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GEOMETRIC ALGEBRA IN SWITCHED SYSTEMS CONTROL

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PHD THESIS SUMMARY
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Abstract

In this thesis, the controllability of 2×2 switched systems with regular matrices is investigated by means of Geometric Algebra for Conics(GAC) as a mathematical framework for analysis and optimization of control strategies. The research demonstrates the efficiency of GAC in the construction of switching points and paths while minimizing the number of switches and numerical errors.

Classification of controllability is provided based on the geometric properties of particular switched systems. For controllable switched systems, the controlling algorithm based on the GAC primitives is introduced in which the symbolic algebra operations are used, more precisely the wedge and inner product. Therefore, no numerical solver to the system of equations is needed. Indeed, the only operation that may bring in a numerical error is a vector normalisation, ie., square root calculation. The proposed approach creates possibility of passing from the classical solution of the controllability problem for switched systems to a geometric one, based on the type of phase trajectory.

Keywords

switched system, geometric algebra, controllability, Clifford algebra

Abstrakt

V této práci je zkoumána řiditelnost switched systémů 2×2 s regulárními maticemi pomocí Geometrické algebry pro kuželosečky (GAC). Je provedena analýza a optimalizace switched strategií. Výzkum ukazuje účinnost GAC při hledání switching bodů a minimalizaci jejich počtu při maximálním omezení numerických chyb.

Na základě geometrických vlastností konkrétních switched systémů je uvedena klasifikace jejich řiditelnosti. Pro řiditelné switched systémy je představen algoritmus na hledání switching cest založený na kuželosečkách reprezentovaných v GAC, v němž jsou použity operace symbolické algebry, přesněji vnější a vnitřní součin. Není tedy zapotřebí numerického řešení soustavy rovnic. Jedinou operací, která může vnést numerickou chybu, je normalizace vektoru, tedy výpočet odmocniny. Navržený přístup vytváří možnost přejít od klasického řešení problému řiditelnosti pro switched systémy ke geometrickému řešení založenému na typu fázové trajektorie.

Klíčová slova

switched systém, geometrická algebra, řiditelnost, Cliffordova algebra

Declaration

I hereby declare that my doctoral thesis topic on the theme of ”Geometric Algebra in Switched Systems Control” represents my own work under the guidance of supervisor, and has not been previously included in a thesis or dissertation submitted to this or any other institution for a degree, diploma or other qualifications.

20. 4. 2024 Ing. Anna Derevianko

Preface

I would like to thank my supervisor doc. Mgr. Petr Vašík, Ph.D. for the excellent guidance and support during this process. I also wish to thank all of the respondents, without whose cooperation I would not have been able to perform this analysis.

Ing. Anna Derevianko

Contents

Introduction

Switched systems form a special case of hybrid dynamical systems with both discrete and continuous dynamics, which are described by a differential equation and a state machine or automaton, respectively. They are widely applied in the cases where a real system cannot be described by one single model. Numerous examples are given by engineering systems of electronics, power systems, traffic control and others, [1]. Since the 1990s, research of switched systems stability has become very popular, see e.g. [2, 3]. The particular case of linear switched systems was considered by Patrizio Colaneri in [4]. More modern literature about switched systems is represented by the works of Yuan Lin, Yuan Sun-Ge Wang, and Jiang-Wang [5], Zhong-Ping, Yuan Wang [6]; the question of stability remains relevant until today.

Another topic related to switched systems is controllability. Nowadays, the most popular approach in searching for a control of a switched system is connected with generalizations of Kalman condition, [7] and Lyapunov functions, [8], which requires complex algorithmic structures. Applications of optimal control for switched dynamical systems can be found in various fields, including robotics, power systems, autonomous vehicles, and manufacturing systems, where systems exhibit complex behavior arising from the interaction between continuous dynamics and discrete events, [9].

The ordinary Lyapunov function is used to test whether a given dynamical system is stable (more precisely asymptotically stable), but does not provide any information about controllability. Particular specificity of switched systems is in the interaction between the continuous variable and the discrete state, which is not present in the standard control systems. In the works of D. Liberzon [10, 11], the author addresses the problems of stability and control for particular types of switched systems, using the analytical approach, ie., Lyapunov function and Brockett's condition for asymptotic stability by continuous feedback and controllability. The stabilization problem for switched positive regular linear systems by state-dependent switching was considered in [12] and anti-bump switching control problem was introduced in [13]. The case of linear switched singular systems is studied, e.g. in [3].

The issue of optimal control has also been addressed several times. The most popular are problems of time- or distance-optimality. For example, finding time-optimal control for dynamical system was considered by Nasir Uddin Ahmed [14] while for switched systems analogical problem was considered in [15], where the author constructs a minimizing sequence and uses compactness property for finding subsequence that minimizes the cost functional. Another way of optimization is construction of a switching path with minimal amount of switches which is of our particular interest.

The novelty of this thesis lies in using Geometric Algebra (GA) for investigation of the switched systems controllability. Geometric Algebra offers a powerful mathematical framework for handling geometric operations with elegance and efficiency. Originating from Grassmann algebras and Clifford algebras, GA unifies geometric primitives and transformations within a single algebraic structure, providing a seamless approach to geometric analysis and manipulation.

Geometric Algebra for Conics (GAC) is an efficient geometric tool to handle both conics and their transformations as elements of a particular Clifford algebra, [16], [17], [18].

The thesis is structured in the following way.

First chapter describes the fundamental concepts of Grassmann and Clifford algebras, highlighting their role in forming the basis of Geometric Algebra.

Second chapter deals with description of switched systems, their place in the theory of dynamical systems, and controllability.

Third chapter deals with the 2×2 switched systems with the special type subsystems. More precisely, those subsystems that have a specific equilibrium point and trajectories intrinsic to Geometric Algebra for Conics. Let us stress that in the Euclidean space, the problem of finding intersection points of two conics is reduced to solving the system of quadratic equations using numerical methods. To avoid the use of the numerical solver we use Geometric Algebra for Conics and demonstrate it on various examples.

Fourth chapter describes the controllability of the 2×2 switched systems with regular matrices of each subsystem, whose phase trajectories are not conic sections.

