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A characterization of sliding vectors by dual numbers, some dual curves and the screw calculus

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Abstract

Moving vectors, motors and screws are introduced and the mathematical background is explained: dual numbers are used for their description. Further, the paper deals with the dual space and curves in it. Some examples (in particular helices) are given. Newly, so called Spivak's dual curve is studied from the point of view of its natural parameterization; it is presented that curvature and torsion at zero are not able to distinguish this curve from the plane analogy again – as in the real case. It is also mentioned the applicability of the theory in mechanics.

Keywords: sliding vector; dual number; dual space; curves in dual space; motor; screw; curvature; torsion

1. Sliding vectors

Let us start with a very apposite introduction to sliding vectors, as written in the book Kinetics of Human Motion of Vladimir M. Zatsiorsky. See Zatsiorsky (2002).

Force is a measure of the action of one body on another. Force is a vector quantity. A force can be treated as either a fixed vector or as a sliding vector. When a force is treated as a fixed vector, it is defined by its (a) magnitude, (b) direction, and (c) point of application. When a force is considered a sliding vector, the line of force action rather than the point of application defines the force. Forces are considered sliding vectors when (a) the body of interest is rigid and (b) the resultant external effects, rather than the internal forces and the deformations, are investigated.

In this paper, we deal with dual numbers that very well represent gliding vectors, dual space, and curves in it.

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1.1. Lines

We consider the affine space with the set of points $A = \mathbf{R}^3$ and the vector space $V = \mathbf{R}^3$ possessing the Euclidean inner product. A line L is usually given by two distinct points $X = [x_1, x_2, x_3]$, $Y = [y_1, y_2, y_3]$ or by a point $X = [x_1, x_2, x_3]$ and a non-zero direction vector $\mathbf{u} = (u_1, u_2, u_3)$, so in total by 6 numbers.

However, one can determine a line quite comfortably with only 4 numbers and an information on the choosen axis. So, let a quintuplet (a, b, c, d, ξ) is given, where a, b, c and d are four real numbers and $\xi \in \{x, y, z\}$. The chosen axis represents an axis in which the direction vector $\mathbf{u}$ has component 1 (there exists as $\mathbf{u}$ is non-zero) and the corresponding component of the point X lying on L is 0 (there exists such X because the point coordinate in the direction of the chosen axis is linearly growing). E.g. $(5, 6, 7, 8, z)$ determines the line with $\mathbf{u} = (5, 6, 1)$ and $X = [7, 8, 0]$.

There is also possibility to represent a line in $\mathbf{R}^3$ by 4 or less numbers. First, let us start with lines in $\mathbf{R}^2$. Let us consider two parallel reference lines L_0 given by $x = 0$ and L_1 given by $x = 1$. Then, any line L not parallel to L_0 and L_1 will intersect the two reference lines and the second coordinates of the points of intersection will determine L uniquely. The only exceptions are lines parallel to parallel to L_0 and L_1. In that case, the *first coordinate* determines the line uniquely. Thus, lines in $\mathbf{R}^2$ are represented by 2 or 1 number. Now, we can extend this to lines in $\mathbf{R}^3$. Consider two reference planes P_0 and P_1 planes given by $x = 0$ and $x = 1$, respectively. Then any line L not parallel to P_0 and P_1 will intersect each of the two reference planes and each of the two points of intersection are given by coordinates of which the first coordinate is already fixed, and the remaining two coordinates of each point uniquely specify the line L with 4 numbers. The only exceptions are lines parallel to P_0 and P_1. In that case, the first coordinate specifies a plane that contains the line L and, in that plane, by the case for lines in $\mathbf{R}^2$, that is uniquely determined by 2 numbers, with exceptions noted already. In summary, lines in $\mathbf{R}^3$ are represented by 4, 3 or 2 numbers by this way. (The author thanks a Stack Exchange user nicknamed Somos that suggested this idea.) E.g. (5, 6, 7, 8) determines the line going through $X = [0, 5, 6]$ and $Y = [1, 7, 8]$, (5, 6, 7) the line going through $X = [5, 0, 6]$ and $Y = [5, 1, 7]$ and (5, 6) the line given by $x = 5$ and $y = 6$.

