Time Domain Analysis of Linear Systems Using Laplace Inversion

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Abstract

An efficient method of time domain analysis of linear systems is proposed. This method is based on the state space approach and on the numerical Laplace inversion. The problems of the sampled-data input signal and steady-state response computation are discussed.

Keywords:

Numerical Laplace inversion, state approach, two-graph modified nodal analysis, steady-state response.

1. Introduction

During the developing of system *NAF* for the computer design and optimization of continuous-time filters [1], the following task had to be solved:

The N samples of continuous-time signal are scanned so that the Nyquist's rate is exceeded minimally 100 times. However, the equidistant sampling need not be necessarily used. The s-domain transfer function coefficients of linear filter are known. They are obtained by means of *CAD* system *NAF*. The simulation of initially-at-rest filter response to given input signal is required. Under the term "initially-at-rest", the filter response to the input signal by zero system initial energy is considered [2].

The necessity to solve this problem led to the developing of the Turbo Pascal unit for time domain analysis. That unit is designed for the precise time-domain analysis of designed filters from their transfer functions. The unit performs following functions:

1) To compute the impulse and step response including Dirac impulses identification.
2) To compute the initially-at-rest filter response on the rectangular and trapezoidal signals.
3) To compute the initially-at-rest filter response on the "customer's" signal whose samples are stored in a data file.
4) To compute the periodic steady state on the assumption that the given signal represents just one period of its periodic continuation.

As mentioned below, the used method of Laplace inversion is also effective when the set of circuit equations describing the filter is known instead of its transfer function.

To realize items 1 to 4, the algorithm of numerical Laplace inversion has appeared to be most efficient [3]. This algorithm operates with the state model of filter, which corresponds with its transfer function. The state approach facilitates steady state response computation. The algorithm leads to highly accurate analysis.

2. The state approach

The foundations of Laplace inversion are described in [3],[4], [5],[6]. Let us consider the transfer function $K(s)$ of n-th order linear system with single input $w(t)$ and single output $y(t)$:

$$K(s)=\frac{L\{y(t)\}}{L\{w(t)\}}=\frac{Y(s)}{W(s)}=\frac{\sum_{i=0}^{m} a_i s^i}{s^n+\sum_{i=1}^{n-1} b_i s^i}\ ,m\le n,\ n\ge 1, \qquad (1)$$

where $L\{\ \}$ is the operator of Laplace transform. As shown in [6], the state space description corresponds with this transfer function. Applying the Laplace transform, the state equations can be rewritten as follows [6]:

$$(s\mathbf{E}-\mathbf{A})\mathbf{X}(s)=\mathbf{B}W(s)+\mathbf{x}(0)\ , \qquad (2)$$

where $\mathbf{X}(s)$ and $W(s)$ are the Laplace transforms of the $(nx1)$ state vector $\mathbf{x}(t)$ and the input signal $w(t)$, respectively. $\mathbf{x}(0)$ is the state vector of initial conditions in time t=0 and $\mathbf{E}$ is the (nxn) identity matrix. The state matrix $\mathbf{A}$ and the vector $\mathbf{B}$ have following structure:

$$\mathbf{A}=\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -b_0 & -b_1 & -b_2 & \dots & -b_{n-1} \end{bmatrix},\ \mathbf{B}=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

The output equation in time domain can be expressed as follows:

$$y(t) = \sum_{i=0}^{n-1} (a_i - a_n b_i) x_{i+1}(t) + a_n w(t) \ , \qquad (3)$$

where x_i, $i=1,2,..,n$ are the elements of state vector.

In this manner, the problem of time-domain analysis is transformed to the problem of the Laplace inversion of matrix equation (2). After Laplace inversion, the time-domain state vector $\mathbf{x}(t)$ is obtained. The output signal is then calculated from equation (3).

3. Initially-at-rest response computation

The so-called "resetting principle" [4] is implied in Fig.1.

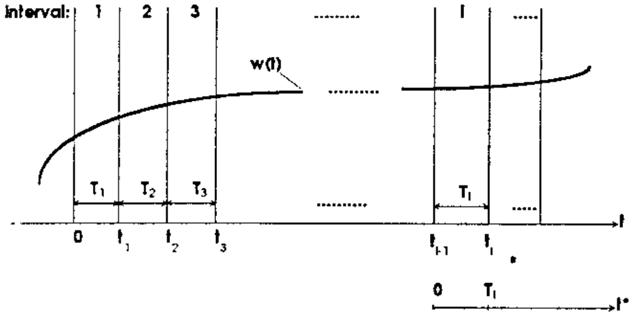

Fig.1
The resetting principle.

