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Research Article

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Absolute Stability of Neutral Systems with Lurie Type Nonlinearity

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Abstract: The paper studies absolute stability of neutral differential nonlinear systems

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf(\sigma(t)), \quad \sigma(t) = c^T x(t), \quad t \geqslant 0$$

where x is an unknown vector, A, B and D are constant matrices, b and c are column constant vectors, $\tau > 0$ is a constant delay and f is a Lurie-type nonlinear function satisfying Lipschitz condition. Absolute stability is analyzed by a general Lyapunov-Krasovskii functional with the results compared with those previously known.

Keywords: Absolute stability, exponential stability, neutral differential system, Lurie type nonlinearity

MSC: 93D22, 93D05, 34K20, 34K25, 34K12

1 Introduction

The paper considers absolute stability of a system of nonlinear differential equations of neutral type

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf(\sigma(t)), \quad t \geqslant 0 \tag{1.1}$$

where $x = (x_1, \dots, x_n)^T : [-\tau, \infty) \to \mathbb{R}^n$, A, B and D are $n \times n$ constant matrices, b is an n-dimensional column constant vector, $\tau > 0$ is a constant delay, $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 0$, is a Lurie-type nonlinear function satisfying Lipschitz condition,

$$\sigma(t) := c^T x(t) \tag{1.2}$$

and c is a column n-dimensional constant vector where the superscript T denotes the transpose of a vector (or a matrix).

1.1 On absolute stability

The problem of absolute stability was formulated in 1944 by Lurie and Postnikov in [39]. In [1, 8], generalizations of the problem are carried out, and the well-known Aizermann hypothesis [1, 49] formulated (some

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scholars still adhere to referring to the problems associated with absolute stability as Lurie nonlinear control system ones). The initial period of studying the theory of absolute stability is associated with the name of its founder [39, 40], who tried to solve the problem with the direct Lyapunov method by means of a Lyapunov function of the "quadratic form plus the integral of nonlinearity" type. This classical method for investigating absolute stability using Lyapunov's functions was used by many authors [1, 40–42, 60]. Since the mid 1960's the intensive development in the theory of absolute stability has included the so-called frequency method, initiated by Popov [21, 22, 43, 45]. In addition to the direct use of so called Popov's criterion for the study of absolute stability, special methods for its application are sought and analogous frequency criteria are formulated [2, 25]. An in-depth analysis of both of the above methods, a study of the relationship between them, and a number of new original approaches are contained in the numerous publications by Yakubovich and his followers [21, 22, 34, 59]. When constructing mathematical models of real systems for a more precise description of the operation of systems, there is a need to consider equations with delay and, in general, neutral type equations (for a recent overview we refer to [48]). Unfortunately, time-delay phenomenon can frequently degrade performance of control systems or even can lead to instability. Therefore, the research of stability in general and, particularly, absolute stability of time-delay Lurie systems is of important significance [10, 11, 17, 19, 26, 28–30, 32, 35, 47]. In [31], Krasovskii replaces Lyapunov functions by what is now called Lyapunov-Krasovskii functionals, which find ample applications in setting out the conditions of stability. Numerous ingenious constructions have been published of such functionals, often using the so-called Razumihin condition [50]. In particular, we refer to [5, 9, 20, 46, 51–53, 56, 58]. Surveys and overviews of the methods and results related to absolute stability achieved since the birth of the theory in 1944 can be found, for example, in [2, 21, 32–34]). Note that a positivity-based approach, closely related to the concept of the Lyapunov functions, which uses the nonnegativity of entries of the Cauchy matrices, is developed in [3, 4, 6, 7, 12, 13, 15]. The above methods are widely used in investigating the stability of neutral differential equations. For recent results we also refer to [14, 16, 24, 36, 54, 55, 57] and to the references therein.

1.2 Preliminaries, the problem considered, and the paper structure

Throughout the paper we use the following restriction on the nonlinearity of f (not mentioning this explicitly when formulating results). Namely, assume that there exists a $k > 0$ such that

$$\left[k\omega - f(\omega)\right] f(\omega) > 0, \quad \forall \omega \neq 0. \tag{1.3}$$

Such a nonlinearity is often said to be of a Lurie type or, based on a geometrical interpretation of (1.3), we also say that f satisfies a sectorial-condition.

Let $\varphi\colon [-\tau, 0] \to \mathbb{R}^n$ be a continuously differentiable vector-function. In addition to system (1.1), consider an initial condition

$$x(t) = \varphi(t), \quad \dot{x}(t) = \dot{\varphi}(t), \quad t \in [-\tau, 0]. \tag{1.4}$$

A solution of the problem (1.1), (1.4) is understood within the meaning of the following definition.

Definition 1. *A function $x\colon [-\tau, \infty) \to \mathbb{R}^n$ is called a solution of* (1.1), (1.4) *if it is continuous on* $[-\tau, \infty)$, *continuously differentiable on* $[-\tau, \infty) \setminus \{\tau s, s = 0, 1, \dots\}$, *satisfying* (1.1) *on* $[0, \infty) \setminus \{\tau s, s = 0, 1, \dots\}$, *and satisfying* (1.4).

The existence of a solution of the problem (1.1), (1.4) on $[-\tau, \infty)$ within the meaning of Definition 1, its uniqueness and continuous dependence on initial values (1.4) are the consequences of the linearity of the first three terms in the right-hand side of (1.1), the Lipschitzeanity of f and the sectorial condition (1.3).

For a vector $y = (y_1, \dots, y_n)^T\colon [-\tau, \infty) \to \mathbb{R}^n$, the following norms are used in the paper

$$|y(t)| = \left(\sum_{i=1}^{n} y_i^2(t)\right)^{1/2}, \quad \text{if} \quad t \in [-\tau, \infty),$$

$$\|y\|_{t,\tau} = \max_{-\tau\leqslant s\leqslant 0}\left\{|y(s+t)|\right\}, \quad \text{if} \quad t \in [0,\infty),$$

$$\|y\|_{t,\tau,\varsigma} = \left(\int_{t-\tau}^{t} \mathrm{e}^{-\varsigma(t-s)}|y(s)|^2 ds\right)^{1/2}, \quad \text{if} \quad t \in [0,\infty),$$

where $\varsigma > 0$ is a constant. If $\mathcal{B}$ is a symmetric positive-definite matrix, denote by $\lambda_{\max}(\mathcal{B})$, $\lambda_{\min}(\mathcal{B})$ its maximal and minimal eigenvalues. We also use a norm of a square matrix $\mathcal{A}$, induced by above Euclidean norm of a vector, computed by formula

$$|\mathcal{A}| = \left(\lambda_{\max}(\mathcal{A}^T\mathcal{A})\right)^{1/2}. \tag{1.5}$$

Listed below are the definitions of stability used.

