QUALITY IMPROVEMENT OF LQG CONTROL USING $\mathcal{H}_\infty$ OPTIMIZATION

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Abstract: This paper describes the method of using $\mathcal{H}_\infty$ optimization for tweaking of the existing structured controller. This method is presented on an example of LQG controller for the classical problem of the inverted pendulum on a cart. This means, that it is applicable not only on controller problems, but also on observer problems.

Keywords: H-infinity, optimization, H-2, norm, control, LQG, hinfstruct, inverted pendulum

1 DESCRIPTION OF CONTROLLED SYSTEM AND USED CONTROLLER

Assumed model of the inverted pendulum on a cart is shown in figure 1. This model has one input, force F, and four states. Those are cart position x, cart acceleration $\dot{x}$, pendulum angle θ and angular acceleration $\dot{\theta}$. Further assume, that the only directly measured states are x and θ. This assumption corresponds to reality, since the acceleration measurement is usually substituted with a differential approximation using measured data of x and θ.

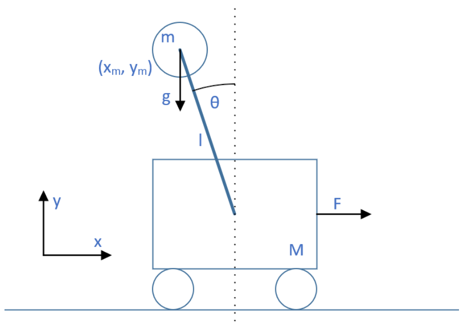

Figure 1: Inverted pendulum scheme

The problem of keeping the pendulum in an upright position has been solved using an LQG control. Details of this design are not concern of this article. As seen in the figure 5, the LQG solution successfully stabilizes the pendulum in the upright position. The cart position x control quality is poor, so it needs to be improved.

2 USING $\mathcal{H}_\infty$ OPTIMIZATION FOR CONTROL QUALITY IMPROVEMENT

Since the LQG control is a special case of $\mathcal{H}_2$ controller synthesis problem [1], the LQG controller tuning using $\mathcal{H}_2$ norm optimization seems to be an obvious solution, but similarity of the LQG and $\mathcal{H}_2$ synthesis is shown to be rather obstruction. Using $\mathcal{H}_2$ synthesis, even using a dynamic weights, weight matrix that is not constant over all frequencies, does not offer any direct advantage over the classical LQG design using weight matrices **W**, **V**,**R** and **Q**.

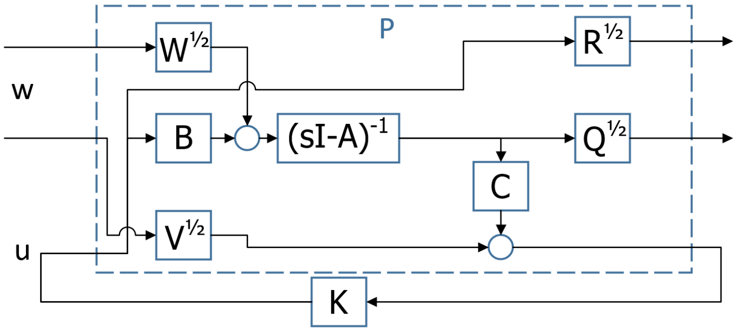

Figure 2: LQG problem defined as $\mathcal{H}_2$ optimization problem [1]

$\mathcal{H}_\infty$ synthesis on the other hand, optimizes peaks of optimized frequency characteristics, and so definition of the desired control characteristics using dynamic weights is simpler. To adjust an existing controller, or observer, the LQG problem first needs to be reformulated to the standard $\mathcal{H}_\infty$ state controller synthesis problem. As a next step, weights that will differ most at the adjustment target frequencies from the original design, to achieve the desired adjusted frequency characteristics, has to be designed. Example of this weights design is to be seen in next chapter.[1], [2]

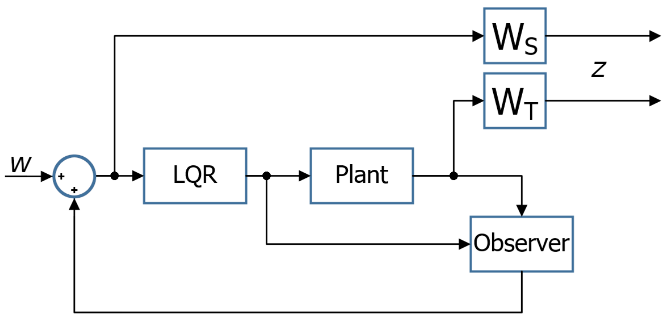

Figure 3: LQG problem reformulated as $\mathcal{H}_\infty$ state controller synthesis problem with sensitivity and complementary sensitivity weights, $\mathbf{W}_S$ and $\mathbf{W}_T$ respectively.

