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Discrete Riccati matrix equation and the order preserving property*

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Abstract

It is known that if a symmetric matrix differential equation has the order preserving property and the matrix dimension is at least 2, then this equation is the Riccati matrix differential equation (see [20]). In this paper we prove that a similar statement holds for discrete matrix equations as well. In the proof we use a new approach, in which we extend a discrete function to a continuous one by using the iteration theory and then apply the known result for the continuous case.

Keywords: Riccati matrix equation, order preserving property, symplectic matrix, iteration

2010 MSC: 39A12, 39A22, 39B12

1. Introduction and motivation

The research presented in this paper was motivated by results about the relation between the Riccati matrix equation and the order preserving property in the continuous and discrete case. By the *Riccati matrix differential equation* we mean the equation

$$Q'(t) + A^T(t)Q(t) + Q(t)A(t) + Q(t)B(t)Q(t) - C(t) = 0, \qquad (1)$$

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where $A(t), B(t), C(t)$ and $Q(t)$ are real $n \times n$ matrix functions of t and $B(t)$, $C(t)$, $Q(t)$ are symmetric. By the *discrete Riccati matrix equation* we mean the difference equation

$$R[Q]_k := Q_{k+1}(\mathcal{A}_k + \mathcal{B}_k Q_k) - (\mathcal{C}_k + \mathcal{D}_k Q_k) = 0, \tag{2}$$

where $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$, and Q_k are real $n \times n$ matrices, Q_k are symmetric and the $2n \times 2n$ matrices $\mathcal{S}_k$ with block entries $\mathcal{A}_k$, $\mathcal{B}_k$, $\mathcal{C}_k$, $\mathcal{D}_k$ are symplectic. This means that

$$\mathcal{S}_k = \begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix}, \quad \mathcal{S}_k^T \mathcal{J} \mathcal{S}_k = \mathcal{J}, \quad \mathcal{J} := \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

By the *symmetric matrix differential equation* we mean the differential equation

$$Q'(t) = G(t, Q),$$

where $Q(t)$ and $G(t, Q)$ are real symmetric $n \times n$ matrix functions.

Further, we say that a symmetric matrix differential equation has the *order preserving property* if for any two solutions $Q(t), \hat{Q}(t)$ of this equation, $Q(t_0) \leq \hat{Q}(t_0)$ implies $Q(t) \leq \hat{Q}(t)$ in some neighborhood of t_0. The inequality $Q \leq \hat{Q}$ means that the symmetric matrix $\hat{Q} - Q$ is nonnegative definite.

Results about the relation between the Riccati matrix equation and the order preserving property were published by Reid in [18] and Coppel in [5] and they say that the continuous matrix Riccati equation (1) has the order preserving property. The result from [5] is formulated in Section 2 of this paper. Later, Stokes, who was a student of Coppel, proved in [20] the converse statement, which says that if a symmetric matrix equation has the order preserving property and the matrix dimension is $n \geq 2$, then it is the Riccati equation. This statement is also recalled in Section 2. See also his dissertation [19].

A corresponding result in the discrete case was formulated in [22], it is also a consequence of [14, Theorem 7.1] or [7, Lemma 3.7]. It says that the discrete matrix Riccati equation has the order preserving property under some assumptions, i.e., for any two solutions X_k, Y_k of this equation, $X_0 \leq Y_0$ implies

$X_1 \leq Y_1$ and $X_{-1} \leq Y_{-1}$, if X_0 is from a given set, see Proposition 5 in Section 5. The discrete Riccati equations appear in many applications, mainly in discrete optimal control theory, which is studied e.g. in [17, 4, 3].

Monotonicity properties for Riccati matrix equation for both continuous and discrete case are also studied in papers [10, 11, 9], where the authors use a Fréchet-derivative-based approach; in the discrete case there is only a special case considered, where the coefficients correspond to a Hamiltonian system instead of a more general symplectic system.

In this paper we prove a modification of the converse statement in the discrete case. That is, we show that a matrix equation with certain order preserving property is the discrete Riccati matrix equation. This new result is formulated and proved in Section 5. The main idea of the proof is that we construct a continuous function from the discrete one. We get a matrix differential equation for which the result for the continuous case holds, and then we translate its conclusion back to the discrete equation. There is necessary to ensure the uniqueness of solutions of Riccati differential equation extended beyond points of discontinuity, which is realized through a projective extension of the real line, see Section 3. The construction of continuous function from a discrete one is realized by using the iteration theory, see Section 4. This theory is studied e.g. in [2, 8, 21].

In the whole paper we denote by $\mathbf{S}$ the set of real symmetric $n \times n$ matrices, where $n \geq 2$ is a given integer. Further we denote by $\mathcal{I}_0$ an open interval in $\mathbb{R}$ such that $0 \in \mathcal{I}_0$.

In the paper we use the matrix functions $\sin(Q), \cos(Q), \tan(Q), \arctan(Q)$ that are applied to symmetric matrices. Therefore we define these matrix functions as

$$f(Q) = V \begin{bmatrix} f(\lambda_1) & 0 & \cdots & 0 \\ 0 & f(\lambda_2) & 0 \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots 0 & f(\lambda_n) \end{bmatrix} V^{-1},$$

where $Q = V \Lambda V^{-1}$ and $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$. For details, see e.g. [13].

2. Riccati equation in continuous case

In this Section we present results from [5, 19, 20] and some of their modifications that we will apply later in the proof of the main result. The first presented result says that continuous matrix Riccati equation has the order preserving property.

Proposition 1 (Proposition 6, Chapter 2 in [5])**.** *Let $Q(t), \hat{Q}(t)$ be symmetric solutions of the Riccati matrix differential equation* (1) *on an interval $\mathcal{I}$. If, for some a in $\mathcal{I}$, $Q(a) \leq \hat{Q}(a)$, then $Q(t) \leq \hat{Q}(t)$ for all t in $\mathcal{I}$. If $Q(a) < \hat{Q}(a)$, then $Q(t) < \hat{Q}(t)$ for all t in $\mathcal{I}$.*

Now we consider any symmetric matrix differential equation of the form

$$Q' = G(t, Q), \tag{3}$$

where $G(t, Q)$ is an $\mathbf{S}$-valued function defined on some domain $\mathbf{D} = \mathbf{D}_R \times \mathbf{D}_S \subset \mathbb{R} \times \mathbf{S}$ which is continuous in Q for each t. We will further assume that $\mathbf{D}_R$ is an open set.