Definition 2. *The zero solution of system* (1.1) *is exponentially stable in the metric C^0 if, for the solution $x\colon [-\tau,\infty) \to \mathbb{R}^n$ of* (1.1), (1.4) *we have*

$$|x(t)| \leqslant \mathcal{F}_0(\varphi,\dot{\varphi})\,\mathrm{e}^{-\eta_0 t}, \quad t \geqslant 0, \tag{1.6}$$

where $\mathcal{F}_0$ is a positive definite functional and $\eta_0 > 0$ is a constant.

Definition 3. *The zero solution of system* (1.1) *is exponentially stable in the metric C^1, if it is stable in the metric C^0 and, for $t \in [0,\infty)\setminus\{\tau s, s = 0,1,\dots\}$,*

$$|\dot{x}(t)| \leqslant \mathcal{F}_1(\varphi,\dot{\varphi})\,\mathrm{e}^{-\eta_1 t}, \tag{1.7}$$

where $\mathcal{F}_1$ is a positive definite functional and $\eta_1 > 0$ is a constant.

Definition 4. *The system* (1.1) *is said to be absolutely stable if its zero solution is exponentially stable in the metric C^1 for an arbitrary function f, satisfying* (1.3).

In the present paper we solve the problem of the absolute stability of system (1.1) within the meaning of Definition 4. The paper is structured as follows. First, the exponential C^0 stability of (1.1) is studied in part 2 with the main result of this part being Theorem 1. Its proof is carried out by the Lyapunov-Krasovskii method of functionals using a special functional V. Such an approach needs the positivity of a matrix (the matrix S below) appearing in the estimate of the derivative of the functional V along solutions of (1.1). Second, the exponential C^1 stability of (1.1) is studied in part 3. The proof of the main result of this part (Theorem 2) is based on the possibility to exclude the delayed derivative term $D\dot{x}(t-\tau)$ from (1.1), considering then (in an admissible domain) only the delayed system derived instead (Lemma 1). Then, the final statement (Theorem 3) is formulated. The results are illustrated by an example in part 4. The paper is concluded by part 5 with comments and conclusions concerning the approach used. Relations with the investigations of other authors are discussed as well. In the formulations of the theorems on C^0 or C^1 exponential stability, functionals $\mathcal{F}_i$ and constants η_i, $i = 0,1$ are defined explicitly with their constructions following from the methods of proofs.

2 Exponential C^0 Stability

Below, we give conditions for C^0 exponential stability of (1.1). From the proof, an estimate of the convergence of solutions follows as well. To solve the problem of absolute stability within the meaning of Definition 4 for system (1.1) under restriction (1.3), the Lyapunov-Krasovskii functional (2.1) will be used of the quadratic type with respect to both the current coordinates and their derivatives,

$$V(x(t)) = x^T(t)\,Hx(t) + \int_{t-\tau}^{t} \mathrm{e}^{-\varsigma_1(t-s)}x^T(s)\,G_1x(s)ds + \int_{t-\tau}^{t} \mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)\,G_2\dot{x}(s)\,ds + \beta\int_0^{\sigma(t)} f(s)ds. \tag{2.1}$$

In (2.1), $t \geqslant 0$, x is a solution of system (1.1), H, G_1, G_2 are positive-definite symmetric matrices, $\varsigma_1 > 0$, $\varsigma_2 > 0$ and $\beta \geqslant 0$. The condition (1.3) and the well-known properties of quadratic forms imply the two-sided estimate of functional (2.1)

$$\begin{aligned}
&\lambda_{\min}(H)\,|x(t)|^2 + \lambda_{\min}(G_1)\|x\|_{t,\tau,\varsigma_1} + \lambda_{\min}(G_2)\|\dot{x}\|_{t,\tau,\varsigma_2} \\
&\leqslant V(x(t)) \leqslant \left(\lambda_{\max}(H) + \frac{1}{2}\beta k\,|c|^2\right)|x(t)|^2 + \int_{t-\tau}^{t} e^{-\varsigma_1(t-s)} x^T(s) G_1 x(s)\,ds \\
&\quad + \int_{t-\tau}^{t} e^{-\varsigma_2(t-s)} \dot{x}^T(s) G_2 \dot{x}(s)\,ds \\
&\leqslant \left(\lambda_{\max}(H) + \frac{1}{2}\beta k\,|c|^2\right)|x(t)|^2 + \lambda_{\max}(G_1)\|x\|_{t,\tau,\varsigma_1} + \lambda_{\max}(G_2)\|\dot{x}\|_{t,\tau,\varsigma_2}.
\end{aligned} \tag{2.2}$$

These will be used in the paper. We also use the notation

$$\lambda_H := \frac{1}{\lambda_{\min}(H)}\left(\lambda_{\max}(H) + \frac{1}{2}\beta k\,|c|^2\right), \quad \lambda_{HG_1} := \frac{\lambda_{\max}(G_1)}{\lambda_{\min}(H)}, \quad \lambda_{HG_2} := \frac{\lambda_{\max}(G_2)}{\lambda_{\min}(H)}.$$

The stability result formulated below depends, among others, on the positive definiteness of an auxiliary $(3n+1) \times (3n+1)$ matrix S. This matrix depends on matrices, vectors and constants used in (1.1)–(1.3), (2.1) and on a parameter $\nu \geqslant 0$, not yet involved. In other words, the elements of S are constructed using β, ς_1, ς_2, k, ν, b, c, A, B, D, H, G_1, G_2, formally

$$S = S(\beta, \varsigma_1, \varsigma_2, k, \nu, b, c, A, B, D, H, G_1, G_2).$$

In its block form, the matrix S is defined as

$$S := \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12}^T & S_{22} & S_{23} & S_{24} \\ S_{13}^T & S_{23}^T & S_{33} & S_{34} \\ S_{14}^T & S_{24}^T & S_{34}^T & S_{44} \end{pmatrix}$$

with block-matrices

$$\begin{aligned}
S_{11} &:= -A^T H - HA - G_1 - A^T G_2 A, \\
S_{12} &:= -HB - A^T G_2 B, \\
S_{13} &:= -HD - A^T G_2 D, \\
S_{14} &:= -Hb - A^T G_2 b - \frac{1}{2}\left(\beta A^T + \nu I\right) c, \\
S_{22} &:= \ e^{-\varsigma_1 \tau} G_1 - B^T G_2 B, \\
S_{23} &:= -B^T G_2 D, \\
S_{24} &:= -B^T \left(G_2 b + \frac{1}{2}\beta c\right), \\
S_{33} &:= \ e^{-\varsigma_2 \tau} G_2 - D^T G_2 D, \\
S_{34} &:= -D^T \left(G_2 b + \frac{1}{2}\beta c\right), \\
S_{44} &:= -b^T G_2 b - \beta c^T b + \frac{\nu}{k}
\end{aligned}$$

where I in S_{14} is the unit $n \times n$ matrix.