1.2. Sliding vectors

A vector $\mathbf{u} \in V$ is also called the *free vector*.

A *bound vector* is a pair $(X, \mathbf{u})$, where $X \in A$ and $\mathbf{u} \in V$. We will write $\mathbf{u}_{(X)}$.

A non-zero *sliding vector* is a pair $(L, \mathbf{u})$ consisting of a line L in A together with a non-zero vector $\mathbf{u} \in V$ that leaves L invariant. We will write $\mathbf{u}_{((L))}$. The zero can be also considered as a sliding vector.

We have natural projections pr_1 sending a bound vector onto a sliding vector, pr_2 sending a sliding vector onto a free vector and pr_3 which is the composition of the previous two projections and sends a bound vector onto a free vector.

In $\mathbf{R}^3$ a sliding vector can be determined by five numbers, e.g., by the coordinates of the point intersection M of one of the coordinate planes and the line containing the vector (two numbers), by the magnitude of the vector (one number) and by two independent angles α and β between the vector and two of coordinate axes (two numbers), see Borisenko and Tarapov (1968).

Let $\mathbf{u}_{(X)}$ be a bound vector. The *moment* $\mathbf{u}^0$ of $\mathbf{u}_{(X)}$ is defined as the cross product

$$\mathbf{u}^0 = \boldsymbol{\rho}(X) \times pr_3(\mathbf{u}_{(X)})$$

where $\boldsymbol{\rho}(X)$ is a (free) radius vector of the point X. The following assertion holds.

$$\boldsymbol{\rho}(X) \times pr_3(\mathbf{u}_{(X)}) = \boldsymbol{\rho}(Y) \times pr_3(\mathbf{u}_{(Y)}) \textit{ if and only if } Y = X + k\mathbf{u}.$$

Proof. ⇐: Let $Y = [x_1+ku_1, x_2+ku_2, x_3+ku_3]$. Then we observe that $\boldsymbol{\rho}(Y) \times pr_3(\mathbf{u}_{(Y)})$ equals $(x_2u_3 - x_3u_2, x_3u_1 - x_1u_3, x_1u_2 - x_2u_1)$.

⇒: Let $(x_2u_3 - x_3u_2, x_3u_1 - x_1u_3, x_1u_2 - x_2u_1) = (y_2u_3 - y_3u_2, y_3u_1 - y_1u_3, y_1u_2 - y_2u_1)$ and let us denote $(v_1, v_2, v_3) = (y_1 - x_1, y_2 - x_2, y_3 - x_3)$. Then components of the equality read as

$$\boldsymbol{v_2 u_3 - v_3 u_2 = 0}$$
$$\boldsymbol{v_3 u_1 - v_1 u_3 = 0}$$
$$\boldsymbol{v_1 u_2 - v_2 u_1 = 0}$$

and hence $(v_1, v_2, v_3) = (ku_1, ku_2, ku_3)$ for an arbitrary $k \in \mathbf{R}$. Q. E. D.

It reads as the bound vectors $\mathbf{u}_{(X)}$ and $\mathbf{u}_{(Y)}$ possess the same moment $\mathbf{u}^0$ if and only if X and Y lie on a line with the direction vector $\mathbf{u}$.

1.3. Motors and screws

By a *motor* we understand a couple $(\mathbf{u}, \mathbf{u}^0)$. "Motor" is a combination of words "moment" and "vector", Dimentberg (1968). It is a representation of a vector system, expressed by the principal vector and principal moment of the system. However, our radius vectors have been so far based on the origin O of the coordinate system, but another point may be such a reference point.

Every motor can be brought to such a reference point that its moment and vector parts become colinear, which turns a motor into an equivalent *screw*. A parallel line through this point is the *screw axis*, see Brodsky and Shoham (1999).

