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ON BOURBAKI-BOUNDED SETS ON QUASI-PSEUDOMETRIC SPACES

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Abstract. In metric spaces, a set is Bourbaki-bounded if and only if every real-valued uniformily continuous function on it is bounded. In this article, we study Bourbaki-boundedness on quasi-pseudometric spaces. It turns out that if a set is Bourbaki-bounded on a symmetrized quasi-pseudometric space, then it is Bourbaki-bounded in the quasi-metric space but the converse need not to be true. We show that an asymmetric normed space is Bourbaki-bounded if and only if it is bounded. Consequently, we prove that every real-valued semi-Lipschitz in the small function on a quasi-metric space is bounded if and only if the quasi-metric is Bourbaki-bounded. This article extends some results from Beer and Garrido's paper [2] from the metric point of view to the context of quasi-metric spaces.

1. Introduction

The theory of Bourbaki-boundedness in metric spaces was introduced by Atsuji in [1] as a generalization of the concept of totally bounded metric spaces. However, the concept of Bourbaki-boundedness attracted a great interest of many scholars (see [5–7]). For instance in [6], the authors introduced new tools for the completeness of metric spaces, called Bourbaki-completeness and cofinal Bourbaki completeness.

In addition, Beer and Garrido [2] proved that in metric space (X, d), a set $B \subseteq X$ is Bourbaki-bounded if and only if $f(B)$ is bounded in metric space (Y, p) whenever $f : (X, d) \to (Y, p)$ is uniformly continuous. Furthermore, they proved that B is Bourbaki-bounded if and only if $f(B)$ is a member of the bornology of bounded subsets of $\mathbb{R}$, where $f : (X, d) \to (\mathbb{R}, |.|)$ is Lipschitz in the small function.

In metric spaces, a family of Bourbaki-bounded sets sits between the family of totally bounded sets and the family of bounded sets. The concept of Bourbaki-boundedness in the framework of quasi-uniform spaces was introduced by Murdeshwar and Theckedath in [10]. They observed that if the set $[0, 1]$ is equipped with the T_0-quasi-metric

$$q(x, y) = \begin{cases} y - x & \text{if } x \leq y \\ 1 & x > y \end{cases}$$

and $\mathcal{U}$ is the quasi-uniformity generated by q on $[0, 1]$ and $\mathcal{U}^{-1}$ is the conjugate quasi-uniformity of $\mathcal{U}$ on $[0, 1]$, then $([0, 1], \mathcal{U})$ and $([0, 1], \mathcal{U}^{-1})$ are Bourbaki-bounded

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quasi-uniform spaces but $([0,1],\mathcal{U}^s)$ is the discrete uniform space which is not Bourbaki-bounded.

Moreover, we observe that for any two quasi-pseudometric spaces (X,q) and (Y,p), if the function $\varphi:(X,q)\to(Y,p)$ is uniformly continuous, then $\varphi:(X,q^s)\to(Y,p^s)$ is also uniformly continuous, but the converse is not true in general (see Example 3.3). All these are great motivations for generalizing the results about Bourbaki-boundedness from the symmetric point of view to asymmetric settings.

In this paper, we first revisit uniformly continuous and semi-Lipschitz functions in asymmetric settings, then we continue the study of the concept of Bourbaki-boundedness in a quasi-metric space. We show, for instance, that in a quasi-metric space (X,q), if a set is q^s-Bourbaki-bounded, then it is q-Bourbaki-bounded (and q^t-Bourbaki-bounded) but the converse need not to be true (see Example 4.9). Furthermore, we characterize Bourbaki-bounded sets in terms of boundedness under uniformily continuous functions and semi-Lipschitz in the small functions (see Theorem 5.2).

2. Preliminaries

A function $q:X\times X\to[0,\infty)$ on a set X will be called a *quasi-pseudometric* on X if, for any $x,y,z\in X$,

(1) $q(x,x)=0$;
(2) $q(x,y)\le q(x,z)+q(z,y)$.
Furthermore, if we also have that
(3) $q(x,y)=0=q(y,x)$ implies $x=y$, then the function q is called a T_0-*quasi-metric* (or *quasi-metric*) on X, and the pair (X,q) is a T_0-quasi-metric space.

If q is a quasi-pseudometric (T_0-quasi-metric) space on X, then the function $q^t:X\times X\to[0,\infty)$ defined by $q^t(x,y)=q(y,x)$ for all $x,y\in X$ is also a quasi-pseudometric (T_0-quasi-metric) on X, often called the *conjugate quasi-pseudometric* of q.

The symmetrized quasi-pseudometric of q is the function $q^s:X\times X\to[0,\infty)$ given by $q^s(x,y)=\max\{q(x,y),q(y,x)\}$ for all $x,y\in X$. It is easy to see that q^s is a pseudometric (metric) on X.

Let (X,q) be a quasi-pseudometric space. Given that $x\in X,\delta>0$ and $F\subseteq X$, we define $\mathrm{dist}_q(x,F)$ by $\mathrm{dist}_q(x,F):=\inf_{f\in F}q(x,f)$ and $\mathrm{dist}_q^t(x,F)$ by $\mathrm{dist}_q^t(x,F)=\mathrm{dist}_q(F,x)=\inf_{f\in F}q(f,x)$. In addition, we have

$$D_q(x,\delta):=\{y\in X:q(x,y)<\delta\}\quad\text{and}\quad D_q[x,\delta]:=\{y\in X:q(x,y)\le\delta\}.$$

Moreover,

$$\mathrm{dist}_q^s(x,F):=\max\{\mathrm{dist}_q(x,F),\mathrm{dist}_q^t(x,F)\}=\inf_{f\in F}q^s(x,f).$$

Definition 2.1. One says that a subset A of a quasi-pseudometric space (X,q) is *q-bounded* if there exists $x\in X$ and $r>0$ and $s>0$ such that $A\subseteq D_q(x,r)\cap D_{q^t}(x,s)$.