We shall demonstrate our results on examples of specific switched systems. We provide outputs of our implementation in Python using a module Clifford for GAC operations.

Chapter 1

Geometric algebras

Geometric Algebra (GA) and its special types, such as Compass Ruler Algebra (CRA) and Geometric Algebra for Conics (GAC), provide powerful mathematical frameworks for handling geometric operations in a unified manner. Originating from Grassmann and Clifford algebras, GA offers a rich algebraic structure that encapsulates geometric primitives and transformations within a single framework. In this chapter, we outline the fundamental concepts of Grassmann and Clifford algebras before dealing with the applications and extensions provided by GA.

Clifford algebras extend Grassmann algebras by incorporating geometric interpretations through the Clifford product. By unifying the outer product and the dot product into a single geometric product, Clifford algebras provide a framework for geometric operations.

1.1 Clifford Algebras

The connection between Clifford algebras and Grassmann algebras arises from the fact that Clifford algebras contain Grassmann algebras as subalgebras, together with the antisymmetric product. Specifically, for a vector space V, the exterior product of vectors in $\Lambda(V)$ is embedded into the corresponding Clifford algebra $\mathcal{C}l(V)$.

The product in the algebra is called the Clifford product. The ideal ensures that the elements of the Clifford algebra square to scalars. The resulting algebra $\mathcal{C}l(V)$ contains elements called multivectors.

Remark. Signature (p,q) of Clifford algebra $\mathcal{C}l(p,q)$ denotes that algebra contains p base vectors that square to $+1$ and q vectors that square to -1.

Let V be a n-dimensional Euclidean space, while p-subspace is a subspace of dimension p. For a multivector $M \in \bigwedge V$, $\quad < M_i >_p \in \bigwedge_p V$ denotes its component of grade p. p-blade is multivector $B = v_1 \wedge \cdots \wedge v_p \in \bigwedge_p V$, with $v_1, \ldots, v_p \in \bigwedge V$. Note, that a scalar is a 0-blade, for more details see [19].

We shall use the term Geometric Algebra to mean the coupling of Clifford algebras with an accompanying geometric interpretation.

1.2 Introduction to Geometric Algebras

Geometric algebra (GA) is a Clifford algebra with a specific embedding of Euclidean space (of arbitrary dimension) in such a way that the intrinsic geometric primitives as well as their Euclidean transformations are viewed as elements of a single vector space, precisely vectors and bivectors, respectively. This concept was introduced by D. Hestenes in [20] and has been used in many mathematical and engineering applications since, see e.g. [21, 22]. The computational advantage of GA lays in that geometric operations, such as intersections, tangents, distances, etc., are linear functions and can therefore be calculated efficiently. To show this, we refer to [23] for the basics of geometric algebras, especially for the conformal representation of Euclidean space. 3D Euclidean space is actually represented in Clifford's algebra $\mathcal{C}l(4,1)$, and the consequent geometric algebra is often denoted as $\mathbb{G}_{4,1}$ with spheres of all types as geometric primitives and Euclidean transformations at hand, see eg. [24]. In the following work the two-dimensional subalgebra $\mathbb{G}_{3,1}$ is also considered. It is called the Compass Ruler Algebra (CRA), [25], which is an analogue of $\mathbb{G}_{4,1}$ for two-dimensional Euclidean space.

Basic elements of n-dimensional Geometric Algebra are represented by blades with grades $0, 1, 2, \ldots, n$, where a scalar is considered as a 0-blade and the 1-blades are the basis vectors $e_1; e_2; \ldots; e_n$. The 2-blades $e_i \wedge e_j$ are blades constructed by two 1-blades, and so on. There exists the only one element of the maximum grade n, $I = e_1 \wedge e_2 \wedge \cdots \wedge e_n$, that is called a pseudoscalar. The products in Geometric Algebra are presented by the outer, the inner and the geometric product, see [25].

Geometric primitives in Geometric Algebra have two algebraic representations, the IPNS (inner product null space) and the OPNS (outer product null space) representation. These representations are duals of each other.

Let $E \subset \mathbb{R}^3$ be one of the geometric entity listed above then we say that a multivector E_{IPNS} is an inner product null space representation of E if

$$\{x \in \mathbb{R}^3 : P(x) \cdot E_{IPNS} = 0\} = E,$$

ie., it is possible to represent a subspace with a blade as the set of all vectors having a zero contraction with the blade. This method is called the Inner Product Null Space (IPNS) representation of subspaces.

Analogously we say that a multivector E_{OPNS} (a dual of E) is an outer product null space representation of E if

$$\{x \in \mathbb{R}^3 : P(x) \wedge E_{OPNS} = 0\} = E,$$

ie., any vector having zero outer product with a blade is inside its subspace. This is called the Outer Product Null Space (OPNS) representation of subspaces.

Further we provide a brief theoretical survey of geometric algebras.

1.3 Geometric Algebra for Conics

Now let us consider Geometric Algebra for Conics (GAC), that is the generalization of $\mathbb{G}_{4,1}$, proposed by C. Perwass, [23], and J. Hrdina, A. Návrat, P. Vašík, [16]. Let us stress that in the current work the notation of [16] is used, ie., $\bar{n}$ and n are taking place of e_0 and e_∞, respectively.

Analogously to the notation in [23], the corresponding basis elements are denoted as

$$\bar{n}_+, \bar{n}_-, \bar{n}_\times, e_1, e_2, n_+, n_-, n_\times. \tag{1.1}$$

This notation suggests that the basis elements e_1, e_2 play the usual role of standard basis of the plane while the null vectors $\bar{n}$, n represent the origin and infinity, respectively. Note that there are three orthogonal 'origins' $\bar{n}$ and three corresponding orthogonal 'infinities' n [16].

$$\mathcal{C}(x, y) = \bar{n}_+ + xe_1 + ye_2 + \frac{1}{2}(x^2 + y^2)n_+ + \frac{1}{2}(x^2 - y^2)n_- + xyn_\times. \tag{1.2}$$

Definition 1.3.1. Geometric Algebra for Conics (GAC) is the Clifford algebra $\mathbb{C}l(5, 3)$ together with the embedding (1.2) in the basis (1.1).