The *screw calculus*, Dimentberg (1968), is based on the basis of the apparatus of modern vector algebra using dual numbers.

1.4. Application: Cosserat media

Traditional mechanics of continua endows particles of a material body with translational degrees of freedom, the Cosserat brothers' approach endows them with both translational and rotational degrees of freedom. In elementary approach a body B of dimension 1 (rods, beams) or 2 (plates, shells) or 3 in $\mathbf{R}^3$ is considered. For each particle of such a body we consider its initial position (a radius vector) and its initial settings of 3 orthonormal directors. Such approach is based on differential geometry theory applied to mechanics and there is no doubt that Cosserat continuum theory is suitable e.g. for describing the kinematics of granular media.

The mathematical description can be based on motors, as stated in the monograph of Vardoulakis (2018) in which the basic theorems used to formulate the Cosserat continuum, together with the appropriate kinematic fields conjugate to the motor vectors; kinematic motors are compound vectors including linear velocity and spin (angular velocity), fully describing a rigid body motion in the new reduced geometric representation.

2. Dual curves

2.1. Dual vectors and dual curves

The dual numbers are defined as numbers of a form $\mathbf{a} = a + A\varepsilon$; a, $A \in \mathbf{R}$, which extend the real numbers by adjoining new ("infinitesimal") element ε with the property $\varepsilon^2 = 0$. The set of dual numbers is denoted by $\mathbf{D}$ and it forms a two-dimensional commutative unital associative R-algebra. The arithmetic of dual numbers has several specifics, so we refer e.g. to the paper Kureš (to appear) for details.

Further, a dual function f: $\mathbf{D} \to \mathbf{D}$ of a dual variable $\mathbf{x} = x + X\varepsilon$ can be represented in the form

$$f(x + X\varepsilon) = \varphi(x, X) + \Phi(x, X)$$

Dual functions which are also differentiable in a neighborhood of a point are called *synektic*. This is satisfied if and only if

$$\frac{\partial \varphi}{\partial x} = \frac{\partial \Phi}{\partial X}$$

and

$$\frac{\partial \varphi}{\partial X} = 0.$$

It is an analogy with the analytic functions over $\mathbf{C}$ which comply with the Cauchy-Riemann conditions. Then we have

$$\frac{\mathrm{d}f}{\mathrm{d}\mathbf{x}} = \frac{\partial\varphi}{\partial x} + \frac{\partial\Phi}{\partial x}\varepsilon.$$

Dual vectors will now be understood as elements of $(\mathbf{D})^3 = \mathbf{D}\times\mathbf{D}\times\mathbf{D}$. Then $(\mathbf{D})^3$ is a free **D**-module, $\dim_{\mathbf{D}} (\mathbf{D})^3 = 3$, and an **R**-vector space, $\dim_{\mathbf{R}} (\mathbf{D})^3 = 6$. We will call $(\mathbf{D})^3$ the *dual space* and its elements will be denoted by

$$\mathbf{a} = (\mathrm{a}_1 + \mathrm{A}_1\varepsilon, \mathrm{a}_2 + \mathrm{A}_2\varepsilon, \mathrm{a}_3 + \mathrm{A}_3\varepsilon).$$

Dual numbers introduced by W. Clifford were deeply investigated by E. Study who used dual numbers and dual vectors in his research on line geometry and kinematics. He devoted special attention to the representation of oriented lines by dual unit vectors and defined the famous mapping: the set of oriented lines in a Euclidean three-dimension space $\mathbf{R}^3$ is one-to-one correspondence with the points of a dual space $\mathbf{D}^3$ of triples of dual numbers. Of course, from the time these classic results came into being, research has expanded from straight lines to more curves. So, let us give a definition.

Synektic curves in the dual space have a form

$$\mathbf{t} = (t + T\varepsilon) \mapsto \mathbf{a}(\mathrm{t} + \mathrm{T}\varepsilon) = (a_1(t,T) + A_1(t,T)\varepsilon, a_2(t,T) + A_2(t,T)\varepsilon, a_3(t,T) + A_3(t,T)\varepsilon),$$

where dual functions (of a dual variable) $a_1(t,T) + A_1(t,T)\varepsilon, a_2(t,T) + A_2(t,T)\varepsilon, a_3(t,T) + A_3(t,T)\varepsilon$ are synektic.

