Vanishing solutions of a second-order discrete non-linear equation of Emden-Fowler type.

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Abstract—The paper discusses a discrete equation of an Emden-Fowler type $\Delta^2 v(k) = -k^3 (\Delta v(k))^3$ where v is a dependent variable, k is an integer-valued independent variable, Δv and $\Delta^2 v$ are the first and second-order forward differences of v, respectively. The paper aims to prove the existence of a nontrivial and vanishing solution for $k \to \infty$. The equation is transformed into a system of two first-order difference equations, which makes it possible to apply previously known results when investigating the system.

Keywords—difference equation, Emden-Fowler type equation, asymptotic behaviour

1. INTRODUCTION

We consider the second-order difference equation

$$\Delta^2 v(k) = -k^3 (\Delta v(k))^3, \tag{1}$$

where the independent variable $k \in \mathcal{Z}_{k_0}^{\infty} = \{k_0, k_0 + 1, \dots\}$, k_0 is a positive integer, the dependent variable $v\colon \mathcal{Z}_{k_0}^{\infty} \to \mathbb{R}$, $\Delta v(k) = v(k+1) - v(k)$ and $\Delta^2 v(k) = v(k+2) + 2v(k+1) - v(k)$ are the first and second-order forward differences. In the paper we prove the existence of a nontrivial volution $v(k)$ such that $\lim_{k\to\infty} v(k) = 0$. It is known that second-order differential equations of Emden-Fowler type can have so called blow-up solutions (that is, solutions with a vertical asymptote). For example, Emden-Fowler differential equation $y''(x) = y^3(x)$ has blow-up solutions $y(x) = \pm\sqrt{2}/(x+K)$ where K is an arbitrary (but fixed) constant. By suitable transformation, this equation can be transferred to another one replacing the problem on the existence of blow-up solutions by one on the existence of vanishing solutions when $t \to \infty$ for a new differential equation. Equation (1) is a discrete variant of such a differential equation. The paper aims to prove that there are solutions $v = v(k)$ to equation (1) such that $\lim_{k\to\infty} v(k) = 0$. Note that differential equations of Emden-Fowler type are widely investigated because they serve as mathematical models of some phenomena appearing in cosmology, astrophysics and other sciences [1].

2. PRELIMINARIES

Definition 1. *Let $f, g\colon \mathcal{Z}_{k_0}^{\infty} \to \mathbb{R}$. We denote $f = O(g)$, if there exists an index $k_1 \geq k_0$ and a constant $M > 0$ such that $|f(k)| \leqslant M|g(k)|$, $\forall k \in \mathcal{Z}_{k_1}^{\infty}$.*

For further investigations we need to refer to main results from [2] and [3]. The following preliminaries are taken from [2, 3]. Consider a system of discrete equations

$$\Delta Y(k) = F(k, Y(k)), \tag{2}$$

where $F\colon \mathcal{Z}_{k_0}^{\infty} \times \mathbb{R}^n \to \mathbb{R}^n$, $Y = (Y_1, \dots, Y_n) \in \mathbb{R}^n$. Throughout the paper assume that F is continuous with respect to Y. Define a set $\Omega \subset \mathcal{Z}_{k_0}^{\infty} \times \mathbb{R}^n$ as

$$\Omega = \{(k, Y)\colon k \in \mathcal{Z}_{k_0}^{\infty}, b_i(k) < Y_i < c_i(k), i = 1, \dots, n\}$$

where $b_i, c_i\colon \mathcal{Z}_{k_0}^{\infty} \to \mathbb{R}$ are auxiliary functions such that $b_i(k) < c_i(k)$, $k \in \mathcal{Z}_{k_0}^{\infty}$, $i = 1, \dots, n$. Let us define functions $B_i, C_i\colon \mathcal{Z}_{k_0}^{\infty} \times \mathbb{R} \to \mathbb{R}$, $i = 1, \dots, n$ by formulas $B_i(k, Y) := -Y_i + b_i(k)$, $\quad C_i(k, Y) := Y_i - c_i(k)$ and sets

$$\Omega_B^i = \{(k, Y)\colon k \in \mathcal{Z}_{k_0}^{\infty}, B_i(k, Y) = 0, B_j(k, Y) \le 0, C_p(k, Y) \le 0,\ \forall j, p = 1, \dots, n, j \ne i\},$$
$$\Omega_C^i = \{(k, Y)\colon k \in \mathcal{Z}_{k_0}^{\infty}, C_i(k, Y) = 0, B_j(k, Y) \le 0, C_p(k, Y) \le 0,\ \forall j, p = 1, \dots, n, p \ne i\}$$

for every $i = 1, \dots, n$.