Definition 1 (Order preserving property)**.** Differential equation (3) has the *order preserving property* on $\mathbf{D}$ if, whenever Q_0, $\hat{Q}_0$ are real symmetric matrices with $Q_0 \leq \hat{Q}_0$ and t_0 is a point for which (t_0, Q_0) and $(t_0, \hat{Q}_0)$ are in $\mathbf{D}$, then there is a neighborhood of t_0, on which the two solutions $Q(t)$ and $\hat{Q}(t)$ of (3) with $Q(t_0) = Q_0$ and $\hat{Q}(t_0) = \hat{Q}_0$ exist and obey $Q(t) \leq \hat{Q}(t)$.

Remark 1. Definition 1 assumes the uniqueness of solution $Q(t)$ of (3) with $Q(t_0) = Q_0$ on a neighborhood of t_0 for any $(t_0, Q_0) \in \mathbf{D}$. This holds true because the function $G(t, Q)$ is continuous in Q. (See e.g. [12]).

Proposition 2 (Theorem 1 in [20])**.** *If* (3) *has the order preserving property on $\mathbb{R} \times \mathbf{S}$ and if $n \geq 2$, then* (3) *is the Riccati matrix differential equation* (1).

Proposition 3 (Theorem 2 in [19])**.** *If* (3) *has the order preserving property on $\mathbf{D}$ and if $n \geq 2$, then* (3) *is the Riccati matrix differential equation* (1) *on $\mathbf{D}$.*

The proofs of these propositions are based on first showing that $x^T G(t,Q)x = g(x,Qx)$, where g is a suitable function and then by manipulating this relation and using that $n \geq 2$ they come to the only possible form of $G(t,Q)$, which is the Riccati form. For the proof see also [1, Section 4.6]. This approach is not possible in the discrete case. We will use modifications of these propositions later in the proof of the main result, where we will have the situation, when the right-hand side of differential equation (3) does not depend on t, i.e., $G(t,Q) = G(Q)$. Thus now let us consider the symmetric matrix differential equation

$$Q' = G(Q), \tag{4}$$

where $G(Q)$ is an $\mathbf{S}$-valued function defined on some domain $\mathbf{M} \subset \mathbf{S}$, which is continuous in Q.

Lemma 1. *Differential equation* (4) *has the order preserving property on* $\mathbb{R}\times\mathbf{M}$ *if, whenever* $Q_0, \hat{Q}_0 \in \mathbf{M}$ *with* $Q_0 \leq \hat{Q}_0$*, then there exists a neighborhood of* 0*, on which two solutions* $Q(t), \hat{Q}(t)$ *of* (4) *with* $Q(0) = Q_0$*,* $\hat{Q}(0) = \hat{Q}_0$ *exist and obey* $Q(t) \leq \hat{Q}(t)$*.*

Proof. In Definition 1 we put $\mathbf{D} = \mathbb{R} \times \mathbf{M}$. Then $(t_0, Q_0) \in \mathbf{D}$ if and only if $Q_0 \in \mathbf{M}$. Now we prove that for all $t_0 \in \mathbb{R}$ if two solutions $Q(t), \hat{Q}(t)$ of (4) with $Q(0) = Q_0$, $\hat{Q}(0) = \hat{Q}_0$ exist and obey $Q(t) \leq \hat{Q}(t)$ on a neighborhood of 0, then also exists a neighborhood of t_0, on which solutions $Q_{t_0}(t), \hat{Q}_{t_0}(t)$ of (3) with $Q_{t_0}(t_0) = Q_0$, $\hat{Q}_{t_0}(t_0) = \hat{Q}_0$ exist and obey $Q_{t_0}(t) \leq \hat{Q}_{t_0}(t)$. First, if $Q(t)$ is a solution of (4) with $Q(0) = Q_0$, then the function $Q_{t_0}(t)$, defined as $Q_{t_0}(t) = Q(t - t_0)$, is solution of (4) with $Q_{t_0}(t_0) = Q(0) = Q_0$. And if $Q(t) \leq \hat{Q}(t)$ for t from a neighborhood of 0, then also $Q(t - t_0) \leq \hat{Q}(t - t_0)$ for t from a neighborhood of t_0. This finishes the proof. □

Now we formulate corollary of Proposition 2.

Corollary 1. *Let for a differential equation* (4) *be true that whenever* $Q_0, \hat{Q}_0 \in \mathbf{S}$ *with* $Q_0 \leq \hat{Q}_0$*, then there is a neighborhood of* 0*, on which two solutions*

$Q(t), \hat{Q}(t)$ of (4) *with $Q(0) = Q_0$, $\hat{Q}(0) = \hat{Q}_0$ exist and obey $Q(t) \leq \hat{Q}(t)$. If $n \geq 2$, then $G(Q)$ is of the form*

$$G(Q) = -A^T Q - QA - QBQ + C, \tag{5}$$

for all $Q \in \mathbf{S}$, where A, B, C are real $n \times n$ matrices and B, C are symmetric.

Proof. By Lemma 1, such differential equation (4) has the order preserving property, hence the form (5) of $G(Q)$ follows from Proposition 2. □

Remark 2. We can also apply Lemma 1 to Proposition 3 and get a stronger statement.

3. Solution of the Riccati differential equation and linear Hamiltonian system

We will consider the linear Hamiltonian system

$$\begin{pmatrix} X(t) \\ U(t) \end{pmatrix}' = \begin{pmatrix} A(t) & B(t) \\ C(t) & -A^T(t) \end{pmatrix} \cdot \begin{pmatrix} X(t) \\ U(t) \end{pmatrix}, \tag{6}$$

where $A(t), B(t), C(t)$ and $X(t), U(t)$ are $n \times n$ matrix-valued functions of t and $B(t), C(t)$ are symmetric. These systems are studied e.g. in [5] and [16]. A connection between solutions of the linear Hamiltonian system and solutions of the Riccati matrix differential equation is showed in the following Proposition.