We present two examples of synektic curves:

$$a_1(t,T) + A_1(t,T)\varepsilon = \mathrm{r}\, cos\, t + (R\, cos\, t - rT\, sin\, t)\varepsilon$$
$$a_2(t,T) + A_2(t,T)\varepsilon = r\, sin\, t + (R\, sin\, t + rT\, cos\, t)\varepsilon$$
$$a_3(t,T) + A_3(t,T)\varepsilon = qt + (qT + Qt)\varepsilon$$

is a *circular helix in the dual space* while

$$a_1(t,T) + A_1(t,T)\varepsilon = r\mathrm{e}^{qt}\cos t + \left(\left(R\mathrm{e}^{qt} + r\mathrm{e}^{qt}(Qt + qT)\right)\cos t - r\mathrm{e}^{qt}T\sin t\right)\varepsilon$$
$$a_2(t,T) + A_2(t,T)\varepsilon = r\mathrm{e}^{qt} sin\, t + \left(\left(R\mathrm{e}^{qt} + r\mathrm{e}^{qt}(Qt + qT)\right)\sin t + r\mathrm{e}^{qt}T\cos t\right)\varepsilon$$
$$a_1(t,T) + A_1(t,T)\varepsilon = r\mathrm{e}^{qt} + \left(R\mathrm{e}^{qt} + r(Qt + qT)\right)\varepsilon$$

is a *conic helix in the dual space*. From the viewpoint of practical applications, the helices in the micro-scale are important and interesting; the fabrication of microhelices from different materials is categorized and their novel properties and applications are summarized and reviewed in Huang and Mei (2015). Smaller helices, i.e. the helices in nano-scale, on the contrary, are even difficult to be fabricated and investigated, although the new sciences in such small scale may suggest even significant potentials in the future. The author's paper Kureš (to appear) highlights the importance of study of helices in dual space and the curvature and torsion of these two types of helices as synektic curves over dual numbers are derived.

2.2. Frenet–Serret formulas

Synectic curves may be parametrized, as in the real case, by the arc length. Details of this reparameterization are described in Navrátil (2017). We denote the dual natural parameter by $\mathbf{s} = s + S\varepsilon$. Then the unit *tangent vector* of a curve $\mathbf{a}(\mathbf{s})$ is the vector

$$\mathbf{g} = \frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\mathbf{s}}$$

and the unit *normal vector* perpendicular to **g** with the same orientation as $\frac{\mathrm{d}\mathbf{g}}{\mathrm{d}\mathbf{s}}$ is denoted by **n**. Furthermore, the unit vector **b** with the same orientation as $\mathbf{g} \times \mathbf{n}$ is called the *binormal vector*. Then for the *dual curvature* κ and for the *dual torsion* τ, the following relations are satisfied:

$$\frac{\mathrm{d}\mathbf{a}}{\mathrm{d}\mathbf{s}} = \mathbf{g}, \quad \frac{\mathrm{d}\mathbf{g}}{\mathrm{d}\mathbf{s}} = \kappa\mathbf{n}, \quad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}\mathbf{s}} = -\kappa\mathbf{g} + \tau\mathbf{b}, \quad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}\mathbf{s}} = -\boldsymbol{\tau}\mathbf{n}.$$

These are Frenet–Serret formulas for dual curves.

2.3. Spivak's dual curve

Michael Spivak has described in his famous book Spivak (1970) a real curve lying in the *xy* plane for $t > 0$ and in the *xz* plane for $t < 0$, effectively switching planes at $t = 0$, while remaining smooth.

We will call this curve *Spivak's (real) curve* for the purposes of this paper.