Definition 2. *The set Ω is called the regular polyfacial set with respect to the discrete system* (2) *if*

$$b_i(k+1) - b_i(k) < F_i(k, Y) < c_i(k+1) - b_i(k), \tag{3}$$

for every $i = 1, \dots, n$ and every $(k, Y) \in \Omega_B^i$ and if

$$b_i(k+1) - c_i(k) < F_i(k, Y) < c_i(k+1) - c_i(k), \tag{4}$$

for every $i = 1, \dots, n$ and every $(k, Y) \in \Omega_C^i$.

Below we use the sets

$$\Omega(k) = \{(k, Y), Y \in \mathbb{R}^n, b_i(k) < Y_i < c_i(k), i = 1, \dots, n\},$$

$$\Omega_i(k) = \{(w), b_i(k) < w < c_i(k)\}, \quad i = 1, \dots, n.$$

We need the following minor modification of Theorem 4 in [2] where it is proved by Liapunov-like reasonings.

Theorem 1. *[2, Theorem 4] Let Ω be regular with respect to the discrete system* (2) *and let the function*

$$G_i(w) := w + F_i(k, Y_1, \dots, Y_{i-1}, w, Y_{i+1}, \dots, Y_n)$$

be monotone on $\overline{\Omega}_i(k)$ for every fixed $k \in \mathcal{Z}_{k_0}^{\infty}$, each fixed $i \in \{1, \dots, n\}$, and every fixed $(Y_1, \dots, Y_{i-1}, Y_{i+1}, \dots, Y_n)$ such that $(k, Y_1, \dots, Y_{i-1}, w, Y_{i+1}, \dots, Y_n) \in \Omega$. Then, every initial problem $Y(k_0) = Y^*$ with $Y^* \in \Omega(k_0)$ defines the solution $Y = Y^*(k)$ of the discrete system (2) *satisfying the relation $Y^*(k) \in \Omega(k)$ for every $k \in \mathcal{Z}_{k_0}^{\infty}$.*

The following theorem is a slight modification of [3, Theorem 1] (see [4, Theorem 2] also) where it is proved by a topological method.

Theorem 2. *Assume that the function $F(k, Y)$ satisfies* (2) *and is continuous with respect to Y. Let inequality*

$$F_i(k, Y) < b_i(k+1) - b_i(k) \tag{5}$$

hold for every $i = 1, \dots, n$ and every $(k, Y) \in \Omega_B^i$. Let, moreover, inequality

$$F_i(k, Y) > c_i(k+1) - c_i(k) \tag{6}$$

hold for every $i = 1, \dots, n$ and every $(k, Y) \in \Omega_C^i$. Then, there exists a solution $Y = Y(k)$, $k \in \mathcal{Z}_{k_0}^{\infty}$ of system (2) *satisfying the inequalities*

$$b_i(k) < Y_{i-1}(k) < c_i(k)$$

for every $k \in \mathcal{Z}_{k_0}^{\infty}$ and $i = 1, \dots, n$.