Note that the above definition is slightly different from the one given in [14]. In the sense of [14], a subset A of X can be q-bounded and not necessarily q^t-bounded.

Obviously, in our context, a subset A is q-bounded if and only if it is q^t-bounded. But q-boundedness does not imply q^s-boundedness in the context of [14]. Moreover, if q is an extended quasi-pseudometric on X (i.e., the distance between two points can be ∞), then a subset B of X can be included in $D_q(x,\epsilon)$ for some $x \in X$, $\epsilon > 0$ but its diameter $\operatorname{diam}(B) = \sup\{q(y,z) : y, z \in B\} = \infty$ (see [14, p. 2022]).

Definition 2.2. [9, Definition 1.1] Let X be a nonempty set. A family $\mathscr{B}$ of subsets of X is called a *bornology* on X, provided the following conditions are satisfied:

(i) $\mathscr{B}$ forms a cover of X, i.e., $X = \bigcup_{B\in\mathscr{B}} B$;
(ii) $\mathscr{B}$ is hereditary under inclusion, i.e., whenever $B \in \mathscr{B}$ and A is a subset of X contained in B, then $A \in \mathscr{B}$;
(iii) $\mathscr{B}$ is stable under finite union, i.e., if $B_1, B_2, ..., B_n \in \mathscr{B}$, then we get $\bigcup_{i=1}^{n} B_i \in \mathscr{B}$.

Given a bornology $\mathscr{B}$ and the set X, a pair $(X, \mathscr{B})$ is called a *bornological universe.*

Let (X,q) be a quasi-pseudometric space. It has been observed in [9,11] that the collection $\mathscr{B}_q(X)$ of all q-bounded subsets of X forms a bornology on X and this bornology is called the *quasi-metric bornology* determined by q. Furthermore, we have:

$$\mathscr{B}_{q^s}(X) = \mathscr{B}_q(X)$$

and bornologies $\mathscr{B}_q(X)$ and $\mathscr{B}_{q^t}(X)$ are equivalent. These observations came from the paper of Olela Otafudu *et al.* [9].

Definition 2.3. ([13, Definition 5]) A quasi-pseudometric space (X,q) is *q-totally bounded* if, for any $\epsilon > 0$, there exist $x_1, x_2, \ldots, x_n \in X$ such that for any $x \in X$, $q^s(x, x_i) < \epsilon$ for some $i \in \{1, \ldots, n\}$.

Let $\mathscr{TB}_q(X)$ be the collection of all q-totally bounded subsets of X. One can easly see that

(i) the singleton $\{x\} \in \mathscr{TB}_q(X)$ whenever $x \in X$;
(ii) if $B \in \mathscr{TB}_q(X)$ and $A \subseteq B \subseteq X$, then $B \in \mathscr{TB}_q(X)$;
(iii) for any $A, B \in \mathscr{TB}_q(X)$, it follows that $A \cup B \in \mathscr{TB}_q(X)$.

Consequently, the collection $\mathscr{TB}_q(X)$ forms a bornology on X that we call the *bornology of q-totally bounded sets.*

Remark 2.4. Let (X,q) be a quasi-pseudometric space. The following

$$\mathscr{TB}_q(X) = \mathscr{TB}_{q^s}(X) = \mathscr{TB}_{q^t}(X)$$

is just a consequence of Definition 2.3.

An *asymmetric norm* on a real vector space X is a function $\|.| : X \to [0,\infty)$ satisfying the following conditions:

(1) $\|x| = \|-x| = 0$ implies $x = 0$;
(2) $\|ax| = a\|x|$;

(3) $\|x+y| \leq \|x| + \|y|$,

for all $x, y \in X$ and $a \geq 0$. Then, the pair $(X, \|.|)$ is called *an asymmetric normed space.*

The *conjugate asymmetric norm* $|.\|$ of $\|.|$ and the *symmetrized norm* $\|.\|$ of $\|.|$ are defined respectively by

$$|x\| := \|-x| \quad \text{and} \quad \|x\| := \max\{|x\|, \|x|\} \text{ for any } x \in X.$$

An asymmetric norm $\|.|$ on X induces a quasi-metric $q_{\|.|}$ on X defined by

$$q_{\|.|}(x,y) = \|x-y| \text{ for any } x, y \in X.$$

If $(X, \|.\|)$ is a normed lattice space, then the function $\|.|$ defined by $\|x| := \|x^+\|$, where $x^+ = \max\{x, 0\}$, is an asymmetric norm on X.

3. Uniformly continuous and semi-Lipschitz functions

Definition 3.1. ([4, p. 146]) Let (X,q) and (Y,p) be quasi-pseudometric spaces. A function $\varphi : (X,q) \to (Y,p)$ is called *quasi-uniformly continuous* (or *uniformly continuous*) if, for any $\epsilon > 0$, there exists $\delta > 0$ such that $q(x,y) \leq \delta$, then $p(\varphi(x), \varphi(y)) < \epsilon$ for all $x, y \in X$.