Proposition 4 (Lemma 7, Chapter 2 in [5])**.** *If $(X(t), U(t))$ is a (matrix-valued) solution of the Hamiltonian system* (6) *on $\mathcal{I}$ such that $X(t)$ is invertible for all $t \in \mathcal{I}$, then $Q(t) = U(t)X(t)^{-1}$ solves the Riccati matrix differential equation* (1) *on $\mathcal{I}$.*

Conversely, if $Q(t)$ solves the Riccati matrix differential equation (1) *on some interval $\mathcal{I}$ and if $X(t)$ is a fundamental matrix solution of the linear system $X'(t) = (A(t) + B(t)Q(t))X(t)$, then $X(t), U(t) = Q(t)X(t)$ is a solution of the Hamiltonian system* (6) *on $\mathcal{I}$.*

For our purposes it will suffice to consider the coefficient matrices $A(t)$, $B(t)$, $C(t)$ to be constant, i.e., $A(t) \equiv A, B(t) \equiv B, C(t) \equiv C$. Then the Riccati matrix differential equation has the form

$$Q'(t) + A^T Q(t) + Q(t)A + Q(t)BQ(t) - C = 0. \tag{7}$$

Lemma 2. *Let*

$$S(t) = \begin{pmatrix} \tilde{X}(t) & \bar{X}(t) \\ \tilde{U}(t) & \bar{U}(t) \end{pmatrix}$$

be the solution of Hamiltonian system

$$S'(t) = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix} \cdot S(t)$$

with the initial condition $S(0) = I$ *and let* $\tilde{X}(t) + \bar{X}(t)Q_0$ *be invertible on* $\mathcal{I}_0$. *Then*

$$Q(t) = (\tilde{U}(t) + \bar{U}(t)Q_0)(\tilde{X}(t) + \bar{X}(t)Q_0)^{-1} \tag{8}$$

is the unique solution of (7) *on* $\mathcal{I}_0$ *with the initial condition*

$$Q(0) = Q_0. \tag{9}$$

Moreover, the matrix $S(t)$ *is symplectic.*

Proof. It is a consequence of Proposition 4 and the uniqueness of solution of (7), (9). We can also directly substitute the right side of (8) into the equation (7) and verify the result. The symplecticity of the matrix $S(t)$ follows from [5, Lemma 1, Chapter 2]. □

Remark 3. The function $S(t)$ from the previous lemma is defined and continuous on the whole $\mathbb{R}$. Thus the function $Q(t)$ of the form (8) is defined and continuous on the whole $\mathbb{R}$ except at a countable number of points. Thus, such function $Q(t)$ solves equation (7) not only on $\mathcal{I}_0$, but on the whole $\mathbb{R}$ except of countable number of points. The question is, whether such "extended solution" is unique, i.e., if there exists another function $\hat{Q}(t)$ such that $\hat{Q}(t)$ solves equation (7) on the same set as $Q(t)$ with $\hat{Q}(0) = Q(0)$, but $\hat{Q}(t) \neq Q(t)$ in some t. If the matrix dimension is $n \geq 2$, the answer is positive, unless $Q(t)$ is defined and continuous on the whole $\mathbb{R}$. See the following Example 1. In the case when $n = 1$, this "extended solution" is always unique, see Corollary 2.

Example 1. Let $n = 2$, $A = 0$, $B = -I$, $C = I$. Then the Riccati equation (7) is $Q' = I + Q^2$ and its solution is $Q(t) = \tan(tI + \arctan(Q_0))$ with $Q(0) = Q_0$. Now if we take $Q_0 = \tan\begin{pmatrix} \pi/2 & \pi/8 \\ \pi/8 & 5\pi/16 \end{pmatrix}$, then the eigenvalues of the solution

$$Q_1(t) = \tan\begin{pmatrix} t + \pi/2 & \pi/8 \\ \pi/8 & t + 5\pi/16 \end{pmatrix}$$

are

$$\lambda_1(t) = \tan(t + \pi/4), \quad \lambda_2(t) = \tan(t + 9\pi/16).$$

This function is not defined at points

$$t_k = \pi/4 + k\pi,\ s_k = 15\pi/16 + k\pi, \quad k \in \mathbb{Z}.$$

If we take $Q_0 = \tan\begin{pmatrix} 5\pi/16 & \pi/8 \\ \pi/8 & \pi/2 \end{pmatrix}$, then we have the solution

$$Q_2(t) = \tan\begin{pmatrix} t + 5\pi/16 & \pi/8 \\ \pi/8 & t + \pi/2 \end{pmatrix},$$

with exactly the same eigenvalues. Now if we take any function $Q(t)$ such that $Q(t) = Q_1(t)$ on the interval $(-\pi/16, \pi/4)$ and on every other interval (t_k, s_k) or (s_k, t_{k+1}) either $Q(t) = Q_1(t)$ for all t from this interval or $Q(t) = Q_2(t)$ for all t from this interval, then such $Q(t)$ solves $Q' = I + Q^2$ on the same set as $Q_1(t)$ and $Q(0) = Q_1(0)$.

In Definition 2 we define the projective continuity of symmetric matrix-valued functions in such way, that the function $Q(t)$ of the form (8) will be the unique solution of (7), (9), such that it is projectively continuous.

By adding a point at infinity to $\mathbb{R}$ we get the set that is called a projectively extended real line. We can define a mapping that maps uniquely all points q of this set to points on the circle with radius $r = \frac{1}{2}$. Now, instead of the values of the original real function $q(t)$, we can investigate the values of the sine and cosine of the position angle φ of its image; see Figure 1.

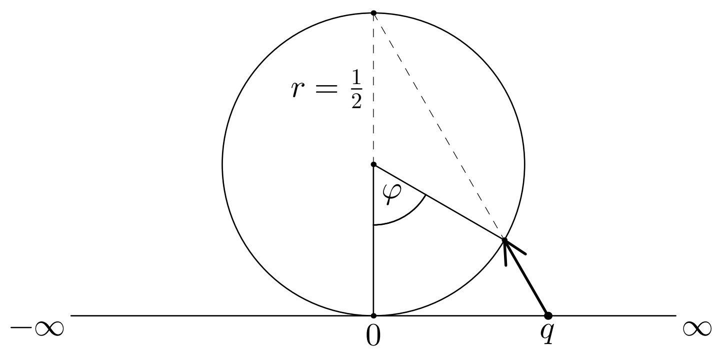

Figure 1: Mapping of a point on the real line to a point on the circle with radius $r = \frac{1}{2}$.

In the following part of this section we apply the same idea to symmetric matrix-valued functions.