3 EXAMPLE OF USING $\mathcal{H}_\infty$ SYNTHESIS FOR LQG CONTROLLER ADJUSTMENT

In the figure 4, there are frequency characteristics of the original and the adjusted LQG controller. It is obvious, that singular values of the original position sensitivity function $\sigma(S_x)$ is inappropriately shaped, because the shape should be similar to low pass filter, and therefore control error should consist of noise only.

At first, the weights were selected such that they differ as little as possible from original singular characteristics, despite being first order dynamic weights. Target of this is to design as similar to the original controller as possible. Then, weight W_{S_x} has been adjusted so that a low frequency gain of $\sigma(S_x)$ was suppressed. That should lead to suppressing the control error.

Synthesis has been done in MATLAB, utilizing command `hinfstruct` from robust control toolbox, that allows $\mathcal{H}_\infty$ synthesis of controllers with predefined structure. Since $\mathcal{H}_\infty$ optimization of such systems is not generally convex problem, the optimization ran repeatedly with random initial conditions to avoid local minimum solution. Controller and observer matrices **L** and **K** has been resynthetized using $\mathcal{H}_\infty$ optimization.[3], [4]

Low frequency gain of $\sigma(S_x)$ is indeed suppressed in $\mathcal{H}_\infty$ solution, which caused shifting of all other singular characteristics to higher frequencies. Also, spike has been created in $\sigma(S_\theta)$ and $\sigma(S_{\dot\theta})$. This could lead to increased noise sensitivity of design. Results can be compared in the figure 4.

Note, that the weights does not bound state observer, nor controller output, which means, that singular characteristics of these signals could be affected by $\mathcal{H}_\infty$ optimization.

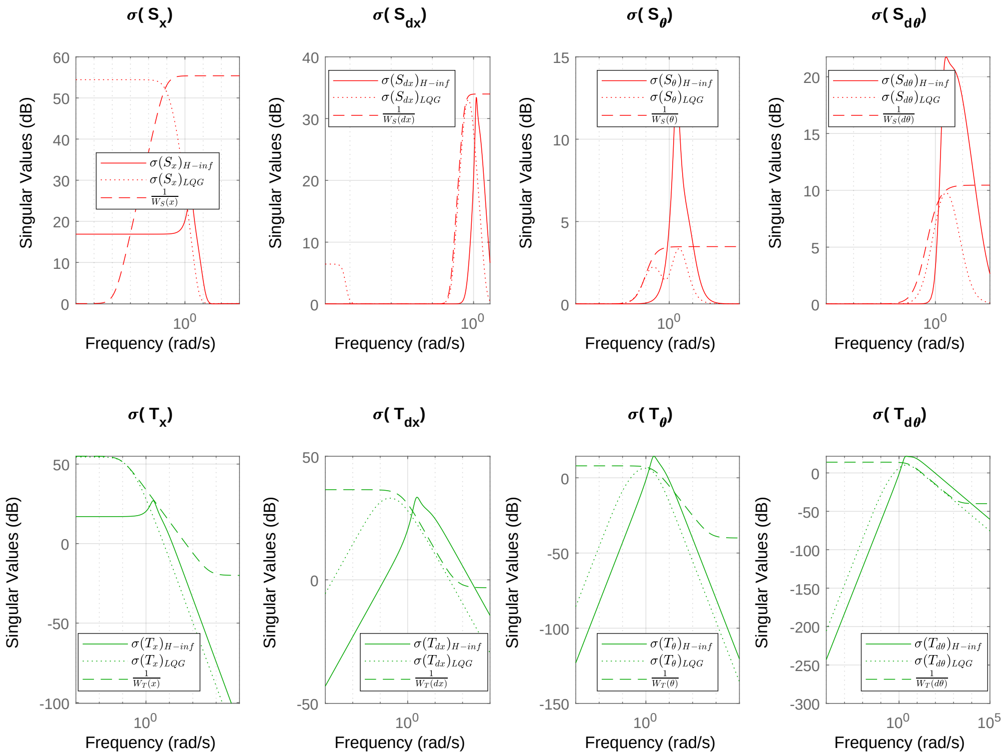

Figure 4: Singular characteristics of individual signals of sensitivity and complementary sensitivity functions compared to weights $\mathbf{W}_S$ and $\mathbf{W}_T$.