Lemma 3.2. *Let (X,q) and (Y,p) be quasi-pseudometric spaces. If the function $\varphi : (X,q) \to (Y,p)$ is uniformly continuous, then the function $\varphi : (X,q^s) \to (Y,p^s)$ is uniformly continuous.*

Proof. Let $\epsilon > 0$. There exists $\delta > 0$ such that if

$$q(x,y) < \delta \quad \text{then} \quad p(\varphi(x), \varphi(y)) \leq \epsilon \quad \text{for all} \quad x, y \in X.$$

Furthermore, from the definition of q^s, it follows easily that

$$q(x,y) \leq q^s(x,y) < \delta \quad \text{, then} \quad p(\varphi(x), \varphi(y)) \leq \epsilon \tag{3.1}$$

and

$$q(y,x) \leq q^s(x,y) < \delta \quad \text{, then} \quad p(\varphi(y), \varphi(x)) \leq \epsilon. \tag{3.2}$$

For all $x, y \in X$, we have $p^s(\varphi(x), \varphi(y)) < \epsilon$ from (3.1) and (3.2).

Therefore, the function $\varphi : (X,q^s) \to (Y,p^s)$ is uniformly continuous. □

Example 3.3. We equip $X = \mathbb{R}_+ = [0, \infty)$ with the quasi-metric q defined by $q(x,y) = (y-x)^+$ for any $x, y \in [0, \infty)$ and equip $Y = \mathbb{R}$ with the T_0-quasi-metric p defined by $p(x,y) = (y-x)^+$ for any $x, y \in \mathbb{R}$. Then, while $f(x) = -\sqrt{x}$ is uniformly continuous from $(\mathbb{R}_+, |.|)$ into $(\mathbb{R}, |.|)$, it is not uniformly continuous from $(\mathbb{R}_+, q)$ into $(\mathbb{R}, p)$.

Proof. One can easily see that f is uniformly continuous from $(\mathbb{R}_+, |.|)$ into $(\mathbb{R}, |.|)$. To show that f is not uniformly continuous from $(\mathbb{R}_+, q)$ into $(\mathbb{R}, p)$, let $\epsilon = 1$ and $\delta > 0$ be chosen arbitrarily.

Then, for any $x > 1$,

$$q(x, 0) = (0-x)^+ = 0 < \delta,$$

but

$$p(f(x), f(0)) = (f(0) - f(x))^+ = (\sqrt{x})^+ = \sqrt{x} > 1 = \epsilon.$$

□

Remark 3.4. It can be noted from Example 3.3 that the converse of Lemma 3.2 does not hold in general. So, uniform continuity of the function $\varphi : (X, q^s) \to (Y, p^s)$ does not imply the uniform continuiy of the function $\varphi : (X, q) \to (Y, p)$.

Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. Then, a function $\varphi : (X, q) \to (Y, \|.|)$ is called a *semi-Lipschitz* if there exists $k \geq 0$ such that

$$\|\varphi(x) - \varphi(y)| \leq kq(x, y) \quad \text{for all } x, y \in X. \tag{3.3}$$

A number k satisfying (3.3) is called *semi-Lipschitz constant* for φ. For more details about semi-Lipschitz functions, we recommend the reader to see for instance [3, 12].

Definition 3.5. Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. Then:

(a) A function $\varphi : (X, q) \to (Y, \|.|)$ is called *locally semi-Lipschitz*, provided that, for all $x \in X$, there exists $\delta(x) > 0$ such that $\varphi|_{D_q(x,\delta(x))}$ is semi-Lipschtz.
(b) A function $\varphi : (X, q) \to (Y, \|.|)$ is called *uniformly locally semi-Lipschitz*, provided that, for all $x \in X$, there exists $\delta > 0$ (δ does not depend on x) such that $\varphi|_{D_q(x,\delta)}$ is semi-Lipschtz.
(c) A function $\varphi : (X, q) \to (Y, \|.|)$ is called *semi-Lipschitz in the small* if there exists $\delta > 0$ and $k \geq 0$ such that $q(x, y) < \delta$, then

$$\|\varphi(x) - \varphi(y)| \leq kq(x, y).$$

We omit the proof of the following since it is a direct application of the definition of q^s.

Lemma 3.6. *Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space. If a function $\varphi : (X, q) \to (Y, \|.|)$ is locally semi-Lipschitz, then $\varphi : (X, q^s) \to (Y, \|.\|)$ is locally semi-Lipschitz.*

The following lemma is easy to prove; therefore, we leave it to the reader.

Lemma
normed space. If function $\varphi : (X, q) \to (Y, \|.|)$ is semi-Lipschitz in the small, then $\varphi : (X, q) \to (Y, \|.|)$ is uniformly continuous.

Remark 3.8. Let (X, q) be a quasi-metric space and $(Y, \|.|)$ be an asymmetric normed space.

(1) If a function $\varphi : (X, q) \to (Y, \|.|)$ is locally semi-Lipschitz, then $\varphi|_{D_q(x,\delta_x)}$ is continuous whenever $x \in X$ and for some $\delta_x > 0$.
(2) Let $F \subseteq X$. If $\varphi_i : (X, q) \to (\mathbb{R}, u)$ is semi-Lipschitz restricted to F for $i = \{1, 2, \ldots, n\}$, then $\max\{\varphi_1, \varphi_2, \ldots, \varphi_n\}$ is semi- Lipschitz restricted to F, where u is the standard quasi-metric defined on $\mathbb{R}$ defined by $u(x, y) = \max\{x - y, 0\}$ for all $x, y \in \mathbb{R}$.