Definition 2 (Projectively continuous function)**.** We say that a matrix function $Q : \mathbb{R} \to \mathbf{S}$ is *projectively continuous* on an interval $\mathcal{I}$ (resp. on $\mathbb{R}$) if $Q(t)$ is defined and continuous at all $t \in \mathcal{I}$ (resp. at all $t \in \mathcal{I}$ for any interval $\mathcal{I} \subset \mathbb{R}$) except of a finite number of points and if the functions $Z_s(t)$, $Z_c(t)$, defined as

$$Z_s(t) := \lim_{\tau \to t} \sin(2\arctan(Q(\tau))) = \lim_{\tau \to t} 2\, Q(\tau)(I + Q^2(\tau))^{-1}, \tag{10}$$

$$Z_c(t) := \lim_{\tau \to t} \cos(2\arctan(Q(\tau))) = \lim_{\tau \to t} (I - Q^2(\tau))(I + Q^2(\tau))^{-1}, \tag{11}$$

are defined and continuous at all $t \in \mathcal{I}$ (resp. at all $t \in \mathbb{R}$).

Now we summarize several identities that we will use later.

Lemma 3. *Let $Q \in \mathbf{S}$ and let $Z_s = 2\, Q(I + Q^2)^{-1}$, $Z_c = (I - Q^2)(I + Q^2)^{-1}$. Then the following identities hold*

$$Q = Z_s(I + Z_c)^{-1} = (I + Z_c)^{-1} Z_s, \tag{12}$$

$$2(I + Q^2)^{-1} = I + Z_c, \tag{13}$$

$$2Q^2(I + Q^2)^{-1} = I - Z_c. \tag{14}$$

Proof. Since Q is a symmetric matrix, then also Z_s and Z_c are symmetric; the matrices commute and we compute directly

$$\begin{aligned}
Z_s(I+Z_c)^{-1} &= 2\,Q(I+Q^2)^{-1}[I+(I-Q^2)(I+Q^2)^{-1}]^{-1}\\
&= 2\,Q(I+Q^2)^{-1}[I+[2I-(I+Q^2)](I+Q^2)^{-1}]^{-1}\\
&= 2\,Q(I+Q^2)^{-1}[2(I+Q^2)^{-1}]^{-1} = Q,\\
(I+Z_c)(I+Q^2) &= I+Q^2+I-Q^2 = 2I,\\
(I-Z_c)(I+Q^2) &= I+Q^2-I+Q^2 = 2Q^2.
\end{aligned}$$

The proof is complete. □

Lemma 4. *Let $\tilde{U}(t), \bar{U}(t), \tilde{X}(t), \bar{X}(t)$ be the functions from Lemma 2. The matrix function $Q(t) = (\tilde{U}(t) + \bar{U}(t)Q_0)(\tilde{X}(t) + \bar{X}(t)Q_0)^{-1}$ is projectively continuous on $\mathbb{R}$.*

Proof. We will show that if $Q(t) = U(t)X^{-1}(t)$ and $X(t), U(t)$ are matrix functions defined and differentiable on the whole $\mathbb{R}$, then the limits in (10) and (11) exist and are finite for all $t \in \mathbb{R}$. We denote the adjugate of the matrix $X(t)$ as $\operatorname{adj}(X(t)) := X^*(t)$ and its determinant $\det(X(t)) := d(t)$. Both $d(t)$ and $U(t)X^*(t)$ are defined and differentiable on the whole $\mathbb{R}$. Further, $U(t)X^*(t)$ is symmetric, thus it can be decomposed as

$$U(t)X^*(t) = V(t)\,\Lambda(t)\,V^T(t),$$

where $V(t)$ is an orthogonal matrix, and $\Lambda(t)$ is a diagonal matrix with the eigenvalues of $U(t)X^*(t)$ on its diagonal. All these eigenvalues are real; we denote them as $\lambda_1(t), \dots, \lambda_n(t)$. The functions $V(t)$ and $\lambda_1(t), \dots, \lambda_n(t)$ are continuous on $\mathbb{R}$, see [15, Theorem 6.1, Chapter II]. Further we have

$$Q(t)(I+Q^2(t))^{-1} = V(t)\,L(t)\,V^T(t), \tag{15}$$

where

$$L(t) = \frac{\Lambda(t)}{d(t)}\left[I + \left(\frac{\Lambda(t)}{d(t)}\right)^2\right]^{-1}$$

is a diagonal matrix and on the diagonal it has the eigenvalues of the matrix $Q(t)(I + Q^2(t))^{-1}$; we denote them as $l_1(t), \dots, l_n(t)$. Further we denote

$$\frac{\lambda_1(t)}{d(t)} =: y_1(t), \ \dots, \ \frac{\lambda_n(t)}{d(t)} =: y_n(t).$$

From the continuity of the functions $\lambda_1(t), \dots, \lambda_n(t)$ and $d(t)$ we have that all the limits

$$\lim_{\tau\to t} l_1(\tau) = \lim_{\tau\to t} \frac{y_1(\tau)}{1 + y_1^2(\tau)}, \dots, \lim_{\tau\to t} l_n(\tau) = \lim_{\tau\to t} \frac{y_n(\tau)}{1 + y_n^2(\tau)},$$

exist and are finite; it is because for each $i = 1, \dots, n$ either $\lim\limits_{\tau\to t} y_i(\tau)$ exists and is finite, or $\lim\limits_{\tau\to t} |y_i(\tau)| = \infty$ and then for $y := y_i(\tau)$ we have

$$\lim_{\tau\to t} l_i(\tau) = \lim_{\tau\to t} \frac{y_i(\tau)}{1 + y_i^2(\tau)} = \lim_{|y|\to\infty} \frac{y}{1 + y^2} = 0.$$

From this and (15) it follows that

$$\lim_{\tau\to t} 2\, Q(\tau)(I + Q^2(\tau))^{-1} = 2 \left(\lim_{\tau\to t} V(\tau)\right) \left(\lim_{\tau\to t} L(\tau)\right) \left(\lim_{\tau\to t} V^T(\tau)\right),$$

where all the limits are finite. Similarly, we have

$$(I - Q^2(t))(I + Q^2(t))^{-1} = V(t)\, M(t)\, V^T(t), \tag{16}$$

where

$$M(t) = \left[I - \left(\frac{\Lambda(t)}{d(t)}\right)^2 \right] \left[I + \left(\frac{\Lambda(t)}{d(t)}\right)^2 \right]^{-1}$$

is a diagonal matrix; we denote as $m_1(t), \dots, m_n(t)$ its entries on the diagonal. Again, all the limits

$$\lim_{\tau\to t} m_1(\tau) = \lim_{\tau\to t} \frac{1 - y_1^2(\tau)}{1 + y_1^2(\tau)}, \dots, \lim_{\tau\to t} m_n(\tau) = \lim_{\tau\to t} \frac{1 - y_n^2(\tau)}{1 + y_n^2(\tau)}$$

exist and are finite; for each $i = 1, \dots, n$ either $\lim\limits_{\tau\to t} y_i(\tau)$ exists and is finite, or $\lim\limits_{\tau\to t} |y_i(\tau)| = \infty$ and then for $y := y_i(\tau)$ we have

$$\lim_{\tau\to t} m_i(\tau) = \lim_{\tau\to t} \frac{1 - y_i^2(\tau)}{1 + y_i^2(\tau)} = \lim_{|y|\to\infty} \frac{1 - y^2}{1 + y^2} = -1.$$

