+b} z_{-1}^{bc} w_0^{ac}, \tag{2.86}$$

$$w_{2n-1} = \alpha^{c(n-1)}\beta^{(1-b)n+b} z_0^c, \tag{2.87}$$

for $n \in \mathbb{N}$, if $ac + b = 1$.

Case $b \neq 0 \neq d$. Let $\lambda_{1,2}$ be the roots of characteristic polynomial (2.59) associated to difference equation (2.42). From the above consideration we see that formulas (2.60)-(2.63) hold and that

$$y_k = 1 + \sum_{j=1}^{k-1} \frac{\lambda_1^{j+1} - \lambda_2^{j+1}}{\lambda_1 - \lambda_2} = \frac{(\lambda_2 - 1)\lambda_1^{k+1} - (\lambda_1 - 1)\lambda_2^{k+1} + \lambda_1 - \lambda_2}{(\lambda_1 - 1)(\lambda_2 - 1)(\lambda_1 - \lambda_2)}, \tag{2.88}$$

for $k \in \mathbb{N}$, if $(ac+b+d)^2 \neq 4bd$, while if $(ac+b+d)^2 = 4bd$, then we have that the following formula holds

$$y_k = 1 + \sum_{j=1}^{k-1} (j+1)\lambda_1^j = \frac{1 - (k+1)\lambda_1^k + k\lambda_1^{k+1}}{(1-\lambda_1)^2}, \quad k \in \mathbb{N}. \tag{2.89}$$

Using formulas (2.60)-(2.63), (2.88) and (2.89) into (2.72) and (2.73) are obtained closed form formulas for sequence w_n in this case, finishing the proof. □

From the proof of Theorem 2.4 we obtain the following corollary.

Corollary 2.5. *Consider system* (1.3) *with $a, b, c, d \in \mathbb{Z}$, $ac \neq 0$. Assume that $z_{-1}, z_0, w_{-1}, w_0 \in \mathbb{C} \setminus \{0\}$. Then the following statements are true.*

(a) *If $b = 0$, $ac + d \neq 1$, then the general solution of system* (1.3) *is given by* (2.48), (2.49), (2.78) *and* (2.79).
(b) *If $b = 0$, $ac + d = 1$, then the general solution of system* (1.3) *is given by* (2.50), (2.51), (2.80) *and* (2.81).
(c) *If $d = 0$, $ac + b \neq 1$, then the general solution of system* (1.3) *is given by* (2.55), (2.56), (2.84) *and* (2.85).
(d) *If $d = 0$, $ac + b = 1$, then the general solution of system* (1.3) *is given by* (2.57), (2.58), (2.86) *and* (2.87).
(e) *If $bd \neq 0$ and $(ac+b+d)^2 \neq 4bd$, then the general solution of system* (1.3) *is given by*

$$z_{2n} = \alpha^{x_n - dx_{n-1}} \beta^{ax_n} z_0^{(ac+b)a_{n-1} - bda_{n-2}} w_{-1}^{ada_{n-1}}, \tag{2.90}$$

$$z_{2n-1} = \alpha^{x_n - dx_{n-1}} \beta^{ax_{n-1}} w_0^{aa_{n-1}} z_{-1}^{ba_{n-1} - bda_{n-2}}, \tag{2.91}$$

$$w_{2n} = \alpha^{cx_n} \beta^{x_n - bx_{n-1}} z_{-1}^{bca_{n-1}} w_0^{(ac+d)a_{n-1} - bda_{n-2}}, \tag{2.92}$$

$$w_{2n-1} = \alpha^{cx_{n-1}} \beta^{x_n - bx_{n-1}} w_{-1}^{da_{n-1} - bda_{n-2}} z_0^{ca_{n-1}}, \tag{2.93}$$

where $(a_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are given by (2.60) *and* (2.64) *respectively.*
(f) *If $bd \neq 0$ and $(ac+b+d)^2 = 4bd$, then the general solution of system* (1.3) *is given by formulas* (2.90)-(2.93), *where $(a_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are given by* (2.61) *and* (2.65) *respectively.*

Proof. Statements (a)–(d) follow directly from the proof of Theorem 2.4.

(e), (f): By using (2.37) and (2.75) in (2.40), (2.41), (2.72) and (2.73), and some calculation formulas (2.90)-(2.93) follow. □

Remark 2.6. A relatively long and tedious calculation shows that formulas (2.90)-(2.93) really present general solution to system (1.3) in the "main" case, that is, $abcd \neq 0$. The authors have verified this, but since such calculations are traditionally not quite suitable for publication we omit the calculation and left it to the reader as an exercise.

Remark 2.7. Bearing in mind that sequence a_n is defined for $n \in \mathbb{N}$, one may think that formulas (2.90)-(2.93) hold only for $n \geq 3$. However, since $b_1 = bd \neq 0$, by using recurrent relation (2.42) we see that sequence a_n can be prolonged for $n \in \mathbb{Z}$. Indeed, for $k = 2$ equation (2.42) becomes $a_2 = a_1 a_1 + b_1 a_0$, from which it follows that $a_0 = (a_2 - a_1 a_1)/b_1 = 1$. In general, if a_{n-1} and a_n are defined for some $n \in \mathbb{Z}$, then a_{n-2} can be calculated/defined by using the following consequence of (2.42)

$$a_{n-2} := \frac{1}{b_1}(a_n - a_1 a_{n-1}). \tag{2.94}$$

By using (2.94) for $n = 1$ is obtained $a_{-1} = 0$, from which along with (2.94) for $n = 0$ is obtained $a_{-2} = -1/(bd)$. Consequently, x_n can be calculated/defined also for every $n \in \mathbb{Z}$, by using the relation $x_{n-1} = x_n - a_{n-1}$. For $n = 1$ is obtained $x_0 = 0$. Using this "prolongation" it is easy to verify that (2.90)-(2.93) hold for every $n \in \mathbb{N}$.

Acknowledgements. Stevo Stević was supported by the Serbian Ministry of Education and Science projects III 41025 and III 44006. Bratislav Iričanin was supported by the Serbian Ministry of Education and Science projects III 41025 and OI 171007. The work of Zdeněk Šmarda was realized in CEITEC - Central European Institute of Technology supported by the Czech Science Foundation under the project 16-08549S.

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