4. Some first results

Definition 4.1. Let (X, q) be a quasi-pseudometric space and $\delta > 0$. For any $\emptyset \neq F \subset X$, we define the δ- *enlargement* $D_q(F, \delta)$ of F by

$$D_q(F, \delta) := \{x \in X : \text{dist}(F, x) < \delta\} = \bigcup_{f \in F} D_q(f, \delta)$$

and

$$D_{q^t}(F,\delta) := \{x \in X : \mathrm{dist}^t(F,x) < \delta\} = \bigcup_{f\in F} D_{q^t}(f,\delta).$$

Furthermore,

$$D_{q^s}(F,\delta) = \max\Big\{D_q(F,\delta), D_{q^t}(F,\delta)\Big\} = \bigcup_{f\in F} D_{q^s}(f,\delta).$$

Lemma 4.2. *Let (X,q) be a quasi-pseudometric space and $\epsilon,\delta > 0$. We have*

$$D_q(D_q(F,\epsilon),\delta) \subseteq D_q(F,\epsilon+\delta).$$

Proof. Let $y \in D_q(D_q(F,\epsilon),\delta) = \bigcup_{v\in D_q(F,\epsilon)} D_q(v,\delta)$. Then, there exists $v \in D_q(F,\epsilon)$ such that $y \in D_q(v,\delta)$. It follows that there exists $f \in F$ such that $q(f,v) < \epsilon$ and $q(v,y) < \delta$. Moreover,

$$q(f,y) \leq q(f,v) + d(v,y) < \epsilon + \delta.$$

Hence, $y \in D_q(f,\epsilon+\delta)$. Thus, $y \in D_q(F,\epsilon+\delta)$. □

Let (X,q) be a quasi-pseudometric space and $\delta > 0$. If $x \in X$ and $n = 0,1,2,\ldots,$ we define the sets $D_q^n(x,\delta)$ by

$$D_q^0(x,\delta) := \{x\} \quad \text{and} \quad D_q^{n+1}(x,\delta) := D_q(D_q^n(x,\delta),\delta).$$

The next remark follows by induction.

Remark 4.3. Let (X,q) be a quasi-pseudometric space and $\delta > 0$. For any $x \in X$ and $n = 0,1,2,\ldots,$ we have

$$D_q^n(x,\delta) \subseteq D_q^{n+1}(x,\delta).$$

Furthermore, note that from Lemma 4.2 we get

$$D_q^n(x,\delta) \subseteq D_q(x,n\delta).$$

Definition 4.4. (compare [8, Definition 2.1]) Let (X,q) be a quasi-pseudometric space. For any given $x,y \in X$ and $\delta > 0$, a δ-*chain* of length n from x to y in (X,q) is a finite sequence of points $x_0,x_1,\ldots,x_n$ such that $x = x_0$, $x_n = y$ and $q(x_{i-1},x_i) < \delta$ for any i with $1 \leq i \leq n$.

Remark 4.5. Let (X,q) be a quasi-pseudometric space and $\delta > 0$. If there exists a δ-chain of length n from x to y in (X,q), then there exists a δ-chain of length n from y to x in (X,q^t) whenever $x,y \in X$.

The following is a consequence of Remark 4.5 and the definition of $D_{q^t}^n(y,\delta)$.

Remark 4.6. Let $\delta > 0$ be a positive real number and x and y be points in a quasi-pseudometric space (X,q). It is easy to check that there exists δ-chain of length n from x to y if and only if $y \in D_q^n(x,\delta)$ if and only if $x \in D_{q^t}^n(y,\delta)$.

Let (X,q) be a quasi-pseudometric space and $\delta > 0$. For any $x,y \in X$ and $\delta > 0$, we define the relation $\asymp_\delta$ on X by $x \asymp_\delta y$ if there exists a δ-chain of some length from x to y.

Lemma 4.7. *For any quasi-pseudometric space (X,q) and any $\delta > 0$, the relation $\asymp_\delta$ is a quasiorder on X.*

Let (X, d) be a quasi-pseudometric space. For $x \in X$, we define the set $x_{\asymp_\delta}$ by

$$x_{\asymp_\delta} = \bigcup_{n=0}^{\infty} D_q^n(x, \delta).$$

Remark 4.8. Let (X, q) be a quasi-pseudometric space. For any $\delta > 0$ and $x, y \in X$, it is easy to see that if $(x_i)_{i=0}^n$ is a δ-chain in (X, q^s) of length n from x to y, then $(x_i)_{i=0}^n$ is also a δ-chain in (X, q) and in (X, q^t) of length n from x to y. Then, with regard to Remark 4.6, we have

$$D_{q^s}^n(x, \delta) \subseteq D_q^n(x, \delta) \tag{4.1}$$

and

$$D_{q^s}^n(x, \delta) \subseteq D_{q^t}^n(x, \delta). \tag{4.2}$$

The following example shows that the inclusions (4.1) and (4.2) cannot be reversed.

