SECOND ORDER DIFFERENTIAL EQUATIONS WITH COMPLEX COEFFICIENTS

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Abstract: The purpose of this paper is to explain a method to solve differential equations with complex coefficients that may be easier than the conventional way, also to show the importance of the complex representation of the solutions in real applications.

Keywords: second order differential equations, complex numbers

1 INTRODUCTION AND CALCULATIONS

Sometimes the task of solving a differential equation can lead to several quantity of algebraic calculations that may become mistakes, and even more when it implies complex numbers. However, it's important to be able to solve these kind of equations with high precision. We deal with an equation of the form:

$$y'' + py' + qy = Ke^{rt}; \quad K, p, q, r \in \mathbb{C}; \quad t \in \mathbb{R}. \tag{1}$$

We first have to solve the homogeneous equation and then find a particular solution. Finally we get the general solution as a sum of these solutions.

The homogeneous equation is described as:

$$y'' + py' + qy = 0. \tag{2}$$

Let's assume a solution of the form: $y(t) = e^{\lambda t}$. Then $y' = \lambda e^{\lambda t}$ and $y'' = \lambda^2 e^{\lambda t}$. Substituting this into (2) we get the characteristic equation that we need to solve for λ:

$$\lambda^2 e^{\lambda t} + p\lambda e^{\lambda t} + qe^{\lambda t} = 0 \quad \rightarrow \quad \lambda^2 + p\lambda + q = 0 \quad \rightarrow \quad \lambda_{1,2} = \frac{-p}{2} \pm \sqrt{\frac{p^2}{4} - q}.$$

To simplify $\lambda_{1,2}$ let's call $\alpha = -\frac{p}{2}$ and $\beta = \sqrt{\frac{p^2}{4} - q}$. Then $\lambda_{1,2} = \alpha \pm \beta$. Depending on the discriminant $\Delta = \frac{p^2}{4} - q$ we have:

- $\Delta = 0 \quad \rightarrow \quad \lambda = \alpha = -\frac{p}{2}$ and the fundamental system is $y_1(t) = e^{\alpha t}$ and $y_2(t) = te^{\alpha t}$. It can be shown, that y_2 solves (2). The Wronskian shows that $W(y_1, y_2) = e^{2\alpha t} \neq 0$, then we have a set of solutions as $y(t) = C_1 e^{\alpha t} + C_2 t e^{\alpha t}$.

- $\Delta \neq 0 \quad \rightarrow \quad \lambda_1 \neq \lambda_2$ then we have two solutions $y_1 = e^{\lambda_1 t}$ and $y_2 = e^{\lambda_2 t}$. The Wronskian shows that $W(y_1, y_2) = \lambda_1 - \lambda_2 \neq 0$, then the general solution is $y(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$.

If p and q are complex numbers, the discriminant Δ can be complex. In that case the problem of solving the characteristic equation reduces to find the square root of a complex number.

$$\beta=\sqrt{\frac{p^2}{4}-q}=\sqrt{a+ib}=x+iy \quad \rightarrow \quad a+ib=x^2+i2xy-y^2 \begin{cases} a=x^2-y^2 \\ b=2xy \end{cases} \quad ,$$

$$\left. y=\frac{b}{2x}, \qquad 4ax^2=4x^4-b^2 \right\} \quad \rightarrow \quad x^2=\frac{1}{2}\left(a\pm\sqrt{a^2+b^2}\right).$$

We don't consider the negative sign because x has to be real, therefore:

$$x=\pm\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)}, \qquad y=\pm\frac{b}{\sqrt{2\left(a+\sqrt{a^2+b^2}\right)}},$$

$$\beta=\sqrt{\frac{p^2}{4}-q}=\sqrt{a+ib}=\pm\left(\sqrt{\frac{1}{2}\left(a+\sqrt{a^2+b^2}\right)}+\frac{ib}{\sqrt{2\left(a+\sqrt{a^2+b^2}\right)}}\right).$$

Now we need to find the particular solution of the equation. Let's assume on the right side of the equation a function of the form $f(t)=Ke^{rt}; \quad K,r\in\mathbb{C}; \quad t\in\mathbb{R}$. In this case the particular solution of the equation will have the form $y_p(t)=At^k e^{rt}$, where k is the number of times where r appears as a root of the characteristic equation.