From this and (16) it follows that

$$\lim_{\tau\to t} (I - Q^2(\tau))(I + Q^2(\tau))^{-1} = \left(\lim_{\tau\to t} V(\tau)\right) \left(\lim_{\tau\to t} M(\tau)\right) \left(\lim_{\tau\to t} V^T(\tau)\right),$$

where all the limits are finite. □

Lemma 5. *Let the matrix functions* $Q, \hat{Q} : \mathbb{R} \to \mathbf{S}$ *be projectively continuous on* $\mathcal{I}_0$*. Let* Q *satisfy the Riccati equation* (7) *for all* $t \in \mathcal{I}_0$ *such that* $Q(t)$ *is defined at* t *and let* $\hat{Q}$ *satisfy the Riccati equation* (7) *for all* $t \in \mathcal{I}_0$ *such that* $\hat{Q}(t)$ *is defined at* t*. If* $Q(0) = \hat{Q}(0)$ *then* $Q(t) = \hat{Q}(t)$ *for all* $t \in \mathcal{I}_0$ *such that both* $Q(t)$ *and* $\hat{Q}(t)$ *are defined at* t*.*

Proof. Let $Z_s(t)$ and $Z_c(t)$ be the functions defined in (10) and (11) via $Q(t)$ and let $\hat{Z}_s(t)$ and $\hat{Z}_c(t)$ be the functions defined in (10) and (11) via $\hat{Q}(t)$. We compute the derivatives of $2\,Q(t)(I+Q^2(t))^{-1}$, $(I-Q^2(t))(I+Q^2(t))^{-1}$ with respect to t. For brevity, we do not write the argument t in these computations. We get

$$\begin{aligned}
[2\,Q(I+Q^2)^{-1}]' =& 2\,Q'(I+Q^2)^{-1} - 2\,Q(I+Q^2)^{-1}(QQ'+Q'Q)(I+Q^2)^{-1} \\
=& 2\,(I+Q^2)^{-1}Q'(I+Q^2)^{-1} - 2\,Q(I+Q^2)^{-1}Q'Q(I+Q^2)^{-1} \\
=& 2\,(I+Q^2)^{-1}(-A^TQ - QA - QBQ + C)(I+Q^2)^{-1} \\
& - 2\,Q(I+Q^2)^{-1}(-A^TQ - QA - QBQ + C)Q(I+Q^2)^{-1},
\end{aligned}$$

$$\begin{aligned}
[(I-Q^2)(I+Q^2)^{-1}]' =& (-QQ' - Q'Q)(I+Q^2)^{-1} \\
& - (I-Q^2)(I+Q^2)^{-1}(QQ'+Q'Q)(I+Q^2)^{-1} \\
=& -2\,(I+Q^2)^{-1}(QQ'+Q'Q)(I+Q^2)^{-1} \\
=& -2\,(I+Q^2)^{-1}Q(-A^TQ - QA - QBQ + C)(I+Q^2)^{-1} \\
& - 2\,(I+Q^2)^{-1}(-A^TQ - QA - QBQ + C)Q(I+Q^2)^{-1}.
\end{aligned}$$

Now from the definition of $Z_s(t)$ and $Z_c(t)$ and from identities (13) and (14) we get

$$\left.\begin{aligned}
2Z_s' =& -(I+Z_c)A^TZ_s - Z_sA(I+Z_c) - Z_sBZ_s + (I+Z_c)C(I+Z_c) \\
& + Z_sA^T(I-Z_c) + (I-Z_c)AZ_s + (I-Z_c)B(I-Z_c) - Z_sCZ_s, \\
2Z_c' =& Z_sA^TZ_s + (I-Z_c)A(I+Z_c) + (I-Z_c)BZ_s - Z_sC(I+Z_c) \\
& + (I+Z_c)A^T(I-Z_c) + Z_sAZ_s + Z_sB(I-Z_c) - (I+Z_c)CZ_s.
\end{aligned}\right\} \quad (17)$$

These equations hold for all $t \in \mathcal{I}_0$, even where $Q(t)$ is not defined, it is because of the limit in the definition of $Z_s(t)$ and $Z_c(t)$. Similarly we can get that the functions $\hat{Z}_s(t)$ and $\hat{Z}_c(t)$ are also satisfying equations (17) for all $t \in \mathcal{I}_0$. System (17) with the initial conditions

$$Z_s(0) = 2\,Q(0)(I + Q^2(0))^{-1}, \quad Z_c(0) = (I - Q^2(0))(I + Q^2(0))^{-1}$$

has a unique solution, it follows from the well-known Picard–Lindelöf Theorem (see e.g. [12, Theorem 1.1, Chapter 2]). Since the functions $Z_s(t)$, $Z_c(t)$, $\hat{Z}_s(t)$, $\hat{Z}_c(t)$ are continuous on $\mathcal{I}_0$ and $\hat{Z}_s(0) = Z_s(0)$, $\hat{Z}_c(0) = Z_c(0)$, from the uniqueness of the solution we get that $\hat{Z}_s(t) = Z_s(t)$, $\hat{Z}_c(t) = Z_c(t)$ on $\mathcal{I}_0$. Now, from (12) we get that

$$Q(t) = Z_s(t)(I + Z_c(t))^{-1} = \hat{Z}_s(t)(I + \hat{Z}_c(t))^{-1} = \hat{Q}(t)$$

for all $t \in \mathcal{I}_0$ such that both $Q(t), \hat{Q}(t)$ are defined at t. □

Lemma 6. *Let the matrix function $Q(t) : \mathbb{R} \to \mathbf{S}$ be projectively continuous on $\mathcal{I}_0$, let Q satisfy the Riccati equation* (7) *for all $t \in \mathcal{I}_0$ such that $Q(t)$ is defined at t and let $Q(0) = Q_0$. Then*

$$Q(t) = (\tilde{U}(t) + \bar{U}(t)Q_0)(\tilde{X}(t) + \bar{X}(t)Q_0)^{-1}$$

for all t such that $Q(t)$ is defined at t, where $\tilde{U}(t), \bar{U}(t), \tilde{X}(t), \bar{X}(t)$ are the functions from Lemma 2.