Example 4.9. Let X be a set of four points $\{1, 2, 3, 4\}$. If we equip X with the T_0-quasi-metric q defined by the distance matrix

$$Q = \begin{pmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{pmatrix}$$

, that is, $q(i, j) = q_{i,j}$ whenever $i, j \in X$, the symmetrized metric q^s of q is induced by the matrix

$$Q^s = \begin{pmatrix} 0 & 1 & 2 & 2 \\ 1 & 0 & 1 & 2 \\ 2 & 1 & 0 & 1 \\ 2 & 2 & 1 & 0 \end{pmatrix}$$

Let $\delta = 2$. If $(f_i)_{i=0}^2 := (4, 2, 1)$ is a sequence in X, then we have

$$q(f_0, f_1) = q(4, 2) = 1 = q(f_1, f_2) = q(2, 1) < \delta.$$

Hence, the sequence $(f_i)_{i=0}^2 := (4, 2, 1)$ is a δ-chain in (X, q) of length 2 from 4 to 1. But the same sequence $(f_i)_{i=0}^2 := (4, 2, 1)$ is not a δ-chain in (X, q^s) of length 2 from 4 to 1 because $q^s(f_0, f_1) = q^s(4, 2) = 2 \geq \delta$.

The motivation of the following definition comes from [11, Definition 1.5] and Remark 4.6.

Definition 4.10. (compare [8, Definition 2.1 and Remark 2.2]) Let (X, q) be a quasi-pseudometric space and $F \subseteq X$. We say that F is *q-Bourbaki-bounded* if, for any $\delta > 0$, there exists a finite subset $\{f_1, f_2, \ldots, f_k\}$ of X and for some positive integer n such that

$$F \subseteq \bigcup_{i=1}^{k} D_q^n(f_i, \delta).$$

The converse of the next lemma does not hold from Example 4.9.

Lemma 4.11. *Let (X, q) be a quasi-pseudometric space and $F \subseteq X$. If F is q^s-Bourbaki-bounded, then F is q-Bourbaki-bounded and q^t-Bourbaki-bounded.*

Proposition 4.12. *Let (X,q) be a quasi-pseudometric space. If F is a subset of X and for $\delta>0$, then we have*

(a) *If F is q-totally bounded, then F is q-Bourbaki-bounded.*
(b) *If F is q-Bourbaki-bounded and q^t-Bourbaki-bounded , then F is q^s-bounded.*
(c) *If F is q-Bourbaki-bounded, then $Cl_{\tau(q)}F$ is also q-Bourbaki-bounded.*

Proof. (a) It is immediate.

(b) Let δ and ϵ be positive real numbers. From the q-Bourbaki-boundedness of F and q^t-Bourbaki-boundedness, there exist finite sets $\{x_1,x_2,\ldots,x_r\}\subseteq X$ and $\{y_1,\ldots,y_l\}\subseteq X$. Let $\{z_1,\ldots,z_k\}=\{x_1,x_2,\ldots,x_r\}\cup\{y_1,\ldots,y_l\}$. Then, there exists some positive integer n such that

$$F\subseteq\bigcup_{i=1}^{r}D_q^n(z_i,\delta)\text{ and }F\subseteq\bigcup_{i=1}^{l}D_{q^t}^n(z_i,\epsilon),$$

where $r,l\leq k$. We now show that F is q^s-bounded.

Indeed,

$$\begin{aligned}F&\subseteq\bigcup_{i=1}^{r}D_q^n(z_i,\delta)\cap\bigcup_{i=1}^{l}D_{q^t}^n(z_i,\epsilon)\\&=\bigcup_{i=1}^{r}\bigcup_{i=1}^{l}D_q^n(z_i,\delta)\cap D_{q^t}^n(z_i,\epsilon)\\&\subseteq\bigcup_{i=1}^{r}\bigcup_{i=1}^{l}D_q(z_i,n\delta)\cap D_{q^t}(z_i,n\epsilon).\end{aligned}$$

Thus, F is q^s-bounded.

(c) Follows since F is included in the union of q-open balls. □

Remark 4.13. Let (X,q) be a quasi-metric space. In the sequel, we denote by $\mathscr{BB}_q(X)$ the collection of all q-Bourbaki-bounded subsets in (X,q). We observe that $\mathscr{BB}_q(X)$ forms a bornology on X that we call the *bornology of q-Bourbaki-bounded sets* in (X,q).

Remark 4.14. If (X,q) is a quasi-pseudo-metric space, then we have the following inclusions:

$$\mathscr{TB}_q(X)\subseteq\mathscr{BB}_{q^s}(X)\subseteq\mathscr{BB}_q(X)\subseteq\mathscr{B}_q(X).\tag{4.3}$$

The first two inclusions of (4.3) can be found in [2] and [5], for example, and the last inclusion of (4.3) is a consequence of Proposition 4.12. For more details about connections between bornology of q^s-totally bounded sets, bornology of q^s-Bourbaki-bounded sets and bornology of q^s-bounded sets, we recommend, for instance, [2, 5, 7, 8].

5. Bornologies and semi-Lipschitz functions

In this last section, we intend to characterize q-Bourbaki-bounded sets in terms of uniformily continuous functions and $\tau(q^s)$-compact set in terms of semi-Lipschitz functions.

In the sequel, we equip $\mathbb{R}$ with its usual T_0-quasi-metric u given by

$$u(x,y) = (x-y)^+ = \max\{x-y, 0\} \text{ whenever } x, y \in \mathbb{R}.$$

Theorem 5.1. *Let (X,q) be a quasi-metric space and $\emptyset \neq F \subseteq X$. Then, the following conditions are equivalent:*

(1) *$cl_{\tau(q)}(F)$ is $\tau(q)$-compact;*
(2) *if $(Y, \|.|)$ is an asymmetric normed space and $\varphi : (X,q) \to (Y, \|.|)$ is continuous, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;*
(3) *if $(Y, \|.|)$ is an asymmetric normed space and $\varphi : (X,q) \to (Y, \|.|)$ is locally semi-Lipschitz, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;*
(4) *if $\varphi : (X,q) \to (\mathbb{R}, u)$ is locally semi-Lipschitz, then $\varphi(F)$ is a u-bounded set of real numbers.*

Proof. (1) ⇒ (2) Suppose that $\mathrm{cl}_{\tau(q)}(F)$ is $\tau(q)$-compact and $\varphi : (X,q) \to (Y, \|.|)$ is continuous. Then, $\varphi(\mathrm{cl}_{\tau(q)}(F))$ is $\tau(\|.|)$-compact. Thus, $\varphi(F)$ is $q_{\|.|}$-bounded.