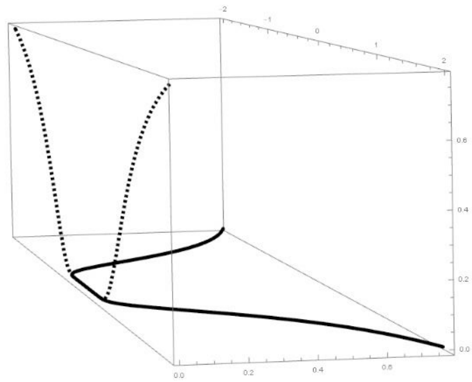

Fig. 1. An illustration of the real Spivak's curve. For the negative t we use the dashed curve and for the positive t we use the full curve. So we start at the top left, switch in the middle and finish at the bottom right. Components are connected in $t = 0$ smoothly.

And now, we will consider this curve in the dual version.

As

$$\mathbf{t}^2 = t^2 + 2tT\varepsilon,$$

$$-\frac{1}{\mathbf{t}^2} = -\frac{1}{t^2} + \frac{2T}{t^3}$$

and

$$\mathrm{e}^{-\frac{1}{\mathbf{t}^2}} = e^{-\frac{1}{t^2}} + e^{-\frac{1}{t^2}}\frac{2T}{t^3}\varepsilon,$$

we generalize Spivak's curve by

$$\mathbf{c}(\mathbf{t}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}) \quad \text{for } t = 0,$$

$$\mathbf{c}(\mathbf{t}) = \left(t + T\varepsilon, e^{-\frac{1}{t^2}} + e^{-\frac{1}{t^2}}\frac{2T}{t^3}\varepsilon, \mathbf{0}\right) \quad \text{for } t > 0 \text{ and}$$

$$\mathbf{c}(\mathbf{t}) = \left(t + T\varepsilon, \mathbf{0}, e^{-\frac{1}{t^2}} + e^{-\frac{1}{t^2}}\frac{2T}{t^3}\varepsilon\right) \quad \text{for } t < 0.$$

This curve will be called *Spivak's dual curve*. (We remark that we obtain Spivak's real curve for $T = 0$.) The original question of Michael Spivak how one can use the curvature and the torsion to distinguish between this curve and the curve

$$\tilde{\mathbf{c}}(\mathbf{t}) = (\mathbf{0}, \mathbf{0}, \mathbf{0}) \quad \text{for } t = 0,$$

$$\tilde{\mathbf{c}}(\mathbf{t}) = \left(t + T\varepsilon, e^{-\frac{1}{t^2}} + e^{-\frac{1}{t^2}}\frac{2T}{t^3}\varepsilon, \mathbf{0}\right) \quad \text{for } t \neq 0$$

is now naturally reformulated for dual space. This question requires the expression of a natural parameter for dual case for a calculation of the curvature and torsion. As a new result, let us conclude with the final expression of this parameter:

$$s + S\varepsilon = \frac{e^{-\frac{2}{t^2}}\left(4t^2(t - 3T\varepsilon) + e^{\frac{2}{t^2}}t^9 + 8T\varepsilon\right)}{\sqrt{1 + \frac{4e^{-\frac{2}{t^2}}}{t^6}t^8}}$$

This expression of the natural parameter is equal for both parts of the curve. But we can deduce from that, quite easily, that this is the property highlighted by Spivak remains in force in the dual case, too.

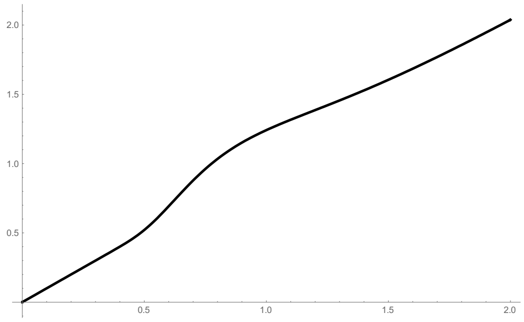

Fig. 2. The natural parameter for $T = 0$ from the previous expression on the inteval (0,2] (monotony and thus invertibility of the function are obvious).

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