More information on the outer (wedge) product, inner product and the duality can be found in the full version of the thesis.

For our purposes, we stress that these operations correspond to sums and products only. Thus, computational error is minimized. Indeed, the wedge product is calculated as the outer product of vectors on each vector space of the same grade

Notation	Meaning
e_1, e_2	2D basis vectors
$\bar{n}_+, \bar{n}_-, \bar{n}_\times$	origins
$n_+, n_-, n_\times$	infinities
AB	geometric product of A and B
$A \wedge B$	outer product of A and B
$A \cdot B$	inner product of A and B
A^*	dual of A
A^{-1}	inverse of A

Table 1.1: Notations of Geometric Algebra for Conics

blades, while the inner product acts on these spaces as the scalar product. The extension of both operations to general multivectors adds no computational complexity due to linearity of both operations.

Let us also recall that if a conic C is considered as a wedge of five different points (which determine a conic uniquely), the appropriate 5-vector E^* is called an outer product null space representation (OPNS) and its dual E, indeed a 1-vector, is called the inner product null space (IPNS) representation. The reason is that if a point P lies on a conic C then

$$P \cdot E = 0 \quad \text{and} \quad P \wedge E^* = 0.$$

Consequently, intersections of two geometric primitives are given as the wedge product of their IPNS representations, ie.,

$$C_1 \cap C_2 = E_1 \wedge E_2$$

for two conics C_1, C_2 and their IPNS representations E_1 and E_2, respectively, see [16].

Let us describe the inner product representation more precisely. An element $A_I \in \mathbb{G}_{5,3}$ is the inner product representation of a geometric object A in the plane if and only if $A = \{\mathbf{x} \in \mathbb{R}^2 : \mathcal{C}(\mathbf{x}) \cdot A_I = 0\}$. The representable objects can be found by examining the inner product of a vector and an embedded point. A general vector in the conic space $\mathbb{R}^{5,3}$ in terms of our basis is of the form

$$v = \bar{v}^+ \bar{n}_+ + \bar{v}^- \bar{n}_- + \bar{v}^\times \bar{n}_\times + v^1 e_1 + v^2 e_2 + v^+ n_+ + v^- n_- + v^\times n_\times$$

and its inner product with an embedded point is then given by

$$\mathcal{C}(x,y) \cdot v = -\frac{1}{2}(\bar{v}^+ + \bar{v}^-)x^2 - \bar{v}^\times xy - \frac{1}{2}(\bar{v}^+ - \bar{v}^-)y^2 + v^1 x + v^2 y - v^+,$$

ie., by a general polynomial of degree two. Thus the objects representable in GAC are exactly conics. We also see that the two-dimensional subspace generated by infinities $n_-, n_\times$ is orthogonal to all embedded points. The classification of conics is well known. There exists 3 types of non–degenerate conics: ellipse, hyperbola, and parabola. Let us briefly describe inner product representations of the conics in GAC. More information can be found in [16] and in the full version of the thesis.

It is well known that the type of a given unknown conic can be read off its matrix representation, which in our case for a conic is given by

$$Q = \begin{pmatrix} -\frac{1}{2}(\bar{v}^+ + \bar{v}^-) & -\frac{1}{2}\bar{v}^\times & \frac{1}{2}v^1 \\ -\frac{1}{2}\bar{v}^\times & -\frac{1}{2}(\bar{v}^+ - \bar{v}^-) & \frac{1}{2}v^2 \\ \frac{1}{2}v^1 & \frac{1}{2}v^2 & -v^+ \end{pmatrix}. \tag{1.3}$$

The entries of (1.3) can be easily computed by means of the inner product:

$$\begin{aligned} q_{11} &= Q_I \cdot \tfrac{1}{2}(n_+ + n_-), \\ q_{22} &= Q_I \cdot \tfrac{1}{2}(n_+ - n_-), \\ q_{33} &= Q_I \cdot \bar{n}_+, \\ q_{12} = q_{21} &= Q_I \cdot \tfrac{1}{2}n_\times, \\ q_{13} = q_{31} &= Q_I \cdot \tfrac{1}{2}e_1, \\ q_{23} = q_{32} &= Q_I \cdot \tfrac{1}{2}e_2. \end{aligned}$$

It is also well known how to determine the internal parameters of an unknown conic and its position and the orientation in the plane from the matrix (1.3), [26]. Hence all this can be determined from GAC vector Q_I by means of the inner product.

The main advantage of GAC compared to other models (for instance, $\mathbb{G}_6$) is that it is fully operational in the sense that it allows all Euclidean transformations, ie., rotations and translations. But not just that, it also allows scaling, for more details see full version of the thesis. Hence, like in the case of CGA (or $\mathbb{G}_{3,1}$), one obtains all conformal transformations: rotation, translation and scaling.

Example 1. Let us consider the IPNS representation of the axis-aligned ellipse with the semi-axes $a = 2$, $b = 4$ centred in $(u, v) = (0, 2)$:

$$E = 0.8e_2 - 0.5e_3 - 0.3e_4 + 0.5e_6 + 0.3e_7.$$

The rotor for a rotation around the origin by the angle $\frac{\pi}{3}$ is given by $R = R_+(R_1 \wedge$

R_2), where

$$R_+ = \cos(\tfrac{\pi}{6}) + \sin(\tfrac{\pi}{6}) e_1 \wedge e_2 = \tfrac{\sqrt{3}}{2} + \tfrac{1}{2} e_1 \wedge e_2, \tag{1.4}$$
$$R_1 = \cos(\tfrac{\pi}{3}) + \sin(\tfrac{\pi}{3}) \bar{n}_\times \wedge n_- = \tfrac{1}{2} + \tfrac{\sqrt{3}}{2} \bar{n}_\times \wedge n_-, \tag{1.5}$$
$$R_2 = \cos(\tfrac{\pi}{3}) - \sin(\tfrac{\pi}{3}) \bar{n}_- \wedge n_\times = \tfrac{1}{2} - \tfrac{\sqrt{3}}{2} \bar{n}_- \wedge n_\times. \tag{1.6}$$