(2) ⇒ (3) Follows from the continuity of locally semi-Lipschitz functions and (3) ⇒ (4) follows without doubt.

(4) ⇒ (1) Suppose that $\mathrm{cl}_{\tau(q)}(F)$ is not $\tau(q)$-compact. Then, we can find a sequence $(f_n)_{n\in\mathbb{N}}$ in F with $f_j \neq f_i$ for $i \neq j$ and the sequence $(f_n)_{n\in\mathbb{N}}$ in F does not converge with respect to $\tau(q)$.

For any $n \in \mathbb{N}$, let $\mu_n := q(f_n, \{f_j : j \neq n\}) > 0$ and $\epsilon_n := \left\{\frac{1}{n}, \frac{\mu_n}{3}\right\}$.

It follows that the family $\{D_q(f_n, \epsilon_n) : n \in \mathbb{N}\}$ is such that whenever $i \neq k$, we have $D_q(f_i, \epsilon_i) \neq D_q(f_k, \epsilon_k)$ and $\epsilon_i + \epsilon_k < \max\{\mu_i, \mu_k\}$.

For any $n \in \mathbb{N}$, let $\phi_n : (X,q) \to (\mathbb{R}, u)$ be a function defined by

$$\phi_n(x) := n - \frac{n}{\epsilon_n} q(f_n, x) \text{ for any } x \in X.$$

Then, for any $x, y \in X$, we have two cases. The case $u(\phi_n(x), \phi_n(y)) = 0$ is obvious. Otherwise, we have

$$\begin{aligned} u(\phi_n(x), \phi_n(y)) = \phi_n(x) - \phi_n(y) &= \left[n - \frac{n}{\epsilon_n} q(f_n, x)\right] - \left[n - \frac{n}{\epsilon_n} q(f_n, y)\right] \\ &= \frac{n}{\epsilon_n}\left[q(f_n, y) - q(f_n, x)\right] \\ &\leq \frac{n}{\epsilon_n}\left[q(f_n, x) + q(x,y) - q(f_n, x)\right] \\ &= \frac{n}{\epsilon_n} q(x,y). \end{aligned}$$

Hence, $\phi_n : (X,q) \to (\mathbb{R}, u)$ is k-semi-Lipschitz with $k = \dfrac{n}{\epsilon_n}$. Observe that $\phi_n(x) > 0$ if and only if $q(f_n, x) < \epsilon_n$ whenever $n \in \mathbb{N}$.

Let $\varphi : (X,q) \to (\mathbb{R}, u)$ be defined by

$$\varphi(x) = \begin{cases} \phi_n(x) & \text{if } \quad x \in D_q(f_n, \epsilon_n) \\ 0 & \text{otherwise.} \end{cases}$$

Since $\varphi(f_n) = \phi_n(f_n) = n - \dfrac{n}{\epsilon_n} q(f_n, f_n) = n$, it follows that $\varphi(F)$ is u-unbounded.

To complete the proof, we need to show that φ is locally semi-Lipschitz. Let us consider an arbitrary point $x_0 \in X$. Since $\epsilon_n < \frac{1}{n}$ for any $n \in \mathbb{N}$ and the sequence (f_n) does not $\tau(q)$- converge to x_0, there exists $\delta > 0$ such that $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) \neq \emptyset$ for at most finitely many n. Let us say $n_1, n_2, \dots, n_k$.

Case 1. If $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) = \emptyset$, then $\varphi|_{D_q(x_0,\delta)} = 0$.

Case 2. If $D_q(x_0, \delta) \cap D_q(f_n, \epsilon_n) \neq \emptyset$, then whenever $q(x_0, x) < \delta$, we have

$$\varphi(x) = \max\{0, \phi_{n_1}(x), \phi_{n_2}(x), \dots, \phi_{n_k}(x)\}.$$

Either way, $\varphi|_{D_q(x_0,\delta)}$ is semi-Lipschitz. □

We point out that one can also generalize [2, Theorem 3.3] in our context.

Theorem 5.2. *Let (X, q) be a quasi-metric space and $\emptyset \neq F \subseteq X$. Then, the following conditions are equivalent:*

(1) *F is q-Bourbaki-bounded;*
(2) *if $(Y, \|.|)$ is an asymmetric normed space and $\varphi : (X, q) \to (Y, \|.|)$ is uniformly continuous, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;*
(3) *if $(Y, \|.|)$ is an asymmetric normed space and $\varphi : (X, q) \to (Y, \|.|)$ is semi-Lipschitz in the small, then $\varphi(F) \in \mathscr{B}_{q_{\|.|}}(Y)$;*
(4) *if $\varphi : (X, q) \to (\mathbb{R}, u)$ is semi-Lipschitz in the small, then $\varphi(F)$ is a u-bounded set of real numbers.*