Theorem 1. *If there exist non-negative constants β, ν, positive constants ς_1, ς_2 and positive definite symmetric matrices H, G_1 and G_2 such that the matrix S is positive definite, then the zero solution of system* (1.1) *is exponentially stable in the metric C^0 and the solution of the problem* (1.1), (1.4) *satisfies*

$$|x(t)| \leqslant \left(\sqrt{\lambda_H}\,|\varphi(0)| + \sqrt{\lambda_{HG_1}}\,\|\varphi\|_{0,\tau,\varsigma_1} + \sqrt{\lambda_{HG_2}}\,\|\dot{\varphi}\|_{0,\tau,\varsigma_2}\right) \mathrm{e}^{-\gamma t/2}, \tag{2.3}$$

where $t \in [0,\infty)$ and fixed number γ satisfies

$$0 < \gamma \leqslant \gamma^* := \min\{\varsigma_1, \varsigma_2, \delta\}, \quad \delta := \frac{\lambda_{\min}(S)}{\lambda_{\max}(H) + \dfrac{1}{2}\beta k\,|c|^2}. \tag{2.4}$$

Proof. The proof splits into two parts. In part *i*), an estimate of the total derivative of the functional (2.1) along solutions of the system (1.1) is computed. Part *ii*) deals with the exponential stability of solutions to system (1.1) in the metrix C^0.

i) Let us calculate the total derivative of the functional (2.1) along solutions of (1.1). Assuming $t \in [0,\infty) \setminus \{\tau s, s = 0, 1, \dots\}$, we have

$$\begin{aligned}\frac{d}{dt}V\left(x(t)\right) = {} & \left[Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf\left(\sigma(t)\right)\right]^T Hx(t) \\ & + x^T(t)\,H\left[Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf\left(\sigma(t)\right)\right] \\ & + x^T(t)G_1x(t) - \mathrm{e}^{-\varsigma_1\tau}x^T(t-\tau)\,G_1x(t-\tau) \\ & + \dot{x}^T(t)G_2\dot{x}(t) - \mathrm{e}^{-\varsigma_2\tau}\dot{x}^T(t-\tau)\,G_2\dot{x}(t-\tau) \\ & - \varsigma_1\int_{t-\tau}^{t}\mathrm{e}^{-\varsigma_1(t-s)}x^T(s)\,G_1x(s)ds - \varsigma_2\int_{t-\tau}^{t}\mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)\,G_2\dot{x}(s)ds \\ & + \beta f(\sigma(t))\,c^T\left[Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf(\sigma(t))\right].\end{aligned}$$

Before writing this derivative in a form with the matrix S, we rewrite it using system (1.1) as follows

$$\begin{aligned}\frac{d}{dt}V\left(x(t)\right) = {} & \left[Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf\left(\sigma(t)\right)\right]^T Hx(t) \\ & + x^T(t)\,H\left[Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf\left(\sigma(t)\right)\right] \\ & + x^T(t)G_1x(t) \\ & - \mathrm{e}^{-\varsigma_1\tau}x^T(t-\tau)\,G_1x(t-\tau) \\ & + (Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf(\sigma(t)))^T\,G_2 \\ & \quad\cdot (Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf(\sigma(t))) \\ & - \mathrm{e}^{\varsigma_2\tau}\dot{x}(t-\tau)\,G_2\dot{x}(t-\tau) \\ & - \varsigma_1\int_{t-\tau}^{t}\mathrm{e}^{-\varsigma_1(t-s)}x^T(s)\,G_1x(s)ds - \varsigma_2\int_{t-\tau}^{t}\mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)\,G_2\dot{x}(s)ds \\ & + \beta f(\sigma(t))\,c^T\left[Ax(t) + Bx(t-\tau) + D\dot{x}(t-\tau) + bf(\sigma(t))\right].\end{aligned} \tag{2.5}$$

Define an auxiliary vector

$$X(t) := \begin{pmatrix} x^T(t) & x^T(t-\tau) & \dot{x}^T(t-\tau) & f(\sigma(t)) \end{pmatrix}.$$

Now, all the non-integral terms in the right-hand side of the last expression can be written using this vector and the matrix S. Omitting technicalities, it can be verified term by term that the identity

$$
\begin{aligned}
&\left[Ax(t)+Bx(t-\tau)+D\dot{x}(t-\tau)+bf(\sigma(t))\right]^T Hx(t)\\
&+x^T(t)H\left[Ax(t)+Bx(t-\tau)+D\dot{x}(t-\tau)+bf(\sigma(t))\right]\\
&+x^T(t)G_1x(t)\\
&-\mathrm{e}^{-\varsigma_1\tau}x^T(t-\tau)G_1x(t-\tau)\\
&+(Ax(t)+Bx(t-\tau)+D\dot{x}(t-\tau)+bf(\sigma(t)))^T G_2\\
&\quad\cdot(Ax(t)+Bx(t-\tau)+D\dot{x}(t-\tau)+bf(\sigma(t)))\\
&-\mathrm{e}^{\varsigma_2\tau}\dot{x}(t-\tau)G_2\dot{x}(t-\tau)\\
&+\beta f(\sigma(t))c^T\left[Ax(t)+Bx(t-\tau)+D\dot{x}(t-\tau)+bf(\sigma(t))\right]\\
\equiv&-X(t)SX^T(t)-\nu\left(\sigma(t)-\frac{1}{k}f(\sigma(t))\right)f(\sigma(t))
\end{aligned}
$$

holds. Then, (2.5) can be written as

$$
\begin{aligned}
\frac{d}{dt}V(x(t))=&-X(t)SX^T(t)-\nu\left(\sigma(t)-\frac{1}{k}f(\sigma(t))\right)f(\sigma(t))\\
&-\varsigma_1\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_1(t-s)}x^T(s)G_1x(s)ds-\varsigma_2\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)G_2\dot{x}(s)ds.
\end{aligned}
\tag{2.6}
$$