2 EXAMPLE

Let's solve the equation $y''-(3+2i)y'+(5+i)y=34e^{(1-i)t}$ to see how it works:

$$\lambda^2-(3+2i)\lambda+(5+i)=0 \quad \rightarrow \quad \lambda_{1,2}=\frac{(3+2i)\pm\sqrt{-15+8i}}{2}$$

$$\sqrt{-15+8i}=\pm(1+4i) \quad \rightarrow \quad \lambda_1=2+3i, \quad \lambda_2=1-i \quad \rightarrow \quad y_h=C_1e^{(2+3i)t}+C_2e^{(1-i)t}$$

$$y_p=Kte^{(1-i)t}, \qquad y_p'=Ke^{(1-i)t}[1+(1-i)t], \qquad y_p''=K(1-i)e^{(1-i)t}[2+(1-i)t]$$

$$K(1-i)[2+(1-i)t]-(3+2i)K[1+(1-i)t]+(5+i)Kt=34 \rightarrow K=-2+8i \rightarrow y_p=(-2+8i)te^{(1-i)t}$$

$$y=y_h+y_p \quad \rightarrow \quad y=C_1e^{(2+3i)t}+C_2e^{(1-i)t}+(-2+8i)te^{(1-i)t}$$

3 APPLICATION

The solution of the differential equation that describes the behavior of the current on a RLC circuit in serie has the form $LI''+RI'+\frac{1}{C}I=f(t)$, where $f(t)=E_0\omega e^{i\omega t}$ on the complex domain and on the real domain $f(t)=E_0\omega\cos(\omega t)$, see [1].

Solving the homogeneous equation $LI_h''+RI_h'+\frac{1}{C}I_h=0$ we get

$$\lambda^2+\frac{R}{L}\lambda+\frac{1}{LC}=0 \quad \rightarrow \quad \lambda_{1,2}=-\frac{R}{2L}\pm\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}} \quad \rightarrow \quad I_h=C_1y_1(t)+C_2y_2(t).$$

The complex particular solution satisfies $L\tilde{I}_p''+R\tilde{I}_p'+\frac{1}{C}\tilde{I}_p=E_0\omega e^{i\omega t}$. We assume $\tilde{I}_p=Ke^{i\omega t}$. Then $\tilde{I}_p'=i\omega Ke^{i\omega t}$ and $\tilde{I}_p''=-\omega^2Ke^{i\omega t}$. Finally we get

$$\left(-L\omega^2+iR\omega+\frac{1}{C}\right)K=E_0\omega \quad ,$$

$$K=\frac{E_0\omega}{-L\omega^2+iR\omega+\frac{1}{C}}=\frac{E_0}{i\left[R+i\left(L\omega-\frac{1}{\omega C}\right)\right]}=\frac{E_0}{i\,(R+iS)}=\frac{E_0}{iZ}, \quad S=L\omega-\frac{1}{\omega C},$$

$$Z=R+iS=\sqrt{R^2+S^2}e^{i\theta}, \quad \theta=\arctan\left(\frac{S}{R}\right).$$

In the last line we have bumped into concepts as the reactance S that represents the complex part of the impedance Z, these two concepts are fundamental in the analysis of circuits that help to understand the behavior of the resistance to the flux of current made by some electrical components.

$$\tilde{I}_p = \frac{E_0}{i\sqrt{R^2+S^2}} e^{i(\omega t - \theta)}$$

Now we solve the same circuit in the real domain. The particular solution I_p satisfies the equation $LI_p'' + RI_p' + \frac{1}{C} I_p = E_0 \omega \cos(\omega t)$. Then we have

$$I_p = A\cos(\omega t) + B\sin(\omega t), \quad I_p' = \omega\left[-A\sin(\omega t) + B\cos(\omega t)\right], \quad I_p'' = -\omega^2[A\cos(\omega t) + B\sin(\omega t)];$$

$$-L\omega^2[A\cos(\omega t) + B\sin(\omega t)] + R\omega[-A\sin(\omega t) + B\cos(\omega t)] + \frac{1}{C}[A\cos(\omega t) + B\sin(\omega t)] = E_0\omega\cos(\omega t).$$

