Every nonlinear element from Chua's table can generate pinched hysteresis loops: generalised homothety theorem

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The pinched hysteresis loops drawn in the voltage–current, voltage–charge, and flux–current characteristics belong to well-known fingerprints of the memory elements known as the memristor, memcapacitor, and meminductor. It is shown that generating such loops is in fact a natural attribute of all nonlinear elements from Chua's table, and that the hysteresis need not necessarily be a manifestation of the internal memory of the element. A generalised homothety theorem is introduced, which describes the regularities of such a hysteresis and its frequency dependence.

Introduction: The so-called (α, β) higher-order elements, organised in Chua's table [1], a part of which is shown in Fig. 1, are the fundamental building blocks for modelling nonlinear systems. The (α, β) element is defined by a generally nonlinear constitutive relation (CR) between the circuit variables denoted by the symbols $v^{(\alpha)}$ and $i^{(\beta)}$, where α and β are integers. The variables represent multiple time-domain derivatives (for positive integers) or integrals (for negative integers) of voltage v and current i. For a voltage-controlled (α, β) element, the CR can be described by the nonlinear function

$$i^{(\beta)} = F(v^{(\alpha)}). \tag{1}$$

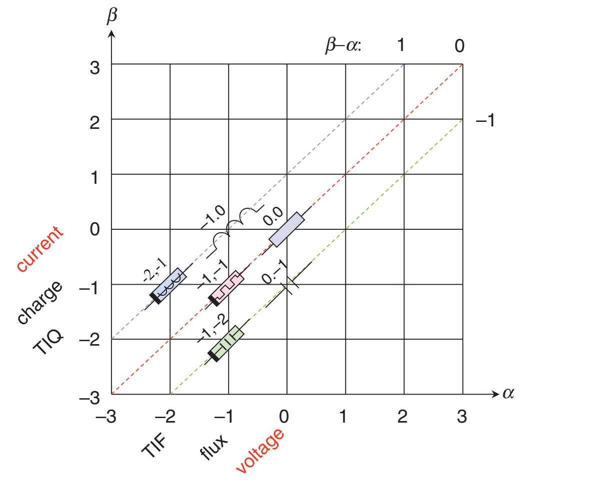

Fig. 1 *Chua's table of fundamental elements (cutout). Location of resistor (0, 0), capacitor (0, −1), inductor (−1, 0), memristor (−1, −1), memcapacitor (−1, −2), and meminductor (−2, −1) in table. TIQ = time-domain integral of charge, TIF = time-domain integral of flux*

In the case of the current-controlled element, the analysis and results are dual, and therefore they will not be shown in this Letter.

For the memristor, which is the (−1, −1) element, the CR (1) represents a relationship between the charge $q = i^{(-1)}$ and the flux $\varphi = v^{(-1)}$

$$q = F(\varphi). \tag{2}$$

The equivalent description is state-dependent Ohm's law

$$i = g(\varphi)v, \quad \varphi = \int v\,\mathrm{d}t \tag{3}$$

where $g(\varphi) = \mathrm{d}F(\varphi)/\mathrm{d}\varphi$ is a memductance, which is dependent on the flux as the time-domain integral of voltage. It is well known that such dependence is the cause of the v–i pinched hysteresis loop of the memristor. Several fingerprints of the loops were enunciated, the best known of them concerning the disappearance of the loop area of the memristor with CR (2), when the frequency of periodical driving voltage or current with limited signal levels tends to infinity. This fingerprint follows directly from state-dependent Ohm's law (3): For zero voltage, the current must also be zero (the loop is then pinched at the origin). If the frequency of the voltage waveform increases towards infinity, the swing of the flux, i.e. the integral of voltage, must decrease towards zero, and the memductance g approaches the constant value (this is a sign of the linear resistor, which does not exhibit hysteretic behaviour).

As shown in [2], memcapacitors and meminductors exhibit formally the same mathematical models as the models (2) and (3) for a memristor. That is why the above fingerprint also holds for these elements after interchanging the appropriate constitutive variables.

Another view of the hysteretic effects of memristors is introduced in [3], whereas the classical fingerprint assumes memristor excitation via voltage or current, thus in the (v, i) space, the memristor in [3] is driven by a signal of the charge or flux type, thus in the (φ, q) space of its constitutive variables. Then, an n-times accelerated signal with its amplitude preserved generates the (v, i) pinched hysteresis loop, which is a homothetic entity with respect to the original loop, with the homothetic centre at the v–i origin and the scale factor n. The loop area increases with the square of the frequency.

The aim of this Letter is to show that the theorem of the homothety can be generalised to the behaviour of an arbitrary (α, β) element from Chua's table. This may be surprising at the first glance, because, according to the current knowledge, the effect of pinched hysteresis loops is typical just of memristors, memcapacitors, and meminductors, the notion being that such a hysteresis differentiates these elements from the classical resistors, capacitors, and inductors.