Proof. It is a consequence of Lemmas 4–5. □

Corollary 2. *Let the real function $q : \mathbb{R} \to \mathbb{R}$ satisfy the Riccati equation* (7) *with $n = 1$, i.e.,*

$$q'(t) + 2A\,q(t) + B\,q^2(t) - C = 0$$

for all $t \in \mathcal{I}_0$ such that $q(t)$ is defined at t and let $q(0) = q_0$. Then $q(t)$ is projectively continuous on $\mathcal{I}_0$ and

$$q(t) = \frac{\tilde{U}(t) + \bar{U}(t)\,q_0}{\tilde{X}(t) + \bar{X}(t)\,q_0} \tag{18}$$

for all t such that $q(t)$ is defined at t, where $\tilde{U}(t), \bar{U}(t), \tilde{X}(t), \bar{X}(t)$ are the (real) functions from Lemma 2 *with $n = 1$.*

Proof. We have

$$Z_s(t) = \lim_{s\to t} \frac{2\,q(s)}{1+q^2(s)}, \quad Z_c(t) := \lim_{s\to t} \frac{1-q^2(s)}{1+q^2(s)},$$

and since $q(t)$ is a solution of (7), we also have that either $\lim_{s\to t} q(s)$ exists and is finite or $\lim_{s\to t} |q(s)| = \infty$. Hence, both limits in the definition of Z_s and Z_t exist and are finite. From Lemma 6 we obtain (18). □

Remark 4. The interval $\mathcal{I}_0$ in Lemmas 5–6 and in Corollary 2 can be replaced by $\mathbb{R}$.

4. Relation between discrete and continuous case

First we show a connection between the Riccati form (5) and the form in the discrete version (2). The discrete Riccati matrix equation (2) can be written as

$$Q_{k+1} = (\mathcal{C}_k + \mathcal{D}_k Q_k)(\mathcal{A}_k + \mathcal{B}_k Q_k)^{-1}. \tag{19}$$

This is possible because $\mathcal{A}_k + \mathcal{B}_k Q_k$ is invertible for all solutions of (2), see [22, Lemma 2.1].

Remark 5. When we compare equation (19) with (8), we can see that if $Q(t)$ is a solution of the Riccati matrix differential equation (7) defined on $[0, N]$, then

$$Q_0 = Q(0), Q_1 = Q(1), \ldots, Q_N = Q(N)$$

solves the discrete Riccati matrix equation (19) with

$$\begin{pmatrix} \mathcal{A}_k & \mathcal{B}_k \\ \mathcal{C}_k & \mathcal{D}_k \end{pmatrix} = \begin{pmatrix} \tilde{X}(1) & \bar{X}(1) \\ \tilde{U}(1) & \bar{U}(1) \end{pmatrix}, \quad k = 0, \ldots, N-1.$$

Remark 6. A converse relation also exists. Let $\mathcal{S} = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$ be a $2n \times 2n$ symplectic matrix and define the function $f:\ \mathbf{S} \to \mathbf{S}$ as

$$f(Q) = (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1}, \tag{20}$$

i.e., $Q_0, Q_1 = f(Q_0)$ solves a discrete Riccati matrix equation (19). We take a matrix H such that $\mathcal{S} = \exp(H) = I + H + \frac{H^2}{2!} + \cdots + \frac{H^k}{k!} + \cdots$. Such matrix

H is logarithm of the symplectic matrix $\mathcal{S}$. The solution $S(t)$ of the differential equation

$$S'(t) = H \cdot S(t)$$

with the initial condition $S(0) = I$ satisfies $S(1) = \left(\begin{smallmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{smallmatrix}\right)$. If the matrix H is hamiltonian, i. e.

$$H = \begin{pmatrix} A & B \\ C & -A^T \end{pmatrix}, \tag{21}$$

where A, B, C are real $n \times n$ matrices and B, C are symmetric, then, by Lemma 2, the solution of Riccati differential equation (7) with the coefficients A, B, C defined in (21) and with $Q(0) = Q_0$ satisfies $Q(1) = f(Q_0)$.

For details on the matrix exponential and the matrix logarithm, see [13]. The logarithm of a symplectic matrix always exists, but may be complex. Conditions, under which a real symplectic matrix has a real Hamiltonian logarithm, are given in [6].

Now we consider the general case; for a function $f : \mathbf{S} \to \mathbf{S}$, we want to find a symmetric matrix differential equation (3) such that its solution $Q(t)$ with $Q(0) = Q_0$ has $Q(1) = f(Q_0)$ for all Q_0 such that $f(Q_0)$ is defined. This can be done with the use of the iteration theory. The matrix $Q(1)$ is the first iteration of Q_0, and $Q(t)$ is the t-th iteration of Q_0. Non-integer iterations $f^t(Q)$ can be represented by an iteration function $F(Q, t) = f^t(Q)$.

It seems natural to ask this iteration function F to be defined and continuous in t on the whole interval $[0, 1]$, as it is e.g. in [21], but then some of the functions of the form (20) would not have an iteration function. Hence we will just ask the iteration function to be projectively continuous.

Definition 3 (Real iteration function)**.** Let the function $f : \mathbf{S} \to \mathbf{S}$ be defined on $\mathbf{M} \subset \mathbf{S}$ and let $\mathcal{I}$ be an open interval such that $[0, 1] \subset \mathcal{I}$. A function $F :\ \mathbf{S} \times \mathbb{R} \to \mathbf{S}$ defined on $\mathbf{D} \subset (\mathbf{S} \times \mathcal{I})$ is called a *real iteration function* of the function f, if it is for all $Q \in \mathbf{S}$ projectively continuous on $\mathcal{I}$, the equation

$$F(Q, 0) = Q \tag{22}$$

holds for all $Q \in \mathbf{S}$, the equation

$$F(Q,1) = f(Q) \tag{23}$$

holds for all $Q \in \mathbf{M}$ and if $(Q,t) \in \mathbf{D}$, then $(F(Q,t),h) \in \mathbf{D}$ if and only if $(Q,t+h) \in \mathbf{D}$ and for such Q,t,h the translation equation

$$F(F(Q,t),h) = F(Q,t+h) \tag{24}$$

holds.