3. AUXILIARY DISCRETE SYSTEM

Below we use the change of variables

$$v(k) = \pm \frac{\sqrt{2}}{k}(1 + Y_1(k)), \tag{7}$$

$$\Delta v(k) = \pm\Delta\left(\frac{\sqrt{2}}{k}\right)\cdot(1+Y_2(k)), \tag{8}$$

$$\Delta^2 v(k) = \pm\Delta^2\left(\frac{\sqrt{2}}{k}\right)\cdot(1+Y_3(k)) \tag{9}$$

where $Y_i(k)$, $i = 1, 2, 3$ are new dependent variables. Formulas (7)–(9) transform (1) to the system of discrete equations

$$\begin{cases} \Delta Y_1(k) = & \dfrac{1}{k}\cdot(Y_1(k)-Y_2(k)), \\ \Delta Y_2(k) = & \left(\Delta\left(\dfrac{1}{k+1}\right)\right)^{-1}\Delta^2\left(\dfrac{1}{k}\right)\cdot\left(-2k^3\left(\Delta^2\left(\dfrac{1}{k}\right)\right)^{-1}\cdot\left(\Delta\left(\dfrac{1}{k}\right)\right)^3\cdot(1+Y_2(k))^3-1-Y_2(k)\right). \end{cases} \tag{10}$$

or, assuming $|Y_2(k)| < 1$ and simplifying expression in the second equation, to

$$\begin{cases} \Delta Y_1(k) = \dfrac{1}{k}\cdot(Y_1(k)-Y_2(k)), \\ \Delta Y_2(k) = \left(-\dfrac{2}{k}+O\left(\dfrac{1}{k^3}\right)\right)\left(2Y_2(k)+3Y_2^2(k)+Y_2^3(k)+O\left(\dfrac{1}{k^2}\right)+O\left(\dfrac{Y_2(k)}{k^2}\right)\right). \end{cases} \tag{11}$$

4. MAIN RESULTS

Consider the second equation in system (11) separately, that is, consider the equation

$$\Delta Y_2(k) = \left(-\frac{2}{k}+O\left(\frac{1}{k^3}\right)\right)\left(2Y_2(k)+3Y_2^2(k)+Y_2^3(k)+O\left(\frac{1}{k^2}\right)\right). \tag{12}$$

Let $b_i(k) := -\varepsilon_i$, $c_i(k) := \gamma_i$ where ε_i, γ_i are fixed positive numbers. Set

$$B_i(k, Y_1, Y_2) := -Y_i - \varepsilon_i, \quad C_i(k, Y_1, Y_2) := Y_i - \gamma_i, \quad i = 1, 2$$

Auxiliary sets Ω_B^2, Ω_C^2 are reduced as follows

$$\Omega_B^2 = \{(k, Y_2)\colon k \in \mathcal{Z}_{k_0}^\infty, Y_2 = -\varepsilon_2\}, \quad \Omega_C^2 = \{(k, Y_2)\colon k \in \mathcal{Z}_{k_0}^\infty, Y_2 = \gamma_2\}.$$

We are going to apply Theorem 1 to the equation (12). This means that we need to show that (3) and (4) hold for $i = 2$ where

$$F_2(k, Y_1, Y_2) = F_2(k, Y_2) = \left(-\frac{2}{k}+O\left(\frac{1}{k^3}\right)\right)\left(2Y_2(k)+3Y_2^2(k)+Y_2^3(k)+O\left(\frac{1}{k^2}\right)\right). \tag{13}$$

Inequality (3) is now modified to

$$0 = b_2(k+1) - b_2(k) < F_2(k, Y_2)|_{(k, Y_2)\in\Omega_B^2} < \gamma_2 + \varepsilon_2. \tag{14}$$

The function

$$F_2(k, Y_2)|_{(k, Y_2)\in\Omega_B^2} = F_2(k, -\varepsilon_2) = \left(-\frac{2}{k}+O\left(\frac{1}{k^3}\right)\right)\left(-2\varepsilon_2+3\varepsilon_2^2-\varepsilon_2^3+O\left(\frac{1}{k^2}\right)\right)$$

assumes positive values for all sufficiently large k if

$$2\varepsilon_2 - 3\varepsilon_2^2 + \varepsilon_2^3 = \varepsilon_2(\varepsilon_2-1)(\varepsilon_2-2) > 0$$

that is if $\varepsilon_2 \in (0, 1) \cup (2, +\infty)$. Because, in the derivation of system (11), we assumed $|Y_2(k)| < 1$, only values $\varepsilon_2 \in (0, 1)$ can be used. The left inequality in (3) holds. The right inequality in (3) holds for all sufficiently large k as well since the function $F_2(k, \varepsilon_2)$ is vanishing.