Hysteretic manifestations of (α, β) element in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space: Differentiating the CR (1) of the (α, β) element and a simple rearrangement yield a mathematical model of its behaviour in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space

$$i^{(\beta+1)} = g\left(v^{(\alpha)}\right)v^{(\alpha+1)}, \quad v^{(\alpha)} = \int v^{(\alpha+1)}\,\mathrm{d}t \tag{4}$$

where $g(v^{(\alpha)}) = \mathrm{d}F(v^{(\alpha)})/\mathrm{d}v^{(\alpha)}$ is a differential parameter of the (α, β) element, which depends on the integral variable $v^{(\alpha)}$. This parameter is therefore analogous to the flux-dependent memductance in (3).

It is obvious that the memristor model (3) is a special case of the model (4) for (α, β) = (−1, −1), and that the structure of the model (4) directly results in the ability to generate hysteresis loops pinched at the origin of ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) coordinates: (4) resemble state-dependent Ohm's law, because the quantities $v^{(\alpha+1)}$ and $i^{(\beta+1)}$ are coupled via a function g, which is analogous to memductance (3), and g is modulated by the integral variable $v^{(\alpha)}$, which resembles the flux in (3).

Excitation of the (α, β) element in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space: Fig. 2*a* shows a block diagram derived from the model (4), illustrating the mechanism of driving the (α, β) element by the signal $v^{(\alpha+1)}$ and generating the response $i^{(\beta+1)}$. Since this structure is equivalent to the structure of ideal memristor (3), the $v^{(\alpha+1)}$ and $i^{(\beta+1)}$ waveforms must exhibit all the known memristor fingerprints including the feature of diminishing hysteresis in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space if the frequency of the excitation $v^{(\alpha+1)}$ increases towards infinity.

Excitation of the (α, β) element in the ($v^{(\alpha)}$, $i^{(\beta)}$) space: The block diagram in Fig. 2*b*, which corresponds to this excitation, is derived directly from the CR (1) of the element: the response $i^{(\beta)}$ to the excitation $v^{(\alpha)}$ is given by the algebraic relation $F()$. The quantities $v^{(\alpha+1)}$ and $i^{(\beta+1)}$ are then derived from $v^{(\alpha)}$ and $i^{(\beta)}$ via differentiating with respect to time.

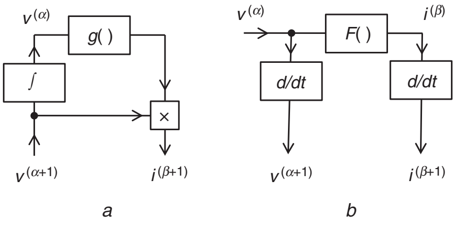

Fig. 2 *Block diagram for driving (α, β) element in space*
a ($v^{(\alpha+1)}$, $i^{(\beta+1)}$)
b ($v^{(\alpha)}$, $i^{(\beta)}$)

The driving signal $v^{(\alpha)}(t)$ causes a pinched hysteresis loop in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space. Now consider the driving signal $v^{(\alpha)}(nt)$, where the number $n > 0$ models the contraction (for $n > 1$) or expansion (for $n < 1$) of the time axis, alias the acceleration or deceleration of the signal $v^{(\alpha)}$. Let us denote the transition from the excitation $v^{(\alpha)}(t)$ to the excitation $v^{(\alpha)}(nt)$ as $v^{(\alpha)}(t) \rightarrow v^{(\alpha)}(nt)$.

The response $i^{(\beta)}$ will be $i^{(\beta)}(nt) = F(v^{(\alpha)}(nt))$ or $i^{(\beta)}(t) \rightarrow i^{(\beta)}(nt)$.

Differentiating the signals $v^{(\alpha)}$ and $i^{(\beta)}$ with respect to time yields the respective relations for the signals $v^{(\alpha+1)}$ and $i^{(\beta+1)}$: $v^{(\alpha+1)}(t) \rightarrow n\, v^{(\alpha+1)}(nt)$, $i^{(\alpha+1)}(t) \rightarrow n\, i^{(\alpha+1)}(nt)$.

This means, for example, that doubling the frequency of the driving signal results in doubling the levels of both signals $v^{(\alpha+1)}$ and $i^{(\beta+1)}$. This follows from the frequency-dependent gain of ideal differentiating elements from Fig. 2*b*. The frequency therefore affects the hysteresis loops in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space such that the scales of the axes are modified. From the geometric point of view, when the frequency changes, an arbitrary point on the original loop shifts along a line which goes through this point and the origin of the coordinates, in the direction from the origin (for $n > 1$), or towards the origin (for $n < 1$). The ratio of the distances of the new and original points from the origin is given by the ratio of frequencies. Then, the corresponding hysteresis loops are homothetic with the homothetic centre at the origin and with the scale factor n. The ratio of the areas of two homothetic entities is n^2 [3]. Starting from this knowledge, we can state *the homothety theorem for (α, β) elements*:

The movement of the operating point in the ($v^{(\alpha)}$, $i^{(\beta)}$) space along the nonlinear CR of an (α, β) element from Chua's table is accompanied by the movement of the operating point in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space along a pinched hysteresis loop. The loops that correspond to the driving signals of the same levels but various frequencies are homothetic entities with the homothetic centre at the origin of the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space, and the scale factor equal to the ratio of frequencies. The areas within the lobes of the loops increase with the square of the frequency.