Let us divide time axis to partial intervals. The "single-step" Laplace inversion will be performed in each i-th time interval to obtain state vector $\mathbf{x}(t)$ from equation (2). This equation is rewritten for the i-th interval as follows:

$$(s_i \mathbf{E} - \mathbf{A}) \mathbf{X}^i = \mathbf{B} W^i(s_i) + \mathbf{x}(t_{i-1}) \ . \qquad (4)$$

On the right hand side of this equation, the $\mathbf{x}(t_{i-1})$ is the state vector, computed in the last step. W^i is the Laplace transform of input signal, considering the origin of time coordinate at the beginning of solved time interval and $w(t)$=0 for $t<t_{i-1}$. The corresponding shift of time origin of time axis $t'=t-t_{i-1}$ is in Fig.1.

In accordance with [4], the equation (4) must be solved for the set of precalculated complex constants s_i. In the simplest case, the single constant

$$s_i = (1+j)/T_i$$

is assumed for i-th interval.

An efficient method for the calculation of vector $\mathbf{X}^i$ is described in [6]. This method is based on the LU decomposition of matrix $s_i\mathbf{E} - \mathbf{A}$. In case of constant step $T_i = T$, the LU decomposition is performed only once. The solution of (4) is then very economical for various right hand sides. As soon as the vector $\mathbf{X}^i$ is found, the time-domain solution is given by inverse Laplace transform

$$\mathbf{x}(t_i) = -2\mathrm{Im}\{\mathbf{X}^i\}/T_i \ . \qquad (5)$$

The output signal $y(t_i)$ at $t'= t_i$ is calculated from the output equation (3).

By aforementioned procedure, the output signal values in the time instants $t_1, t_2, t_3, \ldots$ can be computed. The time steps T_i must be chosen sufficiently short with the aim to make "single-step" Laplace inversion (5) errors negligible.

4. Initial value and Dirac impulse identification

It is often necessary to compute precise initial value of response $y(0_+)$. This value can be found by the initial value theorem. For Laplace transform of rational fractions, this approach can be implemented without problems. After some algebraic manipulation, the incidental Dirac impulse can be separated and its strength found.

Tab.1 Various types of interpolation.

interpolation	time domain: $w^{int}(t')$	Laplace domain: $W^i(s)$ $W^i(s_i), s_i = (1+j)/T_i$
impulse "interpolation"	$w(t_{i-1})T_i\delta(t')$	$w(t_{i-1})T_i$ $w(t_{i-1})T_i$
step interpolation	$w(t_{i-1})$	$w(t_{i-1})/s$ $w(t_{i-1})T_i(1-j)/2$
linear interpolation	$w(t_{i-1}) + [w(t_i) - w(t_{i-1})]t'/T_i$	$w(t_{i-1})/s + [w(t_i) - w(t_{i-1})]T_i/2$ $[w(t_{i-1}) - jw(t_i)]T_i/2$
cubic spline interpolation	$a_0^i + a_1^i t' + a_2^i t'^2 + a_3^i t'^3$	$a_0^i/s + a_1^i/s^2 + 2a_2^i/s^3 + 6a_3^i/s^4$ $\{a_0^i - a_2^i T_i^2 - 3a_3^i T_i^3 - j[a_0^i + a_1^i T_i + a_2^i T_i^2]\}T_i/2$

Remark: sampled signal $w(t_i)$, $i = 0,1,2,\ldots$ step i: $t' = t - t_{i-1}$

However, the limiting approach is not suitable for computer implementation in case of Laplace inversion of matrix equation. To compute initial value, the two-step procedure may be used [5]. In the first step, we choose the integration step T_1 forward to compute vector $\mathbf{x}(T_1)$. Next we take the same integration step back to obtain correct initial vector $\mathbf{x}(0_+)$.

The strength of incidental Dirac impulse for t=0 is computed also by two-step method. For more details, see [5].

5. Sampled-data input signal

The input signal $w(t)$ is often available in samples, obtained by measurement or by other way. It is necessary to perform interpolation to construct continuous-time signal as the input signal of continuous-time system. Four basic types of interpolation are set out in the Tab.1. It should be noted that the "impulse interpolation" is not interpolation in the sense as it is generally understood, because the identity $w^{int}(t_i) = w(t_i)$, i=0,1,.. is not fulfilled.

The choice of interpolation will depend both on the required accuracy of time analysis and on the actual character of input signal the samples of which are at disposal. It is also necessary to take the transmission properties of the simulated circuit into account. The impulse and step interpolations give good results if the Nyquist's condition is fulfilled with a satisfactory reserve. The system must not have a "high-pass character" because the fast changes of input signal must not be amplified. The computation of the Laplace transform $W^s(s)$ is then very simple.

The impulse (step) interpolation can be chosen if the original input signal has the character of narrow return-to-zero impulses (sample-hold impulses).

The linear interpolation provides very good results in case of various types of circuits.

The cubic spline interpolation is appointed for the precision calculation on the assumption that the original input signal is smooth but its sampling was performed with relative low rate near the Nyquist's rate. The inverse Laplace transform with this interpolation is time-consuming (see Tab.1).