Now, we show that (4) holds for $i = 2$. This inequality reduces to

$$-\varepsilon_2 - \gamma_2 = b_2(k+1) - c_2(k) < F_2(k, Y_2)|_{(k,Y_2)\in\Omega_C^2} < \gamma_2 - \gamma_2 = 0, \tag{15}$$

where

$$F_2(k, Y_2)|_{(k,Y_2)\in\Omega_C^2} = F_2(k, \gamma_2) = \left(-\frac{2}{k} + O\left(\frac{1}{k^3}\right)\right)\left(2\gamma_2 + 3\gamma_2^2 + \gamma_2^3 + O\left(\frac{1}{k^2}\right)\right).$$

The function $F_2(k, \gamma_2)$ is negative for $\gamma_2 \in (0, 1)$ and for all sufficiently large k. Therefore the right inequality in (15) holds. The left inequality in (15) holds too because the function $F_2(k, \gamma_2)$ is vanishing as $k \to \infty$. Finally, we need to show that the function

$$G_2(w) := w + F_2(k, w) \tag{16}$$

is monotone on

$$\overline{\Omega}_2(k) = \{(w)\colon w \in \mathbb{R}, b_2(k) \le w \le c_2(k)\} = \{(w)\colon w \in \mathbb{R}, -\varepsilon_2 \le w \le \gamma_2\}$$

for every fixed $k \in \mathcal{Z}_{k_0}^{\infty}$. We will verify the monotony property by computing $G'(w)$. Since a direct computation of the derivative of the function $F_2(k, w)$ expressed by (13) is not possible, we use its definition by the right-hand side of the second equation in system (10). We have

$$G_2(w) = w + F_2(k, w) = w + \left(\Delta\left(\frac{1}{k+1}\right)\right)^{-1}\Delta^2\left(\frac{1}{k}\right)\cdot\left(-2k^3\left(\Delta^2\left(\frac{1}{k}\right)\right)^{-1}\cdot\left(\Delta\left(\frac{1}{k}\right)\right)^3\cdot(1+w))^3 - 1 - w\right)$$

and, therefore,

$$G_2'(w) = 1 + \left(\Delta\left(\frac{1}{k+1}\right)\right)^{-1}\Delta^2\left(\frac{1}{k}\right)\cdot\left(-6k^3\left(\Delta^2\left(\frac{1}{k}\right)\right)^{-1}\cdot\left(\Delta\left(\frac{1}{k}\right)\right)^3\cdot(1+w)^2 - 1\right).$$

Because

$$\left(\Delta\left(\frac{1}{k+1}\right)\right)^{-1}\Delta^2\left(\frac{1}{k}\right) = -\frac{2}{k} + O\left(\frac{1}{k^3}\right) \quad \text{and} \quad -2k^3\left(\Delta^2\left(\frac{1}{k}\right)\right)^{-1}\left(\Delta\left(\frac{1}{k}\right)\right)^3 = 1 + O\left(\frac{1}{k^2}\right),$$

we have

$$G_2'(w) = 1 + \left(-\frac{2}{k} + O\left(\frac{1}{k^3}\right)\right)\cdot\left(3\left(1 + O\left(\frac{1}{k^2}\right),\right)(1+w^2) - 1\right)$$

and, for all sufficiently large k, $G'(w) > 0$. The function G_2 is monotone, Theorem 1 is applicable and, therefore, there exists a solution $Y_2 = Y_2^*(k)$ to equation (12) satisfying inequality

$$-\varepsilon_2 < Y_2^*(k) < \gamma_2, \quad k \in \mathcal{Z}_{k_0}^{\infty} \tag{17}$$

where k_0 is sufficiently large and positive numbers $\varepsilon_2, \gamma_2, \varepsilon_2 < 1, \gamma_2 < 1$ are fixed. Note that this solution is not trivial. Now we use Theorem 2 to analyse the first equation of the system (11), that is, the equation