Due to (1.3),

$$
-\nu\left(\sigma(t)-\frac{1}{k}f(\sigma(t))\right)f(\sigma(t))<0. \tag{2.7}
$$

As S is positive definite, we have

$$
\begin{aligned}
\frac{d}{dt}V(x(t))\leqslant&-\lambda_{\min}(S)\left(\left|x(t)\right|^2+\left|x(t-\tau)\right|^2+\left|\dot{x}(t-\tau)\right|^2+f^2(\sigma(t))\right)\\
&-\varsigma_1\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_1(t-s)}x^T(s)G_1x(s)ds-\varsigma_2\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)G_2\dot{x}(s)ds\\
\leqslant&-\lambda_{\min}(S)\left|x(t)\right|^2\\
&-\varsigma_1\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_1(t-s)}x^T(s)G_1x(s)ds-\varsigma_2\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)G_2\dot{x}(s)ds.
\end{aligned}
\tag{2.8}
$$

ii) Prove the exponential stability of the solutions to system (1.1) in the metrix C^0. Below we analyze two cases dealing with $\varsigma\geqslant\delta$ and $\varsigma<\delta$ where $\varsigma:=\min\{\varsigma_1,\varsigma_2\}>0$.

ii_1) Let $\varsigma\geqslant\delta$. Then, from inequality (2.2), we get

$$
-\left(\lambda_{\max}(H)+\frac{1}{2}\beta k\left|c\right|^2\right)\left|x(t)\right|^2\leqslant -V(x(t))+\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_1(t-s)}x^T(s)G_1x(s)ds+\int\limits_{t-\tau}^{t}\mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)G_2\dot{x}(s)\,ds
$$

and

$$-|x(t)|^2 \leqslant -\frac{1}{\lambda_{\max}(H)+\frac{1}{2}\beta k|c|^2} V\left(x(t)\right)$$
$$+\frac{1}{\lambda_{\max}(H)+\frac{1}{2}\beta k|c|^2}\left(\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_1(t-s)}x^T(s)\,G_1x(s)ds+\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)\,G_2\dot{x}(s)ds\right).$$

Inequality (2.8) yields

$$\frac{d}{dt}V\left(x(t)\right) \leqslant \lambda_{\min}(S)\left(-\frac{1}{\lambda_{\max}(H)+\frac{1}{2}\beta k|c|^2}V(x(t))\right.$$
$$\left.+\frac{1}{\lambda_{\max}(H)+\frac{1}{2}\beta k|c|^2}\left(\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_1(t-s)}x^T(s)\,G_1x(s)ds+\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)\,G_2\dot{x}(s)ds\right)\right)$$
$$-\varsigma_1\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_1(t-s)}x^T(s)\,G_1x(s)ds-\varsigma_2\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)\,G_2\dot{x}(s)ds$$

or

$$\frac{d}{dt}V\left(x(t)\right) \quad\leqslant\quad -\delta V(x(t)) \;-\; (\varsigma_1-\delta)\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_1(t-s)}x^T(s)G_1x(s)ds \;-\; (\varsigma_2-\delta)\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)G_2\dot{x}(s)ds.$$

We get,

$$\frac{d}{dt}V\left(x(t)\right) \leqslant -\delta V(x(t)).$$

Integrating this inequality over $(0, t)$, which is a correct operation because the set of isolated points $\{t = \tau s, s = 0, 1, \dots\}$ is countable, for $t \geqslant 0$, we obtain

$$V(x(t)) \leqslant V(\varphi(0))\mathrm{e}^{-\delta t}. \tag{2.9}$$

ii_2) Let $\varsigma < \delta$. Rewrite the right-hand part of the inequality (2.2) as follows

$$-\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_1(t-s)}x^T(s)G_1x(s)ds - \int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)G_2\dot{x}(s)ds \;\leqslant\; -V(x(t)) + \left(\lambda_{\max}(H)+\frac{1}{2}\beta k|c|^2\right)\left|x(t)\right|^2.$$

Now, using (2.8), we get

$$\begin{aligned}
\frac{d}{dt}V(x(t)) &\leqslant -\lambda_{\min}(S)\,|x(t)|^2 \\
&\quad -\varsigma\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_1(t-s)}x^T(s)\,G_1x(s)ds-\varsigma\int\limits_{t-\tau}^{t} \mathrm{e}^{-\varsigma_2(t-s)}\dot{x}^T(s)\,G_2\dot{x}(s)ds \\
&\leqslant -\lambda_{\min}(S)\,|x(t)|^2+\varsigma\left(-V(x(t))+\left(\lambda_{\max}(H)+\frac{1}{2}k|c|^2\right)|x(t)|^2\right) \\
&= -\varsigma V(x(t))-\left(\lambda_{\min}(S)-\varsigma\left(\lambda_{\max}(H)+\frac{1}{2}\beta k|c|^2\right)\right)|x(t)|^2 \\
&= -\varsigma V(x(t))-(\delta-\varsigma)\left(\lambda_{\max}(H)+\frac{1}{2}\beta k|c|^2\right)|x(t)|^2.
\end{aligned}$$

Then,

$$\frac{d}{dt}V(x(t)) \leqslant -\varsigma V(x(t))$$

and integrating this inequality over $(0, t)$, we obtain (as in ii_1))

$$V(x(t)) \leqslant V(\varphi(0))\, e^{-\varsigma t}, \quad t \geqslant 0. \tag{2.10}$$

From (2.9) and (2.10) we deduce that, in both cases ii_1), ii_2) considered, we have

$$V(x(t)) \leqslant V(\varphi(0))\, e^{-\gamma t}, \quad t \geqslant 0. \tag{2.11}$$