Remark 7. There are many functions that do not have a real iteration function as defined above, because their non-integer iterations are complex-valued (see Remark 6). It is possible to define a complex iteration function in the same way. The next two lemmas hold for such complex-valued (iteration) function $F(t,Q)$ as well.

Under suitable differentiability conditions the "Jabotinsky differential equations" hold for such functions F (see (1–4) in [2]). The following Lemma is an analogue of the second one of these equations.

Lemma 7. *Let $F: \mathbf{D} \subset \mathbf{S} \times \mathbb{R} \to \mathbf{S}$ be a function differentiable with respect to t on $\mathbf{D}$ and satisfy* (24) *whenever both sides are defined. Then*

$$\frac{\partial F(Q,t)}{\partial t} = G(F(Q,t)), \tag{25}$$

$$\textit{where} \quad G(Q) = \frac{\partial F(Q,t)}{\partial t}\Big|_{t=0} \tag{26}$$

holds for all Q,t such that $(Q,t+h) \in \mathbf{D}$ for all h from an interval $\mathcal{I}_0$.

Proof. If $(Q,t+h) \in \mathbf{D}$ for all $h \in \mathcal{I}_0$, then equation (24) holds for Q,t and all such h. We differentiate this equation with respect to h and then put $h = 0$. □

Lemma 8. *Let function $F: \mathbf{S} \times \mathbb{R} \to \mathbf{S}$ defined on $\mathbf{D} \subset (\mathbf{S} \times \mathcal{I})$ be a real iteration function of the function $f: \mathbf{S} \to \mathbf{S}$ defined on $\mathbf{M} \subset \mathbf{S}$. Further let F be differentiable with respect to t on $\mathbf{D}$. Let $Q_0 \in \mathbf{S}$. Then the function $Q(t) := F(Q_0,t): \mathbb{R} \to \mathbf{S}$ is such that*

$$Q(0) = Q_0 \quad \textit{and} \quad Q' = G(Q) \tag{27}$$

on $\mathrm{D} = \{t : (Q_0, t) \in \mathbf{D}\}$*, where* $G(Q)$ *is the function from* (26)*. Moreover, if* $Q_0 \in \mathbf{M}$*, then* $Q(1) = f(Q_0)$*.*

Proof. The function F is differentiable in t on the whole $\mathbf{D}$, it is defined in $(Q_0, 0)$ and $Q(0) = Q_0$, by (22). Further, $F(Q_0, t)$ is projectively continuous on $\mathcal{I}$, thus if $t \in \mathcal{I}$ and $(Q_0, t) \in \mathbf{D}$, there exists an open interval $\mathcal{I}_0$ such that $(Q_0, h+t) \in \mathbf{D}$ for all $h \in \mathcal{I}_0$. Then, by Lemma 7, the function $Q(t) := F(Q_0, t)$ satisfies the differential equation from (27) for all $(Q_0, t) \in \mathbf{D}$. If $Q_0 \in \mathbf{M}$, then F is defined in $(Q_0, 1)$ and $Q(1) = f(Q_0)$, by (23). □

5. The order preserving property in discrete case

Now we formulate the main result of this paper. It says that if f has real iterations such that $Q \leq \hat{Q}$ implies $f^t(Q) \leq f^t(\hat{Q})$ for all t from an interval $\mathcal{I}_0$, then the function f is of the form (20).

Theorem 1. *Let* $f : \mathbf{S} \to \mathbf{S}$ *be defined on* $\mathbf{M} \subset \mathbf{S}$ *and let* f *have a real iteration function* $F : \mathbf{S} \times \mathbb{R} \to \mathbf{S}$*, defined and continuously differentiable with respect to* t *on a set* $\mathbf{D} \subset \mathbf{S} \times \mathcal{I}$ *and with the following order preserving property*

$$\begin{aligned}&\forall Q, \hat{Q} \in \mathbf{S} \ \ \textit{such that} \ \ Q \leq \hat{Q} \ \ \exists \mathcal{I}_0 \ \ \forall t \in \mathcal{I}_0 : \\ &\qquad\qquad\qquad (Q, t), (\hat{Q}, t) \in \mathbf{D} \ \ \textit{and} \ \ F(Q, t) \leq F(\hat{Q}, t).\end{aligned} \tag{28}$$

Then for all $Q \in \mathbf{M}$*,* $f(Q)$ *is of the form* $f(Q) = (\mathcal{C} + \mathcal{D}Q)(\mathcal{A} + \mathcal{B}Q)^{-1}$*, where the matrix* $\left(\begin{smallmatrix}\mathcal{A} & \mathcal{B}\\ \mathcal{C} & \mathcal{D}\end{smallmatrix}\right)$ *is symplectic.*

Proof. Consider the matrix differential equation

$$Y'(t) = G(Y(t)), \tag{29}$$

where $G(Q)$ is the function from (26). First we show that this equation is Riccati differential equation, with use of Corollary 1. Let $Q, \hat{Q} \in \mathbf{S}$ with $Q \leq \hat{Q}$. Define the function $Q(t) := F(Q, t)$ for all t such that $(Q, t) \in \mathbf{D}$ and the function $\hat{Q}(t) := F(\hat{Q}, t)$ for all t such that $(\hat{Q}, t) \in \mathbf{D}$. Then, by Lemma 8, the functions $Q(t)$, respective $\hat{Q}(t)$, are solutions of (29) with initial condition $Q(0) = Q$, resp. $\hat{Q}(0) = \hat{Q}$, on an interval $\mathcal{I}_0$ and from (28) we get that $Q(t) \leq \hat{Q}(t)$ on a (possibly different) interval $\mathcal{I}_0$. Thus, by Corollary 1, it follows that the differential equation (29) is the Riccati differential equation, that is, for all $Q \in \mathbf{S}$, $G(Q) = -A^TQ - QA - QBQ + C$, where A, B, C are real $n \times n$ matrices and B, C are symmetric.