$$\Delta Y_1(k) = (Y_1(k) - Y_2(k))/k. \tag{18}$$

We have proved that there exists a solution $Y_2 = Y_2^*(k)$ of the second equation in the system (11) with asymptotic behaviour described by inequality (17). Let in (18) be such a solution assumed. Then

$$\Delta Y_1(k) = \left(Y_1(k) - Y_2^*(k)\right)/k.$$

Seting

$$F_1(k, Y_1, Y_2) = F_1(k, Y_1) = (Y_1(k) - Y_2^*(k))/k$$

in Theorem 2, the auxiliary sets Ω_B^1, Ω_C^1 are reduced to

$$\Omega_B^1 = \{(k, Y_2)\colon k \in \mathcal{Z}_{k_0}^{\infty}, Y_1 = -\varepsilon_1\}, \quad \Omega_C^1 = \{(k, Y_2)\colon k \in \mathcal{Z}_{k_0}^{\infty}, Y_1 = \gamma_1\}.$$

Then, for $k \in \mathcal{Z}_{k_0}^{\infty}$, inequality (5) becomes

$$F_1(k, Y_1)|_{(k,Y_1)\in\Omega_B^1} = F_1(k, -\varepsilon_1) = \left(-\varepsilon_1 - Y_2^*(k)\right)/k < b_1(k+1) - b_1(k) = 0. \tag{19}$$

Due to (17), we derive

$$\left(-\varepsilon_1 - Y_2^*(k)\right)/k < (-\varepsilon_1 + \varepsilon_2))/k$$

and inequality (19) will hold if $\varepsilon_2 < \varepsilon_1$. For $k \in \mathcal{Z}_{k_0}^{\infty}$, inequality (6) will be now

$$F_1(k, Y_1)|_{(k,Y_1)\in\Omega_C^1} = F_1(k, \gamma_1) = \left(\gamma_1 - Y_2^*(k)\right)/k > c_1(k+1) - c_1(k) = 0. \tag{20}$$

By (17), we derive

$$\left(\gamma_1 - Y_2^*(k)\right)/k > (\gamma_1 - \gamma_2)))/k$$

and inequality (20) will hold if $\gamma_1 > \gamma_2$. Theorem 2 is applicable and, therefore, there exists a solution $Y_1 = Y_1^*(k)$ to equation (19) satisfying inequality $-\varepsilon_1 < Y_1^*(k) < \gamma_1$, $k \in \mathcal{Z}_{k_0}^{\infty}$ where k_0 is sufficiently large and positive number and $\varepsilon_2 < \varepsilon_1 < 1$, $\gamma_2 < \gamma_1 < 1$ are fixed. Note that this solution is not trivial because $Y_2^*(k)$ is not trivial. Summarizing the above investigations, we conclude that the below theorem holds.

Theorem 3. *Let ε_i, γ_i, $i = 1, 2$ be fixed positive numbers such that $\varepsilon_2 < \varepsilon_1 < 1$, $\gamma_2 < \gamma_1 < 1$. Then there exists a solution $Y(k) = Y^*(k) = (Y_1^*(k), Y_2^*(k))$ to the system* (11) *such that*

$$-\varepsilon_i < Y_i^*(k) < \gamma_i, \quad i = 1, 2, \quad \forall k \in \mathcal{Z}_{k_0}^{\infty} \tag{21}$$

provided that k_0 is sufficiently large.

5. CONCLUSION

By transformation (7), Theorem 3 implies that there exist two solutions $v = v_{\pm}(k)$ of equation (1) such that

$$v_{\pm}(k) = \pm\sqrt{2}k^{-1}(1 + Y_1^*(k)), \quad \forall k \in \mathcal{Z}_{k_0}^{\infty}. \tag{22}$$

From (22) we have $\lim_{k\to\infty} v_{\pm}(k) = 0$. Moreover, from (21) we derive

$$\sqrt{2}(1-\varepsilon_1) < kv_+(k) < \sqrt{2}(1+\gamma_1), \quad -\sqrt{2}(1+\gamma_1) < kv_-(k) < -\sqrt{2}(1-\varepsilon_1).$$

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