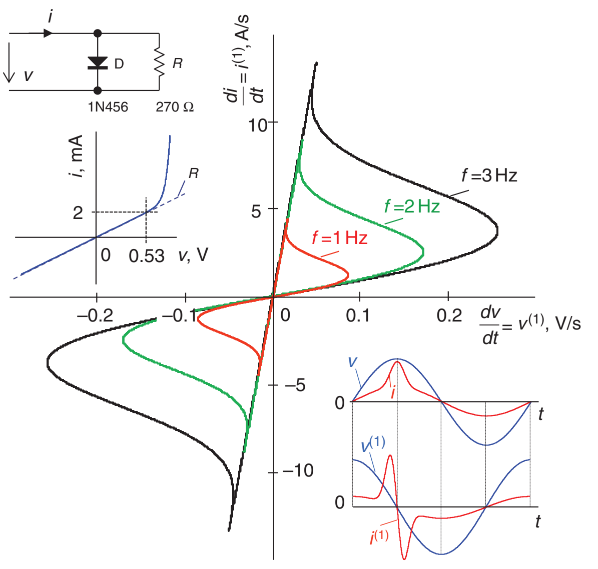

Fig. 3 *i–v Characteristic of nonlinear resistive two-terminal device, waveforms of voltage, current, and their time derivatives, and homothetic hysteresis loops pinched at ($v^{(1)}$, $i^{(1)}$) origin for various frequencies of sinusoidal excitation v*

Note that the hysteresis loops of the (α, β) element appear in a space which corresponds to the CR of a different ($\alpha+1$, $\beta+1$) element: namely, the element which is located in Chua's table at the nearest subsequent position from the original element along the diagonal with the parameter β–α in the direction which corresponds to increasing indices α and β (see Fig. 1). That is why the hysteretic behaviour of memristor, memcapacitor, and meminductor is observable in the space of the CRs of their classical variants (resistor, capacitor, and inductor). According to this rule, the hysteresis, for example, of the classical resistor, must appear in the space of the (1, 1) element, thus within the ($v^{(1)}$, $i^{(1)}$) coordinates.

Simulation: Consider the nonlinear resistive two-terminal device in Fig. 3 consisting of a linear resistor and an Si diode in parallel. If the diode is in the off state, which is true for any voltage of less than ca 0.5 V, then the resulting *i–v* characteristic copies the linear characteristic of the resistor. For higher voltages, the current is intensely increasing. The *i–v* characteristic in Fig. 3 was obtained via the simulation program with integrated circuit emphasis (SPICE) simulation for 0.7 V sinusoidal excitation with a very low repeating frequency of 1 Hz, when the waveforms are not affected by the dynamical parameters of the diode. The ($v^{(1)}$, $i^{(1)}$) pinched hysteresis loops for frequencies of 1, 2, and 3 Hz in Fig. 3 confirm the validity of the generalised homothety theorem. The waveforms of the derivatives of the voltage and current illustrate the reasons for the appearance of the hysteresis: at time instants when the signal $v^{(1)}$ repeatedly intersects a concrete level, the signal $i^{(1)}$ takes different values. The $i^{(1)}$ against $v^{(1)}$ relationship cannot be therefore unambiguous.

When driving a two-terminal device by the $v^{(1)}$ signal with fixed amplitude and variable frequency, then the increasing frequency would result in decreasing the amplitude of the terminal voltage. If the voltage decreased below ca 0.5 V, the device would behave as a linear resistor without hysteresis. This confirms that, when driving a nonlinear resistor in the ($v^{(1)}$, $i^{(1)}$) space, the classical fingerprint about diminishing hysteresis applies when the frequency increases beyond all bounds.

Conclusion: The movement of the operating point of an arbitrary (α, β) element from Chua's table in the ($v^{(\alpha)}$, $i^{(\beta)}$) space, i.e. in the space of its nonlinear CR, is automatically accompanied by drawing a hysteresis loop pinched at the origin of the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space. The true cause of the hysteresis is the nonlinearity of the CR of the element, not the memory effect, which is known for memristors, memcapacitors, and meminductors, and is a mere consequence of this nonlinearity. Driving an arbitrary (α, β) element by a signal with a fixed level whose frequency increases up to infinity, then, for excitation in the ($v^{(\alpha)}$, $i^{(\beta)}$) space, the hysteresis loops are governed by the classical fingerprint known for the memristors (diminishing hysteresis), whereas when driving the element in the ($v^{(\alpha+1)}$, $i^{(\beta+1)}$) space, the generalised homothety theorem is applied.

Acknowledgments: Research described in this paper was financed by the Czech Ministry of Education in frame of National Sustainability Program under grant LO1401. For research, infrastructure of the SIX Center was used.

Submitted: *12 August 2016* E-first: *27 September 2016*
doi: 10.1049/el.2016.2961
One or more of the Figures in this Letter are available in colour online.

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