Utilizing (2.2) and (2.11), we get (the below inequalities hold for $t \geqslant 0$ as well)

$$\lambda_{\min}(H)\,|x(t)|^2 + \lambda_{\min}(G_1)\,\|x\|^2_{t,\tau,\varsigma} + \lambda_{\min}(G_2)\,\|\dot{x}\|^2_{t,\tau,\varsigma} \leqslant V(x(t)) \leqslant V(\varphi(0))\, e^{-\gamma t}$$

$$\leqslant \left[\left(\lambda_{\max}(H) + \frac{1}{2}\beta k\,|c|^2\right)|\varphi(0)|^2 + \lambda_{\max}(G_1)\,\|\varphi\|^2_{0,\tau,\varsigma} + \lambda_{\max}(G)\,\|\dot{\varphi}\|^2_{0,\tau,\varsigma}\right] e^{-\gamma t},$$

hence, it follows that

$$|x(t)|^2 \leqslant \frac{1}{\lambda_{\min}(H)}\left[\left(\lambda_{\max}(H) + \frac{1}{2}\beta k\,|c|^2\right)|\varphi(0)|^2 + \lambda_{\max}(G_1)\,\|\varphi\|^2_{0,\tau,\varsigma} + \lambda_{\max}(G_2)\,\|\dot{\varphi}\|^2_{0,\tau,\varsigma}\right] e^{-\gamma t}.$$

Now, after a simple estimation of the right-hand side,

$$|x(t)| \leqslant \mathcal{F}_0(\varphi, \dot{\varphi}) e^{-\gamma t/2}$$

where

$$\mathcal{F}_0(\varphi, \dot{\varphi}) := \left(\sqrt{\lambda_H}\,|\varphi(0)| + \sqrt{\lambda_{HG_1}}\,\|x\|_{0,\tau,\varsigma} + \sqrt{\lambda_{HG_2}}\,\|\dot{\varphi}\|_{0,\tau,\varsigma}\right).$$

Put $\eta_0 = \gamma/2$. Inequality (2.3) is proved. The zero solution of (1.1) is C^0 exponentially stable within the meaning of Definition 2 since (1.6) holds. □

3 Exponential C^1 Stability and Absolute Stability

To prove the exponential stability in metric C^1, we need the following lemma providing a formula transforming neutral system (1.1) to a delayed one of the Volterra type.

Lemma 1. *Let $m \in \{1, 2, \dots\}$ be fixed. Then, for $t \in ((m-1)\tau, m\tau)$, the solution of the problem* (1.1), (1.4) *satisfies the equation*

$$\dot{x}(t) = D^m \dot{x}(t - m\tau) + Ax(t) + \sum_{i=1}^{m-1} D^{i-1}(DA + B)\,x(t - i\tau) + D^{m-1}Bx(t - m\tau) + \sum_{i=0}^{m-1} D^i bf(\sigma(t - i\tau)). \tag{3.1}$$

Proof. We will prove the lemma by induction. Let $x(t)$ be the solution of problem (1.1), (1.4). Then, for $m = 1$, the lemma holds because formula (3.1) coincides with initial system (1.1). Let us recall that, if we deal with a sum of the form $\sum_{i=1}^{0} \dots$, then, by customary definitions, we set such sum equal to zero. This concerns the first sum in (3.1). Assuming that the conclusion of the lemma holds for $m = j-1$ where $j \geqslant 2$ is a fixed natural number, we will show that, then, it holds for $m = j$ as well. For $m = j-1$, $t \in ((j-2)\tau, (j-1)\tau)$, we get from (3.1)

$$\dot{x}(t) = D^{j-1}\dot{x}(t - (j-1)\tau) + Ax(t) + \sum_{i=1}^{j-2} D^{i-1}(DA + B)\,x(t - i\tau) + D^{j-2}Bx(t - (j-1)\tau) + \sum_{i=0}^{j-2} D^i bf(\sigma(t - i\tau)). \tag{3.2}$$

If $m = j$ and $t \in ((j-1)\tau, j\tau)$, then, using (1.1), we get for the term $\dot{x}(t - (j-1)\tau)$ in (3.2):

$$\dot{x}(t - (j-1)\tau) = D\dot{x}(t - j\tau) + Ax(t - (j-1)\tau) + Bx(t - j\tau) + bf\left(\sigma\left(t - (j-1)\tau\right)\right). \tag{3.3}$$

Finally, substituting (3.3) into the right-hand side of (3.2), we derive

$$\begin{aligned}\dot{x}(t) =& D^{j-1}\left[D\dot{x}(t-j\tau)+Ax(t-(j-1)\tau)+Bx(t-j\tau)+bf\left(\sigma\left(t-(j-1)\tau\right)\right)\right]+Ax(t)\\ &+\sum_{i=1}^{j-2}D^{i-1}(DA+B)\,x(t-i\tau)+D^{j-2}Bx(t-(j-1)\tau)+\sum_{i=0}^{j-2}D^{i}bf(\sigma(t-i\tau))\\ =& D^{j}\dot{x}(t-j\tau)+Ax(t)+\sum_{i=1}^{j-1}D^{i-1}(DA+B)\,x(t-i\tau)\\ &+D^{j-1}Bx(t-j\tau)+\sum_{i=0}^{j-1}D^{i}bf(\sigma(t-i\tau)),\end{aligned}$$

i.e., the formula (3.1) holds for $m = j$ as well. □

Theorem 2. *Let the hypotheses of Theorem 1, where γ is fixed, hold and let*

$$0<|D|\mathrm{e}^{\gamma\tau/2}<1. \tag{3.4}$$

Then, the zero solution of the system (1.1) *is exponentially stable in the metric C^1 and the solution of the problem* (1.1), (1.4) *satisfies*

$$|\dot{x}(t)| \leqslant \left(M\sqrt{\lambda_H}\,|\varphi(0)|+\frac{|B|}{|D|}\,\|\varphi\|_{0,\tau}+M\sqrt{\lambda_{HG_1}}\,\|\varphi\|_{0,\tau,\varsigma_1}+\|\dot{\varphi}\|_{0,\tau}+M\sqrt{\lambda_{HG_2}}\,\|\dot{\varphi}\|_{0,\tau,\varsigma_2}\right)\mathrm{e}^{-\gamma t/2}, \tag{3.5}$$

where $t\in[0,\infty)\setminus\{\tau s, s=0,1,\dots\}$ and

$$M := |A|+|b|\,|c|\,k+\frac{|DA+B|+|D|\,|b|\,|c|\,k}{1-|D|\,\mathrm{e}^{\gamma\tau/2}}\,.$$