Proof. (1) ⇒ (2) We assume that $\varphi : (X, q) \to (Y, \|.|)$ is uniformly continuous. Then there exists $\delta > 0$ such that whenever $x, y \in X$ with $q(x, y) < \delta$, then

$$q_{\|.|}(\varphi(x), \varphi(y)) = \|\varphi(x) - \varphi(y)| < 1. \tag{5.1}$$

By the q-Bourbaki-boundedness of F, there exists $A := \{a_1, a_2, \dots, a_m\} \subseteq X$ such that

$$F \subseteq \bigcup_{i=1}^{m} D_q^n(a_i, \delta)$$

for some positive integer n. If we take f artbitrarily in F, then there exists k with $1 \leq k \leq m$ such that $f \in D_q^n(a_k, \delta)$. Then, for some k with $1 \leq k \leq m$, there exists a δ-chain $\{f_0, f_1, \dots, f_n\}$ with $f_0 = a_k$ and $f_n = f$ and

$$q(f_{i-1}, f_i) < \delta \text{ whenever } i \text{ with } 1 \leq i \leq m.$$

For some k with $1 \leq k \leq m$, it follows from the uniform continuity of φ and inequality (5.1) that

$$q_{\|.|}(\varphi(f_{i-1}), \varphi(f_i)) < 1 \text{ whenever } i \text{ with } 1 \leq i \leq m.$$

Hence, for some k with $1 \leq k \leq m$, we have

$$\begin{aligned} &q_{\|.|}(\varphi(a_k), \varphi(f)) \\ &= q_{\|.|}(f_0, f_n) \leq q_{\|.|}(f_0, f_1) + q_{\|.|}(f_1, f_2) + \cdots + q_{\|.|}(f_{n-1}, f_n) < n. \end{aligned}$$

Hence, $\varphi(f) \in \bigcup_{i=1}^{m} D_{q_{\|.|}}(\varphi(a_i), n)$ for any $f \in F$. Thus, $\varphi(F) \subseteq D_q(\varphi(A), n)$. Hence, $\varphi(F)$ is $q_{\|.|}$-bounded.

(2) ⇒ (3) Follows from Lemma 3.7.

(3) ⇒ (4) It is obvious.

(4) $\Rightarrow$ (1) Suppose that F is not q-Bourbaki-bounded. Then, there exists δ such that if $\{f_1, f_2, \ldots, f_k\} \subseteq X$ and n is a positive integer, we have $F \not\subseteq \bigcup_{i=1}^{k} D_q^n(f_i, \delta)$. We have two cases on the structure of F.

Case 1. If $f \in F$, there exists a positive integer n such that

$$F \cap D_q^n(f, \delta) = F \cap \bigcup_{i=1}^{k} D_q^n(f_i, \delta).$$

Let f_1 be an arbitrary point of F. We choose a positive integer n_1 such that

$$F \cap D_q^{n_1}(f_1, \delta) = F \cap \bigcup_{n_1=0}^{\infty} D_q^{n_1}(f, \delta).$$

Since F is not q-Bourbaki-bounded, there exists $f_2 \in F$ such that $f_2 \notin D_q^{n_1}(f_1, \delta)$. It follows that $\bigcup_{n_1=0}^{\infty} D_q^{n_1}(f_1, \delta) \neq \bigcup_{n_2=0}^{\infty} D_q^{n_2}(f_2, \delta)$ by the choice of n_1.

Choose another positive integer n_2 such that $n_2 > n_1$ and

$$F \cap D_q^{n_2}(f_2, \delta) = F \cap \bigcup_{n_2=0}^{\infty} D_q^{n_2}(f_2, \delta).$$

Moreover, since $F \not\subseteq \bigcup_{n_j=0}^{\infty} D_q^{n_j}(f_j, \delta)$, we can find

$$f_3 \in F \setminus \left[\bigcup_{n_3=0}^{\infty} D_q^{n_3}(f_3, \delta) \cup \bigcup_{n_2=0}^{\infty} D_q^{n_2}(f_2, \delta) \right].$$

Continuing this procedure by induction, we can find a sequence (f_j) with distinct terms in F such that, for any $i \neq j$, we have $\bigcup_{n_i=0}^{\infty} D_q^{n_i}(f_i, \delta) \neq \bigcup_{n_j=0}^{\infty} D_q^{n_j}(f_j, \delta)$. Therefore, we define a function $\varphi : (X, q) \to (\mathbb{R}, u)$ by

$$\varphi(x) = \begin{cases} j & \text{if } \quad x \asymp_\delta f_j \text{ for some } \; j \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the function φ is constant on $D_q(x, \delta)$ and it is unbounded on F since $\varphi(f_j) = j$. Therefore, the function φ is semi-Lipschitz in the small function.

Case 2. If there exists $f \in F$, then for all positive integers n, there exists $j \in \mathbb{N}$ such that

$$F \cap D_q^n(f, \delta) \subset F \cap D_q^{n+j}(f, \delta).$$

For $x \asymp_\delta f$, let $n(x)$ be the smallest positive integer n such that

$$x \in F \cap D_q^n(f, \delta).$$

We then define the function $\varphi : (X, q) \to (\mathbb{R}, u)$ by

$$\varphi(x) = \begin{cases} (n(x) - 1)\delta + \operatorname{dist}_q(x, D_q^{n(x)-1}(f, \delta)) & \text{if } \quad x \neq f \text{ and } x \asymp_\delta f \\ 0 & \text{otherwise.} \end{cases}$$

By definition, the function φ is unbounded on F. We now have to show that if x is not related to y with respect to $\asymp_\delta$ and $q(x,y) < \delta$, then for $k = 2$,

$$u(\varphi(x), \varphi(y)) \leq kq(x,y).$$

If either x or y is not related to f with respect to $\asymp_\delta$, then we have:

(1) If $y \asymp_\delta f$ but x is not related to f with respect to $\asymp_\delta$, then

$$u(\varphi(x), \varphi(y)) = 0 < 2q(x,y)$$

since $\varphi(x) = 0$.