Rotated ellipse has equation:

$$E_{rotated} = -0.69282e_1 + 0.4e_2 - 0.5e_3 + 0.15e_4 - 0.25981e_5 + 0.5e_6 - 0.15e_7 + 0.25981e_8$$

The scalor for a by $\alpha \in \mathbb{R}^+$ is given by $S = S_+ S_- S_\times$, where

$$S_+ = \tfrac{\alpha+1}{2\sqrt{\alpha}} + \tfrac{\alpha-1}{2\sqrt{\alpha}} \bar{n}_+ \wedge n_+, \tag{1.7}$$
$$S_- = \tfrac{\alpha+1}{2\sqrt{\alpha}} + \tfrac{\alpha-1}{2\sqrt{\alpha}} \bar{n}_- \wedge n_-, \tag{1.8}$$
$$S_\times = \tfrac{\alpha+1}{2\sqrt{\alpha}} + \tfrac{\alpha-1}{2\sqrt{\alpha}} \bar{n}_\times \wedge n_\times. \tag{1.9}$$

For more information see [16].

Consider axis-aligned ellipse with the semi-axes $a = 2, b = 4$ centred in $(u, v) = (0, 2)$.

The result of applying the scaling by 2 is the ellipse

$$E_{scaled} = 3.2e_2 - 1.0e_3 - 0.6e_4 + 1.0e_6 + 0.6e_7$$

and translations by vectors $(0, 2), (2, 0), (-3, 2)$ are:

$$E(0,2) = 2.76923e_1 + 1.23077e_2 + 0.73077e_3 - 0.19231e_4 + 1.73077e_6 + 0.19231e_7,$$

$$E(2,0) = 2.46154e_2 + 1.65385e_3 - 0.19231e_4 + 2.65385e_6 + 0.19231e_7,$$

$$E(-3,2) = -4.15385e_1 + 2.96154e_3 - 0.19231e_4 + 3.96154e_6 + 0.19231e_7,$$

respectively.

In this part we provide a procedure for intersecting two conics, particularly ellipses with common centre in the coordinate origin but in a general mutual position otherwise. Again, the contribution of GAC lies in avoiding the use of a solver which leads to accuracy improvement. More information can be found in [27].

Let us first describe some differences to CRA or its 3–dimensional version CGA (Conformal Geometric Algebra). Crucial difference lies in the type of objects that are intrinsic to respective structures. For CRA (CGA), spheres (circles) are the geometric primitives that may be represented by specific elements. Taking into account that lines and planes are spheres with infinite radii and a point pair is a 1–dimensional sphere, we receive all geometric primitives for analytic geometry. Moreover, intersection still remain such objects, indeed, an intersection of two spheres or two circles are circles or point pairs, respectively. Therefore, intersections that are realised by wedge of IPNS representations remain representatives of Euclidean primitives intrinsic to CRA (CGA). On contrary, in GAC the situation is different. Even if we restrict to the case of co–centric ellipses, their intersection is a ”four point” which has no meaning in the sense of conic–sections. Indeed, a planar conic is generated by five points at least. This leads to an algorithm that may be used for co–centric conics (all types). On the other hand, the algorithm is still geometric–based and may be realised by a sequence of simple operations in GAC, ie., there is no numerical solver involved.

1.4 Summary of Chapter 1

Throughout this chapter, we have explored the fundamental concepts of Grassmann algebras and Clifford algebras, highlighting their role in forming the basis of Geometric Algebra. One of the key features of GA is its ability to represent geometric entities such as points, lines, planes, and volumes using multivectors, which are elements of a geometric algebra. These multivectors capture not only the geometry of objects but also their spatial relationships and transformations. Through the use of the outer, inner, and geometric product operations, GA enables the manipulation and analysis of these geometric entities in a rigorous and intuitive manner.

Chapter 2

Switched systems

2.1 Dynamical system

By dynamical system we mean the triplet $(\mathbb{R}, \mathbb{R}^n, f)$, where $\mathbb{R}^n$ is a metric space, which we call phase or state space, and $f(t, \cdot)$ is a system of evolution operators defined as the mapping $f(t, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$, which maps the initial state $x(0) \in \mathbb{R}^n$ to some state $x(t) \in \mathbb{R}^n$.

Dynamical system can be generated by a form of the system of Ordinary Differential Equations (ODE)

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(t, \boldsymbol{x}), \tag{2.1}$$

in which a function $\boldsymbol{f} \in C(\mathbb{R}^{n+1}, \mathbb{R}^n)$ describes the time dependence of a point $\boldsymbol{x} \in \boldsymbol{X}$ and the derivative is considered w.r.t. time.

Definition 2.1.1. Switched system is the system of differential equations in the vector form of the type

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_{\sigma(t)}(\boldsymbol{x}); \tag{2.2}$$

where $\boldsymbol{x} = (x_1, \dots, x_m) \in \mathbb{R}^m$ is called *a continuous state*, $\sigma : \mathbb{R} \to \mathbb{R}$ is a left continuous piecewise constant function of time with finite amount of pieces, that is called *a discrete state*(switching signal) with values from an index set $M := \{1, \dots, n\}$, and $f_{\sigma(t)} : \mathbb{R}^m \to M$ is a family of functions of class C^k for sufficiently large k.

Along with (2.2) we consider the initial condition $\boldsymbol{x}(0) = \boldsymbol{x}_0$, $\boldsymbol{x}_0 \in \mathbb{R}^m$,

Definition 2.1.2. Let $\boldsymbol{x}$ be a continuous and piecewise differentiable function on $\mathbb{R}$. Then $\boldsymbol{x}$ is a solution of the switched system (2.2) on $\mathbb{R}$ if for all $t \in \mathbb{R}$, $\quad \boldsymbol{x}(t) \in \mathbb{R}^n$ and $\dot{\boldsymbol{x}} = \boldsymbol{f}_{\sigma(t)}(\boldsymbol{x}(t))$, and at the breakpoints, sigma has the right side derivative.

Definition 2.1.3. We say that the switched system

$$\dot{\boldsymbol{x}} = \boldsymbol{f}_{\sigma(t)}(\boldsymbol{x})$$

is *controllable* if for any two points $A, B \in \mathbb{R}^m$ from the state space there exists a switching signal generating a trajectory from A to B.