Proof. By Theorem 1, the zero solution of the system (1.1) is exponentially stable in the metric C^0. We show that the zero solution of (1.1) is also C^1 exponentially stable. As it follows from formula (3.1) in Lemma 1, for $t\in((m-1)\tau, m\tau)$, $m=1,2,\dots$, we have

$$|\dot{x}(t)| \leqslant |D|^m\,\|\dot{x}\|_{0,\tau}+|A|\,|x(t)|+|DA+B|\sum_{i=1}^{m-1}|D|^{i-1}\,|x(t-i\tau)|+|D|^{m-1}\,|B|\,\|x\|_{0,\tau}+\sum_{i=0}^{m-1}|D|^{i}\,|b|\,|f(\sigma(t-i\tau))|.$$

From (1.3) and (1.2) we derive $|f(\sigma(t-i\tau))|\leqslant k\,|\sigma(t-i\tau)|\leqslant k|c||x(t-i\tau)|$ and, therefore,

$$|\dot{x}(t)| \leqslant |D|^m\,\|\dot{x}\|_{0,\tau}+[|A|+|b|\,k\,|c|]\,|x(t)|+\sum_{i=1}^{m-1}|D|^{i-1}\,[|DA+B|+|D||b|k|c|]\,|x(t-i\tau)|+|D|^{m-1}\,|B|\,\|x\|_{0,\tau}\,. \tag{3.6}$$

Let us estimate $|x(t)|$ and $|x(t-i\tau)|$ in (3.6) by formula (2.3). We obtain

$$\begin{aligned}|\dot{x}(t)| \leqslant& |D|^m\,\|\dot{\varphi}\|_{0,\tau}+|D|^{m-1}\,|B|\,\|\varphi\|_{0,\tau}\\ &+[|A|+|b|k\,|c|]\left(\sqrt{\lambda_H}\,|\varphi(0)|+\sqrt{\lambda_{HG_1}}\,\|\varphi\|_{0,\tau,\varsigma_1}+\sqrt{\lambda_{HG_2}}\,\|\dot{\varphi}\|_{0,\tau,\varsigma_2}\right)\mathrm{e}^{-\gamma t/2}\\ &+[|DA+B|+|D|\,|b|\,k\,|c|]|D|^{-1}\cdot\left(\sum_{i=1}^{m-1}|D|^{i}\,\mathrm{e}^{i\gamma\tau/2}\right)\\ &\cdot\left(\sqrt{\lambda_H}\,|\varphi(0)|+\sqrt{\lambda_{HG_1}}\,\|\varphi\|_{0,\tau,\varsigma_1}+\sqrt{\lambda_{HG_2}}\,\|\dot{\varphi}\|_{0,\tau,\varsigma_2}\right)\mathrm{e}^{-\gamma t/2}\\ =& |D|^m\,\|\dot{\varphi}\|_{0,\tau}+|D|^{m-1}\,|B|\,\|\varphi\|_{0,\tau}\end{aligned}$$

$$+ [|A| + |b||k||c|] \left(\sqrt{\lambda_H}\, |\varphi(0)| + \sqrt{\lambda_{HG_1}}\, \|\varphi\|_{0,\tau,\varsigma_1} + \sqrt{\lambda_{HG_2}}\, \|\dot{\varphi}\|_{0,\tau,\varsigma_2} \right) \mathrm{e}^{-\gamma t/2}$$

$$+ \frac{|DA + B| + |D|\,|b|\,k\,|c|}{1 - |D|\,\mathrm{e}^{\gamma\tau/2}} \left[1 - |D|^{m-1}\, \mathrm{e}^{(m-1)\gamma\tau/2} \right]$$

$$\cdot \left(\sqrt{\lambda_H}\, |\varphi(0)| + \sqrt{\lambda_{HG_1}}\, \|\varphi\|_{0,\tau,\varsigma_1} + \sqrt{\lambda_{HG_2}}\, \|\dot{\varphi}\|_{0,\tau,\varsigma_2} \right) \mathrm{e}^{-\gamma t/2}. \tag{3.7}$$

Since $t \in ((m-1)\tau, m\tau)$, then $|D|^m \leqslant \exp((t/\tau) \ln |D|)$. We have

$$|D|^m\, \|\dot{\varphi}\|_{0,\tau} + |D|^{m-1}\, |B|\, \|\varphi\|_{0,\tau} \leqslant \left[\|\dot{\varphi}\|_{0,\tau} + \frac{|B|}{|D|}\, \|\varphi\|_{0,\tau} \right] \exp\left(-\frac{t}{\tau} \ln \frac{1}{|D|} \right)$$

and, by (3.4),

$$|D|^m\, \|\dot{\varphi}\|_{0,\tau} + |D|^{m-1}\, |B|\, \|\varphi\|_{0,\tau} \leqslant \left[\|\dot{\varphi}\|_{0,\tau} + \frac{|B|}{|D|}\, \|\varphi\|_{0,\tau} \right] \mathrm{e}^{-\gamma t/2}.$$

From (3.7), we get

$$|\dot{x}(t)| \leqslant \left[\|\dot{\varphi}\|_{0,\tau} + \frac{|B|}{|D|}\, \|\varphi\|_{0,\tau} \right] \mathrm{e}^{-\gamma t/2}$$

$$+ \left[|A| + |b| k\, |c| + \frac{|DA + B| + |D|\,|b|\,k\,|c|}{1 - |D| e^{\gamma\tau/2}} \left[1 - |D|^{m-1} \mathrm{e}^{(m-1)\gamma\tau/2} \right] \right]$$

$$\cdot \left(\sqrt{\lambda_H}\, |\varphi(0)| + \sqrt{\lambda_{HG_1}}\, \|\varphi\|_{0,\tau,\varsigma_1} + \sqrt{\lambda_{HG_2}}\, \|\dot{\varphi}\|_{0,\tau,\varsigma_2} \right) \mathrm{e}^{-\gamma t/2}. \tag{3.8}$$

Then, for any admissible m, inequality (3.8) with the term

$$-|D|^{m-1} \mathrm{e}^{(m-1)\gamma\tau/2}$$

omitted, yields

$$|\dot{x}(t)| \leqslant \mathcal{F}_1(\varphi, \dot{\varphi}) \mathrm{e}^{-\gamma t/2}$$

where

$$\mathcal{F}_1(\varphi, \dot{\varphi}) := M\sqrt{\lambda_H}\, |\varphi(0)| + \frac{|B|}{|D|}\, \|\varphi\|_{0,\tau} + M\sqrt{\lambda_{HG_1}}\, \|\varphi\|_{0,\tau,\varsigma_1} + \|\dot{\varphi}\|_{0,\tau} + M\sqrt{\lambda_{HG_2}}\, \|\dot{\varphi}\|_{0,\tau,\varsigma_2}.$$

Inequality (3.5) is proved. The zero solution of (1.1) is C^1 exponentially stable within the meaning of Definition 3 where $\eta_1 = \gamma/2$ and (1.7) holds. □

The following theorem on absolute stability of system (1.1) within the meaning of Definition 4 directly follows from Theorem 1 and Theorem 2.