4 RESULTS OF USING $\mathcal{H}_\infty$ SYNTHESIS FOR LQG CONTROLLER ADJUSTMENT

The method is demonstrated on the inverted pendulum on a cart example. Results can be observed in the figure 5, that displays a simulation of original and adjusted controllers. The simulation consists of two parts. First part shows simulation of state controller part only, assuming that all states are measured. In this period, the state observers are stabilized. For second part of the simulation, only states x and θ are measured and the system is controlled by using the LQG controller.

The cart position control quality has been increased compared to the original controller, as visible from figure 5. Since the controller output has not been bounded, it is generally bigger in the adjusted control. Also, the noise sensitivity of the adjusted LQG is visible, which was expected.

Results show, that by using this method the user is capable of adjusting an existing LQG controller.

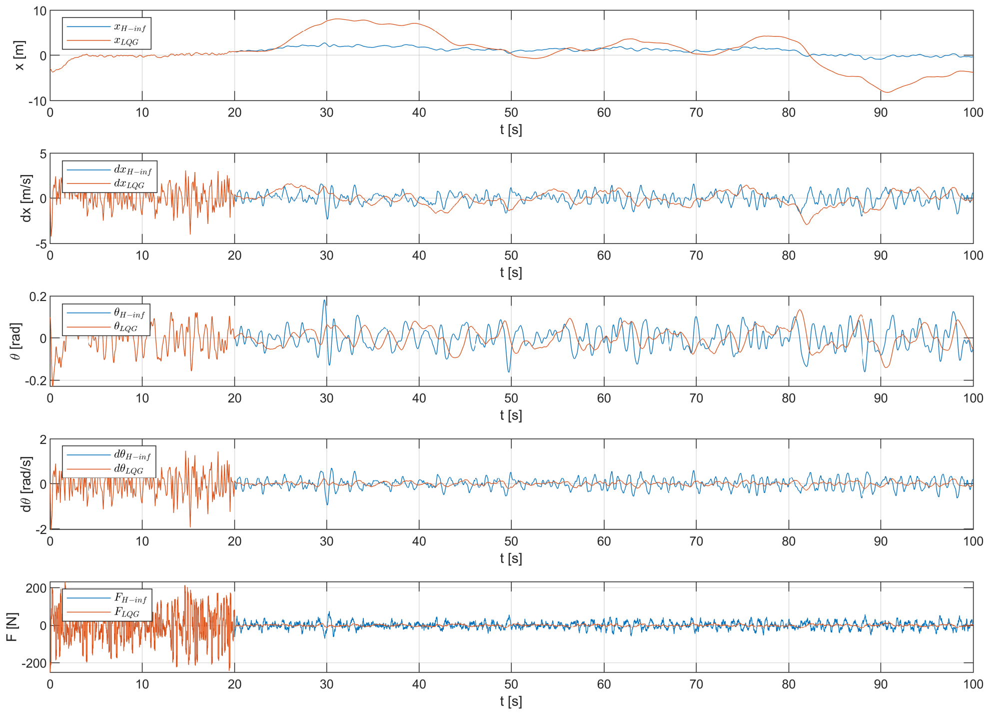

Figure 5: The original and $\mathcal{H}_\infty$ adjusted LQG control comparison. For all the state variables, setpoints are set to zero.

5 CONCLUSION

The main target of this article is to demonstrate the possibility of using $\mathcal{H}_\infty$ optimization to adjust an existing LQG controller. This method is now simple to implement, thanks to `hinfstruct` [4] command in robust control toolbox in MATLAB. The method consists of a few steps. The first step is reformulation to the standard $\mathcal{H}_\infty$ state controller synthesis problem. The optimization weights needs to be designed, so that spikes appear where the singular characteristics needs to be attenuated and valley appears where it needs to be amplified. In order to attenuate all spikes, $\mathcal{H}_\infty$ optimization leads to the desired adjustment. This method has been successfully demonstrated on the inverted pendulum on a cart example. Using this method, cart position control quality has been increased compared to the original controller.

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