Depending on the type of switch the following types of switched systems are distinguished [10]:

- State-dependent versus time-dependent;
- Autonomous (no direct control over the switching mechanism that triggers the discrete events) versus controllable (direct control over the switching mechanism)

Let us now concentrate on one of the subsystems of the switched system. Various methods of studying these systems include using eigenvalues, phase portrait, nullclines etc. Let us consider some of them more precisely.

Classification of equilibrium points in the case when $\det A \neq 0$ is shown in the Table 2.1.

Table 2.1: Classification of equilibrium points in the case $\det A \neq 0$

Roots of characteristic Equation	Point Type
λ_1, λ_2 are real numbers of the same sign $\lambda_1 \lambda_2 > 0$	Node
λ_1, λ_2 are real numbers of the opposite sign $\lambda_1 \lambda_2 < 0$	Saddle
λ_1, λ_2 are complex numbers $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 \neq 0$	Focus
λ_1, λ_2 are complex numbers $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 = 0$	Center

2.2 Summary of Chapter 2

The Chapter provides an overview of dynamical systems, particularly focusing on the theory of switched systems. In this chapter we introduced controllability of switched systems and presented standard methods for analyzing dynamical systems. Finally, the types of 2x2 linear systems depending on equilibrium points were discussed.

Chapter 3

Control of the switched system by means of GAC

The following chapter deals with the 2×2 switched systems with two subsystems of the special type. More precisely, subsystems have a specific equilibrium point and trajectories intrinsic to Geometric Algebra for Conics. Indeed, the trajectories in question are ellipses and hyperbolas, ie., the elements of GAC.

3.1 Center-Center

In the following section, the case of 2×2 matrices with both subsystems having pure imaginary eigenvalues is studied. Further we will refer to this type of the switched system as Center-Center. This case has already been considered in [28], and the main difference lies in using GAC as a suitable space for geometric operations with the ellipses, for elementary notions see Chapter 1. First, consider one dimensional oscillation problem of a spring pendulum under the condition of absence of external and friction forces and its model equation in the form

$$\ddot{x} = -kx, \quad k \in \mathbb{R} \tag{3.1}$$

with a switchable stiffness coefficient $k > 0$ with two possible values. In the spring pendulum problem, this corresponds to joining and removing an additional spring with stiffness coefficient k_2 to the original spring with stiffness coefficient k_1. Two cases can be considered. If the springs are connected in parallel, the parameter k of the system switches between k_1 and $k_1 + k_2$. If the connection is in series, the parameter k of the system switches between k_1 and $\frac{k_1 k_2}{k_1 + k_2}$.

Let us rewrite the differential equation (3.1) of the pendulum oscillations as a switched system in the form of a system of two differential equations of order one by denoting

$$x_1 = x, \quad x_2 = \dot{x}.$$

Consequently, if we set $\boldsymbol{x} = (x_1, x_2)^T$, the resulting system in the matrix form can be written as

$$\dot{\boldsymbol{x}}(t) = A_i \boldsymbol{x}(t), \quad A_i \in \mathcal{M}at_2(\mathbb{R}), \quad i = 1, 2. \tag{3.2}$$

Without the loss of generality, let us assume that we start and end with the first system $i = 1$. Suppose that two nonzero points (starting $S = [a_1, a_2] \in \mathbb{R}^2$ and ending $E = [b_1, b_2] \in \mathbb{R}^2$) are given.

In the case $\text{Tr}A_i = 0$, ie., the case of the spring pendulum without damping, for example,

$$A_i = \begin{pmatrix} 0 & 1 \\ -\alpha_i & 0 \end{pmatrix}, \quad \alpha_i \in \mathbb{R}^+,$$

we have an axis-aligned ellipse, while if $\text{Tr}A_i \neq 0$, then the ellipses are rotated and the given switched system is equivalent to the equation describing the oscillatory system with damping. In this case the rotation angle can be calculated from the elements of the conic matrix Q, (1.3), namely of its 2×2 submatrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ as follows:

$$\theta = \begin{cases} \arctan\left(\frac{1}{q_{12}}\left(q_{22} - q_{11} - \sqrt{(q_{11} - q_{22})^2 + q_{12}}\right)\right), & q_{12} \neq 0 \\ 0, \quad q_{12} = 0, \quad q_{11} < q_{22} \\ \frac{\pi}{2}, \quad q_{12} = 0, \quad q_{11} > q_{22} \end{cases}$$

Note that this enables us to solve efficiently even the systems with rotated ellipses as the trajectories.

3.2 Algorithm for a switching path construction

In the following, we describe the algorithm for finding a control of a switched system, ie., finding a path composed of the systems' integral curves from the starting point S to the endpoint E such that the number of switches is minimal. More information ca be found in [27]. Consider the case $n = 2$, ie., only two systems are included, and both starting and final ellipse belong to the same family. To apply the GAC based calculations, it is necessary to get the exact GAC form of the representatives of both families of ellipses. Thus the system of ODEs is solved numerically (e.g. by Runge-Kutta method) with the initial condition at the starting point A. This will give us a set of points representing the initial ellipse. After applying the GAC conic fitting algorithm, [16], we get the ellipse in IPNS representation. Note that according to [29], the algorithm may be further specified by prescribing the resulting ellipse to be axis-aligned and with its centre placed in the origin.

3.3 Examples and comparison to the numerical methods

Let us now consider the following set of examples, which generalize the system from [30, p. 6]. The following systems describe the oscillatory problem without damping.

Example 2. Consider the switched system (3.2) in its matrix form, ie., $\dot{\boldsymbol{x}} = A_i\boldsymbol{x}$, for $i = 1, 2$, where

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -\frac{1}{2} & 0 \end{pmatrix}.$$

Consider the initial point $S = [2, 5]$ and the ending point $E = [12, 22]$. We need to find the path from S to E composed of the respective system trajectories. For example of the pair of ellipses families see Figure 3.1, left, where the ellipses have a common centre and perpendicular semiaxes. In Figure 3.1, right, the resulting path can be found. Consequently, the set of switching points is calculated with the following result:

```
[0,-5.74456],[8.12404,0],[0,11.48913],[-16.24808,0],
[0,-22.97825],[23.2054167141,12.86501593890354].
```

This is a result of Python code written in a module Clifford according to the algorithm in Chapter 3.2. Note that the red points in Figure 3.1, left, form the set of points generated by Runge-Kutta method and you can see the fitted conic, too.