Theorem 3. *If the hypotheses of Theorems 1 and 2 hold, then the zero solution of the system* (1.1) *is absolutely stable.*

4 Example

We will investigate system (1.1) where $n = 2$, $\tau = 1$,

$$A = \begin{pmatrix} -3 & -2 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0.74035 \\ 0.74035 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix},$$

$$b = c = \begin{pmatrix} 0,01 & -0.01 \end{pmatrix}^T, \quad \sigma(t) = c^T x(t) = 0.01 x_1(t) - 0.01 x_2(t),$$

i.e., the system

$$\dot{x}_1(t) = 0.1\dot{x}_1(t-1) - 3x_1(t) - 2x_2(t) + 0.74035x_2(t-1) + 0.01f(\sigma(t)), \tag{4.1}$$

$$\dot{x}_2(t) = 0.1\dot{x}_2(t-1) + 1x_1(t) \qquad + 0.74035x_1(t-1) - 0.01f(\sigma(t)), \tag{4.2}$$

with initial conditions (1.4). Set $\varsigma_1 = 0.01$, $\varsigma_2 = 0.001$, $k = 1$, $\nu = 0.1$, $\beta = 0.01$ and

$$G_1 = \begin{pmatrix} 0.5 & 0.1 \\ 0.1 & 0.2 \end{pmatrix}, \qquad G_2 = \begin{pmatrix} 0.06 & 0 \\ 0 & 0.06 \end{pmatrix}, \qquad H = \begin{pmatrix} 0.67 & 0.43 \\ 0.43 & 0.67 \end{pmatrix}.$$

Part of the below numerical computations was performed by MATLAB. For the eigenvalues of matrices G_1, G_2 and H, we get $\lambda_{\min}(G_1) \doteq 0.1697$, $\lambda_{\max}(G_1) \doteq 0.5303$, $\lambda_{\min}(G_2) = \lambda_{\max}(G_2) = 0.06$, $\lambda_{\min}(H) = 0.24$, $\lambda_{\max}(H) = 1.1$. The matrix S takes the form

$$S \doteq \begin{pmatrix} 2.0600 & 1.5000 & -0.3627 & -0.3627 & -0.0490 & -0.0490 & -0.0003 \\ 1.5000 & 1.2800 & -0.4960 & -0.2295 & -0.0310 & -0.0670 & 0.0042 \\ -0.3627 & -0.4960 & 0.4671 & 0.0990 & 0 & -0.0044 & 0.0005 \\ -0.3627 & -0.2295 & 0.0990 & 0.1651 & -0.0044 & 0 & -0.0005 \\ -0.0490 & -0.0310 & 0 & -0.0044 & 0.0593 & 0 & -0.0001 \\ -0.0490 & -0.0670 & -0.0044 & 0 & 0 & 0.0593 & 0.0001 \\ -0.0003 & 0.0042 & 0.0005 & -0.0005 & -0.0001 & 0.0001 & 0.0999 \end{pmatrix}$$

and $\lambda_{\min}(S) \doteq 0.00000407$. Because all eigenvalues are positive, matrix S is positive definite. Further we have

$$\begin{aligned} \lambda_H &= \frac{1}{\lambda_{\min}(H)} \left(\lambda_{\max}(H) + \frac{1}{2}\beta k |c|^2 \right) \doteq 4.5833, \\ \lambda_{HG_1} &= \frac{\lambda_{\max}(G_1)}{\lambda_{\min}(H)} \doteq 2.2095, \\ \lambda_{HG_2} &= \frac{\lambda_{\max}(G_2)}{\lambda_{\min}(H)} = 0.25, \\ \delta &= \frac{\lambda_{\min}(S)}{\lambda_{\max}(H) + \frac{1}{2}\beta k |c|^2} \doteq 3.7026 \cdot 10^{-6}, \\ \gamma^* &= \min\{\varsigma_1, \varsigma_2, \delta\} = \delta \end{aligned}$$

Moreover

$$|A| \doteq 3.7025, \quad |B| = 0.74035, \quad |D| = 0.1, \quad |DA + B| \doteq 0.9186, \quad |b| = |c| \doteq 0.0141.$$

Because

$$\gamma^* < -2\tau^{-1} \ln |D| \doteq 4.6052$$

we set $\gamma := \gamma^*$ in (3.4). Then $M \doteq 4.7233$. All hypotheses of Theorem 1 and of Theorem 2 are satisfied and, consequently, the zero solution of (4.1), (4.2) is exponentially stable in the metric C^0 and in the metric C^1. Finally, from (2.3), (2.4) and (3.5), it follows that the inequalities

$$\begin{aligned} |x(t)| &\leqslant \left(\sqrt{\lambda_H}\, |\varphi(0)| + \sqrt{\lambda_{HG_1}}\, \|\varphi\|_{0,\tau,\varsigma_1} + \sqrt{\lambda_{HG_2}}\, \|\dot{\varphi}\|_{0,\tau,\varsigma_2} \right) \mathrm{e}^{-\gamma^* t/2} \\ &\doteq \left(\sqrt{4.5834}\, |\varphi(0)| + \sqrt{2.2095}\, \|\varphi\|_{0,1,0.01} + \sqrt{0.25}\, \|\dot{\varphi}\|_{0,1,0.001} \right) \mathrm{e}^{-0.000001851t}, \\ |\dot{x}(t)| &\leqslant \Bigg(M\sqrt{\lambda_H}\, |\varphi(0)| + \frac{|B|}{|D|}\, \|\varphi\|_{0,\tau} + M\sqrt{\lambda_{HG_1}}\, \|\varphi\|_{0,\tau,\varsigma_1} \\ &\qquad + \|\dot{\varphi}\|_{0,\tau} + M\sqrt{\lambda_{HG_2}}\, \|\dot{\varphi}\|_{0,\tau,\varsigma_2} \Bigg) \mathrm{e}^{-\gamma^* t/2} \\ &\doteq \Big(4.7233\sqrt{4.5834}\, |\varphi(0)| + 7.4\, \|\varphi\|_{0,1} + 4.7233\sqrt{2.2095}\, \|\varphi\|_{0,1,0.01} \end{aligned}$$