(2) If $x \asymp_\delta f$ but y is not related to f with respect to $\asymp_\delta$, then

$$\begin{aligned} u(\varphi(x), \varphi(y)) &= (n(x)-1)\delta + \operatorname{dist}_q(x, D_q^{n(x)-1}(f,\delta)) \\ &\leq (n(x)-1)\delta + q(x,y) \\ &= q(x,y) < 2q(x,y), \end{aligned}$$

since f is not related to y with respect to $\asymp_\delta$, triangle inequality holds and $n(x) \leq 1$.

Now, if $x \asymp_\delta f$ and $y \asymp_\delta f$, then we have some cases on $n(x)$ and $n(y)$.

(a) If one of $n(x)$ or $n(y)$ is zero, then firstly, if $n(x) = 0$ and $n(y) \neq 0$, then we have $x = f$ and $0 < q(x,y) < \delta$, which implies that $y \in D_q(x,\delta)$, thus, $n(y) = 1$. Hence,

$$u(\varphi(x), \varphi(y)) = u(0, \varphi(y)) = 0 < 2q(x,y).$$

Secondly, if $n(y) = 0$ and $n(x) \neq 0$, then by similar arguments $n(x) = 1$. Thus,

$$\begin{aligned} u(\varphi(x), \varphi(y)) &= u((1-1)\delta + \operatorname{dist}_q(x, D_q^0(f,\delta)), 0) \\ &= \operatorname{dist}_q(x, \{y\}) = q(x,y) < 2q(x,y). \end{aligned}$$

(b) If $n(x) = n(y) \geq 1$, then

$$\begin{aligned} u(\varphi(x), \varphi(y)) &= \max\{[\operatorname{dist}_q(x, D_q^{n(x)-1}(f,\delta)) - \operatorname{dist}_q(y, D_q^{n(x)-1}(f,\delta))], 0\} \\ &\leq q(x,y) < 2q(x,y). \end{aligned}$$

(c) If $n(x) > n(y) \geq 1$ (i.e. $n(x) = n(y) + j$ for some $j \geq 1$), then if $\varphi(x) \leq \varphi(y)$, then there is nothing to prove since $u(\varphi(x), \varphi(y)) = 0 < 2d(x,y)$.

Now, if $\varphi(x) > \varphi(y)$, then

$$\begin{aligned} &u(\varphi(x), \varphi(y)) = \varphi(x) - \varphi(y) \\ &= [(n(x)-1)\delta + \operatorname{dist}_q(x, D_q^{n(x)-1}(f,\delta))] - [(n(y)-1)\delta + \operatorname{dist}_q(y, D_q^{n(y)-1}(f,\delta))] \\ &= [(n(y)+j-1)\delta - [(n(y)-1)\delta] - [\operatorname{dist}_q(x, D_q^{n(y)+j-1}(f,\delta))] \\ &\qquad - [\operatorname{dist}_q(y, D_q^{n(y)-1}(f,\delta))]. \end{aligned}$$

Furthermore,

$$\begin{aligned} u(\varphi(x), \varphi(y)) &= j\delta + [\operatorname{dist}_q(x, D_q^{n(y)-1+j}(f,\delta))] - \operatorname{dist}_q(y, D_q^{n(y)-1}(f,\delta))] \\ &\leq j\delta + q(x,y) + \operatorname{dist}_q(y, D_q^{n(y)-1+j}(f,\delta)) - \operatorname{dist}_q(y, D_q^{n(y)-1}(f,\delta)). \end{aligned}$$

Since $n(y)$ is the smallest n such that $y \in F \cap D_q^n(f,\delta)$, then

$$\operatorname{dist}_q(y, D_q^{n(y)-1+j}(f,\delta)) = 0,$$

then we have

$$u(\varphi(x),\varphi(y)) \leq j\delta + q(x,y) - \operatorname{dist}_q(y, D_q^{n(y)-1}(f,\delta)). \tag{5.2}$$

We claim that

$$j\delta - q(x,y) \leq \operatorname{dist}_q(y, D_q^{n(y)-1}(f,\delta)). \tag{5.3}$$

Suppose otherwise that $\operatorname{dist}_q(y, D_q^{n(y)-1+j}(f,\delta)) < j\delta - q(x,y)$. Then,

$$\begin{aligned}\operatorname{dist}_q(x, D_q^{n(y)-1}(f,\delta)) &\leq q(x,y) + \operatorname{dist}_q(y, D_q^{n(y)-1}(f,\delta))\\ &< q(x,y) + j\delta - q(x,y)\\ &< j\delta.\end{aligned}$$

So, $x \in D_q^{n(y)-1+j}(f,\delta)$, which implies that $n(x) \leq n(y) - 1 + j$ but this is a contradiction since $n(x) > n(y)$.

Combining (5.2) and (5.3) we have

$$u(\varphi(x),\varphi(y)) \leq j\delta + q(x,y) - j\delta + q(x,y) \leq 2q(x,y).$$

Therefore, the proof is complete. □

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