In order to compare the result received by the use of GAC with numerical solution, we solve the same system numerically. Instead of GAC conic fitting and searching for intersections, we use Runge-Kutta method for the next ellipse construction and for the last ellipse we get the system of two quadratic equations. For fitting the ellipse we use the least squares fitting of ellipses by Halir and Flusser [31].

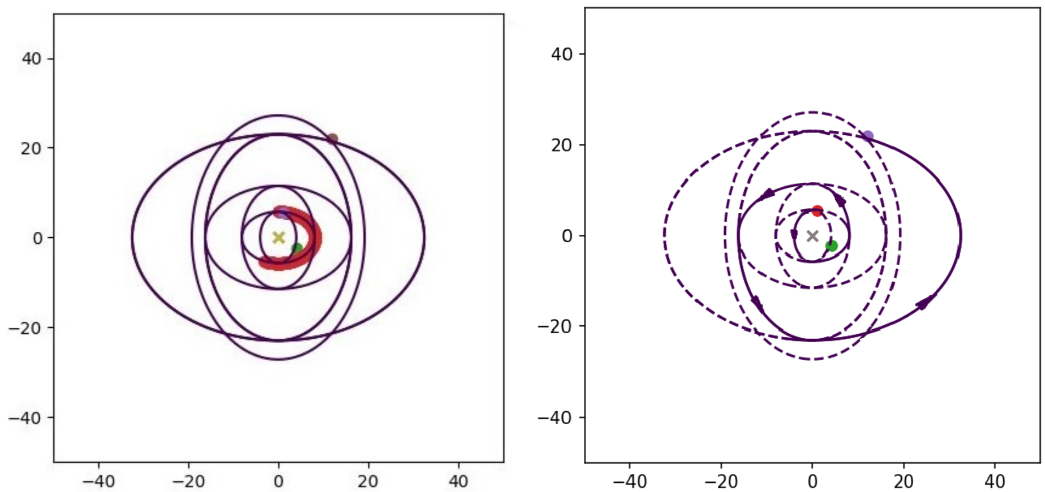


Figure 3.1: Example 2

As a result of the above numerical algorithm we receive the following set of switching points:

```
[0.0, 5.74456265], [8.1240384,0.0], [0.0, 11.48912529],
[16.24807681,0.0],[0.0,22.97825059],
[-21.1344899771933,12.7540843143374].
```

Let us compare the resulting coordinates with the case of using geometric algebra:

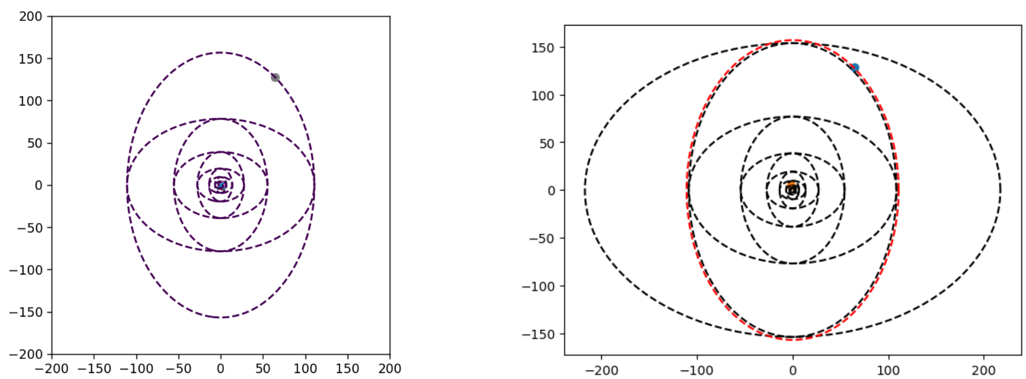
```
[0.0, 5.744562646461688], [8.12404, 0.0], [0.0, 11.48913],
[16.24808, 0.0], [0.0, 22.97825],
[-23.205416714111358,  12.86501593890354].
```

Note that the final switching point is different in the numeric case. This happens due to the fact that the calculations in the numerical solution contains a numerical error in the final numerical calculation of the ellipses intersection, which is completely avoided in the case of GAC where the intersections are obtained geometrically, for more details see full version of the thesis.

Let us note that even numerically we can find the switches of the system, but they may not be optimal due to the amount of switches, see the following example.

Example 3. In some cases even the number of switches can be different. If, for example, the ending point belongs to one of the ellipses of the starting family, and the number of steps is relatively large, the numeric solution can offer extra ellipse, therefore, the set of switching points is growing and the solution is not optimal with respect to the number of switches.

Figure 3.2, right, demonstrates the extra horizontal ellipse leading to new vertical ellipse as a solution, offered by numerical solution. Nevertheless, the num-



(a) GAC solution

(b) Numeric solution

Figure 3.2: Example 3

ber of switches can differ only by 1 because the length of the semiaxes of the circumscribed ellipses grows much faster than the numerical error. As a result we conclude that the numeric solution may not be optimal with respect to the number of switches.

Example 4. Now let us consider the pendulum problem with damping. The corresponding switched system is switched system $\dot{\boldsymbol{x}} = A_i \boldsymbol{x}$, $i = 1, 2$, where matrices of the sybsystems are given as

$$A_1 = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}.$$

The initial point is $[2; 5]$, and we need to find a path to the point $[30; 22]$. Both of the matrices have pure imaginary eigenvalues, so the system is switching between ellipses. Ellipses of the second family are rotated. That means that the second subsystem describes one of the cases of the oscillatory system with damping. The switching path calculated in Python module Clifford can be seen in Figure 3.3.

Note that the presented algorithm is applicable for the switched systems, with subsystems, whose phase curves are intrinsic to GAC elements.