$$+ \|\dot{\varphi}\|_{0,1} + 4.7233\sqrt{0.25}\, \|\dot{\varphi}\|_{0,1,0.001} \Big) \mathrm{e}^{-0.000001851t}$$

hold on $[0, \infty)$ and on $[0, \infty)\setminus\{s, s = 0, 1, \dots\}$ respectively. By Theorem 3, the zero solution of the system (4.1), (4.2) is absolutely stable.

Remark 1. System (4.1), (4.2) is considered in [5, 38, 44, 56] with omitted nonlinearity ($f \equiv 0$), the same matrices A, D and with the matrix

$$B = \alpha \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where α is a constant. In [44] asymptotic stability is established for $|\alpha| < 0.4$ while, in [38], for $|\alpha| < 0.533$ and, in [56] as the authors state, for $|\alpha| \leqslant 1$. In these papers no estimates are given related to the convergence of solutions or their derivatives. Exponential stability with explicit formulas estimating the convergence of solutions and their derivatives is given in [5] for $|\alpha| \leqslant 0.6213$. The present example gives exponential stability and explicit formulas estimating the convergence of solutions and their derivatives for $|\alpha| \leqslant 0.74035$.

5 Comments and Conclusions

The direct Lyapunov method is widely used for the stability analysis of functional differential systems. Its successful application substantially depends on an ingenious construction of Lyapunov-Krasovskii functional, suitable for the problem considered. As a rule, in the investigation of nonlinear neutral type control systems, functionals are used in the form of the sum of a quadratic form of terms on the right-hand side of the given system, a quadratic form of a “prehistory" and an integral of the nonlinearity. For example, in [27, 51], in the case of system (1.1), the following one is used:

$$V(x(t)) = \left[x(t) - Dx(t-\tau)\right]^T H \left[x(t) - Dx(t-\tau)\right] + \int_{t-\tau}^{t} \mathrm{e}^{-\varsigma(t-s)} x^T(s) G x(s) ds + \beta \int_{0}^{\sigma(t)} f(s) ds \tag{5.1}$$

where H and G are positive definite matrices, $\varsigma > 0$ and $\beta > 0$. However, applying functional (5.1) or any such functional type, does not result in an estimate of the norm $|x(t)|$ of solutions, but rather in the integral norm $\|x\|_{t,\tau,\varsigma}$. Many authors use method of functionals or other methods to state the fact of asymptotic stability only because their approach does not give an explicit estimate of the norm $|x(t)|$ or $|\dot{x}(t)|$ (such as formulas (2.3), (3.5) in Theorem 1 and Theorem 2). We refer, e.g., to [7, 18, 23, 37, 56]. Functional (2.1) is more general than some of the previously ones used to investigate neutral equations. For example, functionals used in [5, 35, 52, 53] are, if the systems considered are reduced to the form (1.1), its particular cases.

Let us discuss some specific properties of our approach. By an appropriate choice of the coefficient ν in the blocks S_{43} and S_{44} of the matrix S, in some cases, the positive definiteness of S can be achieved. Such a parameter is involved because of the sectorial condition (1.3) used in the proof (inequality (2.7)). The block-matrix S_{11} contains the expression $-A^T H - HA$. By the well-known result on Lyapunov matrix equations, if system $\dot{x}(t) = Ax(t)$ is asymptotically stable, then, for any positive definite symmetric matrix Q, there exists a unique positive definite symmetric matrix H such that $-A^T H - HA = -Q$. This fact can be, in principle, used when the derivative of the functional V along trajectories is estimated (see (2.5), (2.6)). We did not use such a possibility preferring consideration with the matrix S giving more general results. The parameters ς_1, ς_2 in the exponential multipliers $\exp(-\varsigma_1(t-s))$, $\exp(-\varsigma_2(t-s))$ in the definition of $V(x(t))$ by (2.1) have an impact on diagonal block-matrices S_{22}, S_{33} and, moreover, as suggested by formulas (2.4), (3.5), on the speed of convergence of the vanishing solutions. As it follows from the proof of Theorem 1, part ii_2), formula (2.10) (and computations above this formula), the assumption $\varsigma_1 > 0$, $\varsigma_2 > 0$ cannot be replaced by $\varsigma_1 \geqslant 0$, $\varsigma_2 \geqslant 0$. If $\varsigma_1 = 0$ or if $\varsigma_2 = 0$ then, proceeding by the scheme of the proof, we are not able to prove an exponential stability and construct explicit formulas estimating solutions and their derivatives.

In Theorem 2 we assumed $|D| \neq 0$. This is a quite natural assumption because if $|D| = 0$, then, by (1.5), all eigenvalues of the matrix $D^T D$ are zeroes. Consequently, $D = \Theta$, where Θ is $n \times n$ zero matrix and (1.1) becomes a delayed rather than neutral system and can be investigated with a functional defined by (2.1) if the integral with kernel defined by the derivatives is omitted. In many investigations when neutral systems such as (1.1) are considered, assumption $|D| < 1$ is necessary. In the present paper, this inequality follows from formula (3.4) in Theorem 2. In Theorem 1, a block-structured matrix S is applied. Although the latter assumption is not explicitly formulated (and is not explicitly used in its proof), it is probably implied by the positivity of S.

It is an open question, whether a functional $V_p(x(t)) = e^{pt} V(x(t))$, where p is a suitable real parameter and $V(x(t))$ is defined by formula (2.1), can be used to estimate solutions to system (1.1) in non-stable cases. Further progress can be achieved as well if the approach used is extended to systems (1.1) with the sum of nonlinearities $\sum_{i=1}^{s} b_i f_i(\sigma_i(t))$, $\sigma_i(t) = c_i^T x(t)$ of a Lurie type instead of only one such nonlinearity. Another challenge is a possible generalization of the results to systems with variable matrices and delays and to fractional differential systems as well.

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