In the Turbo-Pascal unit, the step interpolation is preselected as the compromise between the analysis rate and accuracy. The user can choose other types of interpolation if he wishes.

6. Periodic steady-state response

A simple method of periodic steady-state computation is described in [7]. Some improvements were made in [8]. This method is based on the state approach. To approximate the derivatives by difference formulas, the matrix differential state equation is replaced by the set of algebraic equations.

To utilize the algorithm of Laplace inversion, another efficient method of steady-state response computation can be developed.

In case of periodic steady-state response with the period of repetition T_p, the vectors $\mathbf{x}(0)$ and $\mathbf{x}(T_p)$ are equal:

$$\mathbf{x}(0) = \mathbf{x}(T_p) \ . \tag{6}$$

After finding these vectors, computation of the steady-state response is easy.

It is well known that the solution of equation (2) in the time domain consists of the initially-at-rest part $\mathbf{x}_{iar}(t)$ and of the zero-input-response $\mathbf{x}_{zir}(t)$. For the time $t=T_p$,

$$\mathbf{x}(T_p) = \mathbf{x}_{iar}(T_p) + \mathbf{x}_{zir}(T_p) \ . \tag{7}$$

The zero-input-response vector is proportional to the vector of initial conditions $\mathbf{x}(0)$:

$$\mathbf{x}_{zir}(T_p) = \mathbf{g}^*(T_p)\mathbf{x}(0) \ . \tag{8}$$

For the state system (2), the matrix $\mathbf{g}^*(T_p)$ is given by matrix expression

$$\mathbf{g}^*(T_p) = e^{\mathbf{A}T_p} \ . \tag{9}$$

Arranging equations (7), (8) and (9) yields

$$\mathbf{x}(0) = [\mathbf{E} - \mathbf{g}^*(T_p)]^{-1}\mathbf{x}_{iar}(T_p) \ . \tag{10}$$

The vector $\mathbf{x}_{iar}(T_p)$ can be calculated by Laplace inversion of equation (2) during the period T_p under zero initial conditions. The matrix $\mathbf{g}^*(T_p)$ can be also obtained by Laplace inversion of (2) for $W(s)$=0. To calculate the k-th column of $\mathbf{g}^*(T_p)$, we take $x_k(0) = 1$, $x_i(0) = 0$, $i \neq k$.

The matrix $\mathbf{g}^*(T_p)$ can be calculated even more efficiently by expanding the exponential function in (9) into Taylor series and taking limited number of N terms:

$$\mathbf{E} - e^{\mathbf{A}T_p} \approx -\sum_{k=1}^{N} \frac{(\mathbf{A}T_p)^k}{k!} \ . \tag{11}$$

The number of elements N must be chosen with respect to the acceptable error ε. For ε the following expression can be derived:

$$\varepsilon = \left\| \frac{(\mathbf{A}T_p)^N}{N!} \right\| / \left\| \sum_{k=1}^{N} \frac{(\mathbf{A}T_p)^k}{k!} \right\| \ .$$

The symbol $\| \ \|$ designates the matrix norm.

As soon as the calculation of vector $\mathbf{x}(0)$ is finished, the steady-state response is obtained by the Laplace inversion of (2).

7. Time-domain analysis based on two-graph modified nodal approach

To perform the time-domain analysis according to the above described method, we must know the s-domain transfer function of the system. However, the matrix equation based on the two-graph modified nodal approach (2-graph *MNA*)[4] can be used instead of the transfer function. As will be shown, the virtually same method can be used even in this case.

The 2-graph *MNA* leads to the matrix equation

$$(\mathbf{G}+s\mathbf{C})\mathbf{V}(s)=\mathbf{D}w(s)+\mathbf{C}\mathbf{v}(0)\ , \quad (12)$$

where **V** is the vector of nodal voltages and incidental branch currents, w is the input signal, and **v**(0) is the vector of initial conditions. $\mathbf{G}+s\mathbf{C}$ is the modified admittance matrix. **D** is an incidence column vector. The computer formulation of this equation is well known.

Comparing equations (12) and (2), the equation (12) represents a special state-space model of linear system. Applying Laplace inversion, the aforementioned procedures can be used. For the steady-state response computation, however, the procedure (11) can not be used for matrix $\mathbf{g}^*(T_p)$ evaluation. This matrix must be now obtained by Laplace inversion.

8. Conclusions

The basic idea of time-domain computer-oriented analysis of linear systems by Laplace inversion is described in this paper. The problems of initially-at-rest response and periodic steady-state response computation are discussed. Various interpolations of sampled-data user's input signals are considered. Either s-domain transfer function or two-graph modified nodal approach can be assumed as the model of linear system.

The described principles have been implemented in a Turbo Pascal unit used as a part of the *CAD* system *NAF*.

9. References

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