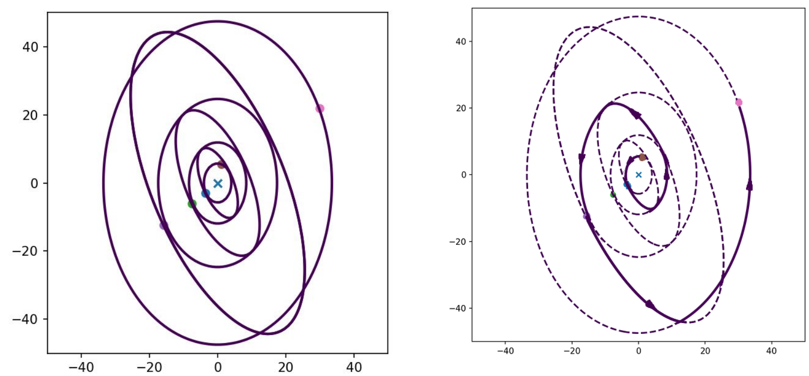


Figure 3.3: Example 4

In this case, there is no set of switching points that allows to get from points in the first and third quadrants to points in the second and fourth quadrants. Therefore, there exists at least one pair of points that cannot be connected by a trajectory.

The Saddle-Saddle type switched system is not controllable. Let us note that in the case when the algorithm for switching path construction, is applied, it will be interrupted by reaching a predescribed maximal number of iterations.

3.4 Center-Saddle

In the next step, we consider a switched system, where one of the matrices has real eigenvalues of different signs (a singular point of the Saddle type), and the other has purely imaginary eigenvalues.

Let us consider switched system (3.2), where matrices take the form

$$A_1 = \begin{pmatrix} 0 & 1 \\ -\alpha & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 1/\alpha & 0 \end{pmatrix}, \quad \alpha > 0,\ \alpha \in \mathbb{R}. \tag{3.3}$$

When switched system consists of subsystems of the Center and Saddle type, the algorithm is greatly simplified. The fact is that in this case, only two switches are enough to find the path. Let us formulate the following theorem.

Theorem 3.4.1. The switched system (3.2) with subsystems having matrices of the type (3.3) is controllable. Moreover, if the movement from the starting point corresponds to the system of the Saddle type, then it is possible to get to an arbitrary end point using two switches.

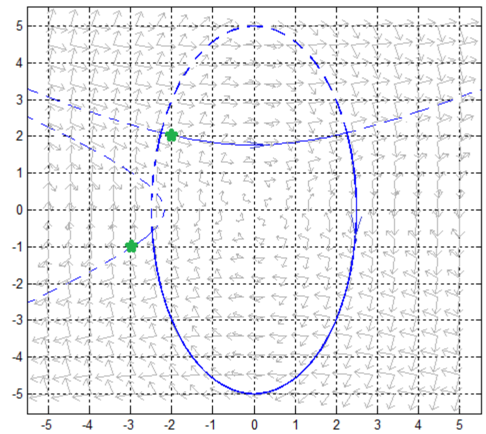


Figure 3.4: Phase portrait for system with Saddle and Center subsystems, $\alpha = 4$, Starting point [-2,2], ending point [-3,-1]

Therefore, the first step is to find the hyperbolas passing through the starting and ending points. Then the intermediate ellipse is used for the motion between them, therefore we find the needed path with 2 switches (see Figure 3.4).

3.5 Summary of Chapter 3

The Chapter describes the controllability of the 2×2 switched systems with regular matrices of each subsystem by the means of Geometric Algebra. It was demonstrated that the use of GAC for construction of switching points of 2D switched systems leads to the solution that is optimal with respect to the number of switches. From the geometric nature of our approach we can see that the number of switches can only differ by 1 from the numerical solution but also the numerical error for particular switches must be taken into account. We provided examples with axes aligned ellipses but from the description of GAC it is clear, that their approach will handle rotated conics of any type as well, in which case numerical solution will carry even larger error.

Chapter 4

Cases laying out of GAC

In the previous chapter we considered the cases where the phase curves of the subsystems of the switched system are intrinsic to GAC, ie., conic sections.

The aim of the following chapter is to describe particular systems whose phase portrait contains 2D curves, not being conic sections (for example, spirals), therefore, laying out of GAC.

Upon further investigation, it turned out that among the switched systems with regular 2x2 matrices, except the systems described in the previous Chapter, controllable can be also switched systems corresponding to switch between matrices with equilibrium point of the type stable and unstable Focuses (λ_1, λ_2 are complex numbers $\mathrm{Re}\lambda_1 = \mathrm{Re}\lambda_2 \neq 0$) and partially controllable systems are systems, which have switches between Saddle (λ_1, λ_2 are real numbers of the opposite sign $\lambda_1\lambda_2 < 0$) and Focus (λ_1, λ_2 are complex numbers $Re\lambda_1 = Re\lambda_2 \neq 0$). Now let us consider the controllable case Stable-Unstable Focus.

Example 5. Let us consider the system (3.2) with the following particular matrices, ie., we choose $\alpha = 2:$

$$A_1 = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -1 & \frac{1}{2} \\ -\frac{1}{2} & -1 \end{pmatrix}.$$

Let the starting point be $S = [-2; -1.1]$ and the ending point $E = [1; 2.4]$. We start from finding the corresponding spirals passing through S and E, respectively. The switching path is then constructed by switching between the spirals in arbitrary points with the final switch given by the intersection with the spiral containing the endpoint.

The corresponding switching path is demonstrated in Figure 4.1.