By Lemma 8 we further have that if $Q(t) = F(Q, t)$ with $Q \in \mathbf{S}$, then

$$Q'(t) = G(Q(t)) = -A^TQ(t) - Q(t)A - Q(t)BQ(t) + C$$

holds for all $t \in \mathrm{D} = \{t : (Q, t) \in \mathbf{D}\}$ and hence, by Lemma 6, $Q(t)$ is of the form

$$Q(t) = (\mathcal{C}(t) + \mathcal{D}(t)Q)(\mathcal{A}(t) + \mathcal{B}(t)Q)^{-1}$$

on the whole interval $[0, 1]$, where $\begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix} = S(t)$ is a symplectic matrix for all $t \in \mathbb{R}$.

Finally, from this and (23) we have that for any $Q \in \mathbf{M}$

$$f(Q) = F(Q, 1) = Q(1) = (\mathcal{C}(1) + \mathcal{D}(1)Q)(\mathcal{A}(1) + \mathcal{B}(1)Q)^{-1},$$

with $\begin{pmatrix} \mathcal{A}(1) & \mathcal{B}(1) \\ \mathcal{C}(1) & \mathcal{D}(1) \end{pmatrix}$ symplectic. □

Our aim was to formulate and prove a result for discrete matrix equation, that is similar to Proposition 2, [20, Theorem 1], and also corresponding to the following converse result from [22].

Proposition 5 (Proposition 2.4 from [22])**.** *Assume that Q and $\hat{Q}$ are symmetric solutions of the Riccati equation* (2) *on* $[0, N]_{\mathbb{Z}}$ *such that* $(\mathcal{A}_k + \mathcal{B}_kQ_k)^{-1}\mathcal{B}_k \geq 0$ *on* $[0, N]_{\mathbb{Z}}$*. If* $Q_0 \leq \hat{Q}_0$ *(*$Q_0 < \hat{Q}_0$*), then* $Q_k \leq \hat{Q}_k$ *(*$Q_k < \hat{Q}_k$*) on* $[0, N+1]_{\mathbb{Z}}$*. Moreover, in this case* $(\mathcal{A}_k + \mathcal{B}_k\hat{Q}_k)^{-1}\mathcal{B}_k \geq 0$ *on* $[0, N]_{\mathbb{Z}}$ *as well.*

Remark 8. The assumption $(\mathcal{A}_k + \mathcal{B}_kQ_k)^{-1}\mathcal{B}_k \geq 0$ on $[0, N]_{\mathbb{Z}}$ is necessary, see [22, Remark 2.5].

Remark 9. It is an open problem, how to formulate condition (28) in Theorem 1, if the iterations $f^t(Q)$ of the function f are not real for $t \notin \mathbb{Z}$ (such functions exist, see Remark 6 and Examples 3 and 5). Then the relation "$f^t(Q) \leq f^t(\hat{Q})$"

is not defined for such t. A question is, whether we can ask instead of the existence of real iteration with given properties only that $f : \mathbf{S} \to \mathbf{S}$ is such that for $Q, \hat{Q} \in \mathbf{M}$ with $Q \leq \hat{Q}$ we have $f^t(Q) \leq f^t(\hat{Q})$ for $t \in \{-1, 1\}$, i.e.,

$$\forall Q, \hat{Q} \in \mathbf{M} : \quad Q \leq \hat{Q} \Longleftrightarrow f(Q) \leq f(\hat{Q}) \tag{30}$$

holds, where $\mathbf{M}$ is a suitable subset of $\mathbf{S}$.

6. Examples

Example 2. Let $f(Q) = \mathcal{A}Q\mathcal{A}^T$, where $\mathcal{A}$ is a real symmetric positive definite $n \times n$ matrix. This function f has the real iteration function

$$F(Q,t) = \mathcal{A}^t Q (\mathcal{A}^t)^T$$

defined on $\mathbf{D} = \mathbf{S} \times \mathbb{R}$ with the order preserving property (28). If $Q \leq \hat{Q}$, then the inequality $F(Q,t) \leq F(\hat{Q},t)$ holds for all t. The function $G(Q)$ from (26) is

$$G(Q) = \ln(\mathcal{A})Q + Q\ln(\mathcal{A}).$$

The property (30) holds for $\mathbf{M} = \mathbf{S}$.

Example 3. Let again $f(Q) = \mathcal{A}Q\mathcal{A}^T$, $n = 2$ and $\mathcal{A} = \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then the function f has the complex iteration function

$$F(Q,t) = \begin{pmatrix} e^{2i\pi t} q_{11} & e^{i\pi t} q_{12} \\ e^{i\pi t} q_{12} & q_{22} \end{pmatrix},$$

where $\left(\begin{smallmatrix} q_{11} & q_{12} \\ q_{12} & q_{22} \end{smallmatrix}\right) = Q$. Further, the function $G(Q)$ from (26) is

$$G(Q) = \begin{pmatrix} 2i\pi q_{11} & i\pi q_{12} \\ i\pi q_{12} & 0 \end{pmatrix} = \begin{pmatrix} i\pi & 0 \\ 0 & 0 \end{pmatrix} Q + Q \begin{pmatrix} i\pi & 0 \\ 0 & 0 \end{pmatrix}.$$

The property (30) again holds for $\mathbf{M} = \mathbf{S}$.

Example 4. Let$f(Q) = -Q^{-1}$. This function f has the real iteration function

$$F(Q,t) = \tan\left(\frac{\pi}{2} tI + \arctan(Q)\right),$$

which is not defined on the whole $\mathbf{S}\times\mathbb{R}$. If $Q \leq \hat{Q}$, then the inequality $F(Q,t) \leq F(\hat{Q},t)$ does not hold for all t where both $F(Q,t), F(\hat{Q},t)$ are defined, unless $Q \equiv \hat{Q}$. This is related to the fact that in this case (30) does not hold on the whole $\mathbf{S}$, but only on its subsets. If we take $\mathbf{M}$ such that (30) holds, then $f(Q) \notin \mathbf{M}$. The function $G(Q)$ from (26) is

$$G(Q) = \frac{\pi}{2}(I + Q^2).$$

Example 5. Let $f(Q) = Q^2$. Then its iteration function and the function $G(Q)$ from (26) are

$$F(Q,t) = Q^{2^t}, \quad G(Q) = Q\ln(Q)\ln(2),$$

both may be complex-valued. For example, if $Q = -I$, then

$$F(Q,t) = [\cos(\pi 2^t) + i\sin(\pi 2^t)]\, I \quad \text{and} \quad G(Q) = -i\pi\ln(2)I.$$

The equivalence (30) does not hold on $\mathbf{S}$.

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