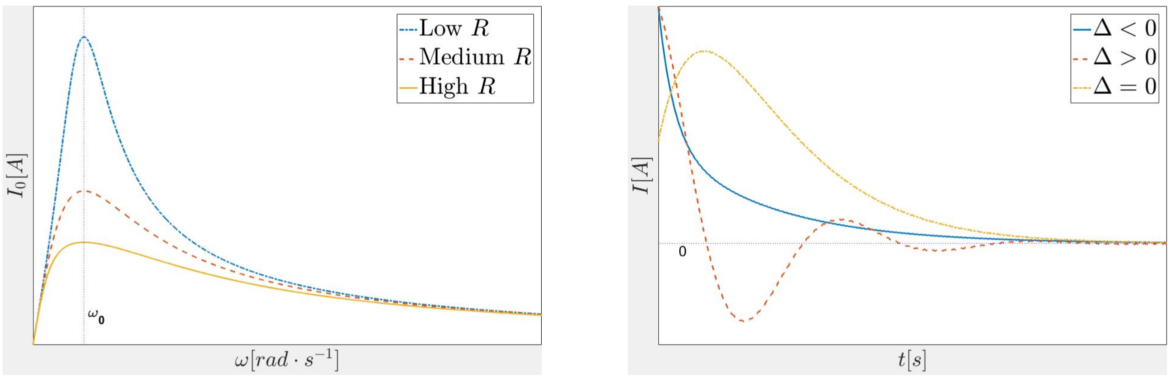
$$\left.\begin{array}{r} -L\omega^2 A + R\omega B + \frac{1}{C} A = E_0 \omega \\ -L\omega^2 B - R\omega A + \frac{1}{C} B = 0 \end{array}\right\} \to \quad A = -\frac{E_0 S}{R^2+S^2}, \qquad B = \frac{E_0 R}{R^2+S^2}; \qquad S = L\omega - \frac{1}{\omega C}.$$

$$I_p = I_0 \sin(\omega t - \theta) = I_0[\sin(\omega t)\cos(\theta) - \sin(\theta)\cos(\omega t)].$$

$$A = -I_0 \sin(\theta), \quad B = I_0 \cos(\theta), \quad I_0^2 = A^2 + B^2 = \frac{E_0^2}{R^2+S^2},$$

$$I_0 = \frac{E_0}{\sqrt{R^2+S^2}}, \quad \theta = \arctan\left(\frac{S}{R}\right), \quad I_p = \frac{E_0}{\sqrt{R^2+S^2}} \sin(\omega t - \theta).$$

Now we can represent the behavior of the solution $I(t) = I_h + I_p$ considering some circumstances. In (a) is shown how the resonance of the system decreases as the resistance decreases with the maximum I_0 in $\omega_0 = \frac{1}{\sqrt{LC}}$. And in (b) we can see how $I(t)$ varies depending on the value of the determinant Δ.



(a) Resonance depending on R

(b) $I(t)$ depending on the discriminant Δ

4 CONCLUSION

Once everything is done we can see that taking the complex or the real way to solve the differential equation, leads us exactly to the same solution. However, if we look in detail, using the complex way we can get a solution in a simpler way avoiding long algebraic calculations, besides that the complex method gives us information that may not be found trivially using the real path, a clear example of this, is the concept of impedance as a sum of a complex and a real quantity that represent a resistance and a reactance, respectively. In the complex solution of the example this relation was found throw the exercise of solving the equation, while in the real solution the relation wasn't even there.

REFERENCE

[1] Kreyszig, E.: Advanced engineering mathematics, 9th ed., John Wiley & Sons Inc., 2006, p. 98, ISBN 0-471-72897-7.