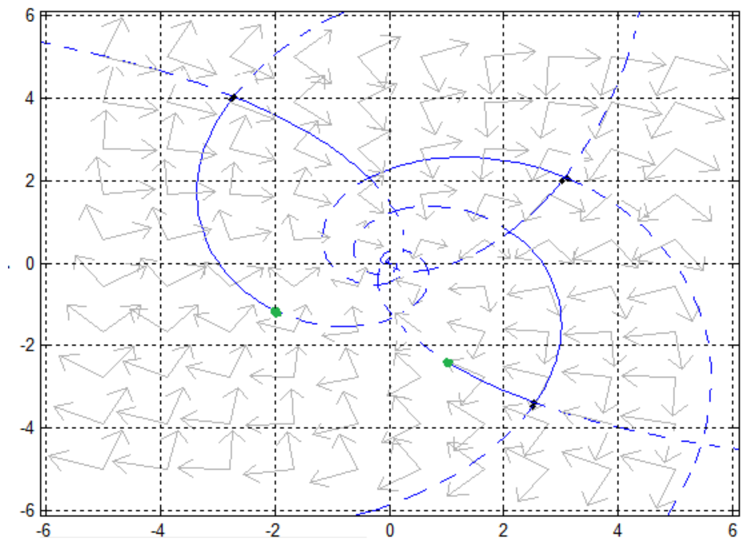


Figure 4.1: Example 5, $\alpha = 2$

Note that due to the different directions of the spirals any two points can be connected by a path containing finite number of switches. Moreover, with respect to the minimal number of switches, any two points may be connected by a switching path containing just one switch. The reason is that two spirals corresponding to different subsystems have infinitely many intersections and therefore it is always possible to choose a switch which will consequently take us to the endpoint. Note that this choice may lead to undesirably long time.

4.1 Summary of Chapter 4

The Chapter describes the controllability of the 2×2 switched systems with regular matrices of each subsystem, whose phase portraits are not conic sections. Concrete examples and phase portraits were utilized to illustrate the controllability characteristics of different system configurations. It was demonstrated that the only controllable systems from cases laying out of GAC are the systems, which are the combination of Stable and Unstable focuses. The switched systems with subsystem matrices's curves of the types Saddle and Focus are "controllable under condition", ie., dependent on the constant. Those systems were found to be controllable only within certain parameter ranges, contingent upon the real part of the Saddle system's eigenvalue.

Chapter 5

Results

As a result we get the following table that shows which type of switched system is controllable. For example, switched systems with matrices of the type Center-Center (both matrices have pure complex eigenvalues), are controllable, while switched systems with matrices of the type Saddle-Saddle(both matrices have real eigenvalues) are not controllable.

Table 5.1: Controllability of the switched 2x2 systems

A_1 n A_2		**Center**	**Saddle**	**Node**		**Focus**	
				Stable	**Unstable**	**Stable**	**Unstable**
Center		+	+	-	-	-	-
Saddle		+	-	-	-	controllable under condition	controllable under condition
Node	**Stable**	-	-	-	-	-	-
	Unstable	-	-	-	-	-	-
Focus	**Stable**	-	controllable under condition	-	-	-	+
	Unstable	-	controllable under condition	-	-	+	-

Therefore, the only controllable types of the switched systems are the combination of Center and Saddle, Center and Center, Stable and Unstable focuses. The switched systems with subsystem matrices's curves of the types Saddle and Focus are ”controllable under condition”.

Chapter 6

Conclusions

The thesis dealt with the controllability of the 2×2 switched systems with regular matrices of each of two subsystems. For the systems whose phase trajectories are conics (which are the elements of GAC), the Geometric Algebra approach was used. It was demonstrated that the use of GAC for construction of switching paths of 2D switched systems leads to the solution that is optimal with respect to the number of switches.

We demonstrated a complete geometric algorithm for a system with two families of axes–aligned, centralized ellipses, where we showed symbolic and Python calculations, respectively. In addition, we used a property of GAC that it contains a two–dimensional conformal geometric algebra CRA, where our calculations were completed. This case corresponds to an oscillatory switched system without damping. But our approach applies also for damped systems where the integral curves are formed by rotated ellipses, ie., non axes–aligned. Also in this case no solver was needed because, in the system of two quadratic equations describing the ellipses' intersections, we replaced an ellipse equation by a line equation which reduced the degree and allowed analytic solution. Note that both approaches exploit the elegance of conics manipulation in GAC by constructing a pair of lines containing the intersecting points and circumscribed ellipses simply calculated by GAC scaling with a factor. Let us point out that even the preparation of initial trajectories is highly geometric. Fitting a conic with prescribed properties in GAC eliminates an error in numerical solution to our switched system. Indeed, all trajectories will be precisely of a given type, ie., co–centred and axes–aligned. Consequent GAC transformations do not change these properties and do not input any numerical errors. Therefore, the only place for a rounding error is the calculation of fractions and square roots because. Using the geometric nature of our approach, we demonstrated that the number of switches can only differ by 1 from the numerical solution but also the numerical error for particular switches must be taken into account. We provided examples with axes aligned ellipses but from the description of GAC it is clear, that this approach will handle rotated conics of any type as well, in which case numerical solution will carry even larger error. Therefore, the advantages of using GAC lies in the following: The GA approach speeds up algorithms, eliminates the need for numerical solvers, reduces computational complexity, minimizes numerical errors, and allows for a coordinate-free formulation.

Classification of switched systems controllability was provided based on their geometric properties. It was proved that the only controllable systems are systems of the type Center-Center, Center-Saddle and Stable-Unstable focuses. The switched systems of type Saddle-Focus are "controlled under condition". For controllable switched systems, the controlling algorithm based on the GAC primitives and their transformations was introduced. The proposed approach creates possibility of passing from the classical solution of the controllability problem for switched systems to a geometric one, using the type of phase trajectory. Geometric Algebra (GA) provides a comprehensive mathematical framework for handling geometric operations efficiently. Originating from Grassmann algebras and Clifford algebras, GA unifies geometric primitives and transformations into a single algebraic structure, facilitating geometric analysis and manipulation. Key concepts include representing geometric entities using multivectors and utilizing outer, inner, and geometric product operations for manipulation.

Overall, the thesis demonstrated the effectiveness of using Geometric Algebra for analyzing and controlling 2×2 switched dynamical systems, offering advantages in efficiency, accuracy, and simplicity of implementation.

The controllability of 2×2 switched systems with regular matrices is addressed using Geometric Algebra. GA aids in constructing optimal switching points, minimizing the number of switches required.

A novel algorithm for optimal control of switched dynamical systems with purely imaginary eigenvalues is proposed. This algorithm utilizes Geometric Algebra for Conics (GAC) to construct switching paths consisting of circumscribed ellipses, minimizing numerical errors and eliminating the need for solvers.

The thesis highlights the geometric nature of the approach, emphasizing its ability to handle various system configurations and minimize computational errors. It also discusses the formulation of controlling algorithms in a coordinate-free form, solely using GAC operations.

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