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Discrete matrix delayed exponential for two delays and its property

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Abstract

In recent papers, a discrete matrix delayed exponential for a single delay was defined and its main property connected with the solution of linear discrete systems with a single delay was proved. In the present paper, a generalization of the concept of discrete matrix delayed exponential is designed for two delays and its main property is proved as well.

Introduction

Throughout the paper, we use the following notation. For integers s, t, $s \leq t$, we define a set $\mathbb{Z}_s^t := \{s, s+1, \dots, t-1, t\}$. Similarly, we define sets $\mathbb{Z}_{-\infty}^t := \{\dots, t-1, t\}$ and $\mathbb{Z}_s^\infty := \{s, s+1, \dots\}$. The function $\lfloor \cdot \rfloor$ used below is the floor integer function. We employ the following property of the floor integer function:

$$x - 1 < \lfloor x \rfloor \leq x, \tag{1}$$

where $x \in \mathbb{R}$.

Define binomial coefficients as customary, *i.e.*, for $n \in \mathbb{Z}$ and $k \in \mathbb{Z}$,

$$\binom{n}{k} := \begin{cases} \frac{n!}{k!(n-k)!} & \text{if } n \geq k \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

We recall that for a well-defined discrete function $f(k)$, the forward difference operator Δ is defined as $\Delta f(k) = f(k+1) - f(k)$. In the paper, we also adopt the customary notation $\sum_{i=i_1}^{i_2} g_i = 0$ if $i_2 < i_1$. In the case of double sums, we set

$$\sum_{i=i_1, j=j_1}^{i_2, j_2} g_{ij} = 0 \tag{3}$$

if at least one of the inequalities $i_2 < i_1$, $j_2 < j_1$ holds.

In [1, 2], a discrete matrix delayed exponential for a single delay $m \in \mathbb{N}$ was defined as follows.

Definition 1 For an $r \times r$ constant matrix B, $k \in \mathbb{Z}$, and fixed $m \in \mathbb{N}$, we define the discrete matrix delayed exponential e_m^{Bk} as follows:

$$\mathrm{e}_m^{Bk} := \begin{cases} \Theta & \text{if } k \in \mathbb{Z}_{-\infty}^{-m-1}, \\ I + \sum_{j=1}^{\ell} B^j \cdot \binom{k-m(j-1)}{j} & \text{if } \ell = 0,1,2,\dots, k \in \mathbb{Z}_{(\ell-1)(m+1)+1}^{\ell(m+1)}, \end{cases}$$

where Θ is an $r \times r$ null matrix and I is an $r \times r$ unit matrix.

Next, the main property (Theorem 1 below) of discrete matrix delayed exponential for a single delay $m \in \mathbb{N}$ is proved in [1].

Theorem 1 *Let B be a constant $r \times r$ matrix. Then, for $k \in \mathbb{Z}_{-m}^{\infty}$,*

$$\Delta \mathrm{e}_m^{Bk} = B\mathrm{e}_m^{B(k-m)}. \tag{4}$$

The paper is concerned with a generalization of the notion of discrete matrix delayed exponential for two delays and a proof of one of its properties, similar to the main property (4) of discrete matrix delayed exponential for a single delay.

Discrete matrix delayed exponential for two delays and its main property

We define a discrete $r \times r$ matrix function e_{mn}^{BCk} called the discrete matrix delayed exponential for two delays $m, n \in \mathbb{N}$, $m \neq n$ and for two $r \times r$ commuting constant matrices B, C as follows.

Definition 2 Let B, C be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$ be fixed integers. We define a discrete $r \times r$ matrix function e_{mn}^{BCk} called the discrete matrix delayed exponential for two delays m, n and for two $r \times r$ constant matrices B, C:

$$\mathrm{e}_{mn}^{BCk} := \begin{cases} \Theta & \text{if } k \in \mathbb{Z}_{-\infty}^{-\max\{m,n\}-1}, \\ I & \text{if } k \in \mathbb{Z}_{-\max\{m,n\}}^{0}, \\ I + (B+C)\sum_{i=0,j=0}^{p(k)-1,q(k)-1} B^i C^j \binom{i+j}{i}\binom{k-mi-nj}{i+j+1} & \text{if } k \in \mathbb{Z}_1^{\infty}, \end{cases}$$

where

$$p(k) := \left\lfloor \frac{k+m}{m+1} \right\rfloor, \qquad q(k) := \left\lfloor \frac{k+n}{n+1} \right\rfloor. \tag{5}$$

The main property of e_{mn}^{BCk} is given by the following theorem.

Theorem 2 *Let B, C be constant $r \times r$ matrices with the property $BC = CB$ and let $m, n \in \mathbb{N}$, $m \neq n$ be fixed integers. Then*

$$\Delta \mathrm{e}_{mn}^{BCk} = B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \tag{6}$$

holds for $k \geq 0$.

Proof Let $k \geq 1$. From (1) and (5), we can see easily that, for an integer $k \geq 0$ satisfying

$$(p_{(k)} - 1)(m + 1) + 1 \leq k \leq p_{(k)}(m + 1) \wedge (q_{(k)} - 1)(n + 1) + 1 \leq k \leq q_{(k)}(n + 1),$$

the relation

$$\Delta \mathrm{e}_{mn}^{BCk} = \Delta \left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - nj}{i + j + 1} \right]$$

holds in accordance with Definition 2 of e_{mn}^{BCk}. Since $\Delta I = \Theta$, we have

$$\Delta \mathrm{e}_{mn}^{BCk} = \Delta \left[(B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - nj}{i + j + 1} \right]. \tag{7}$$

Considering the increment by its definition, *i.e.*,

$$\Delta \mathrm{e}_{mn}^{BCk} = \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk}, \tag{8}$$

we conclude that it is reasonable to divide the proof into four parts with respect to the value of integer k. In case one, k is such that

$$(p_{(k)} - 1)(m + 1) + 1 \leq k < p_{(k)}(m + 1) \wedge (q_{(k)} - 1)(n + 1) + 1 \leq k < q_{(k)}(n + 1),$$

in case two

$$k = p_{(k)}(m + 1) \wedge (q_{(k)} - 1)(n + 1) + 1 \leq k < q_{(k)}(n + 1),$$

in case three

$$(p_{(k)} - 1)(m + 1) + 1 \leq k < p_{(k)}(m + 1) \wedge k = q_{(k)}(n + 1)$$

and in case four

$$k = p_{(k)}(m + 1) \wedge k = q_{(k)}(n + 1).$$

We see that the above cases cover all the possible relations between k, $p_{(k)}$ and $q_{(k)}$.

In the proof, we use the identities

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}, \tag{9}$$

where $n, k \in \mathbb{N}$ and

$$\binom{i}{i} = \binom{i-1}{i-1}, \qquad \binom{j}{0} = \binom{j-1}{0}, \qquad \binom{i+j}{i} = \binom{i+j-1}{i-1} + \binom{i+j-1}{i}, \tag{10}$$

where $i, j \in \mathbb{N}$, which are derived from (2) and (9).

I. $(p_{(k)}-1)(m+1)+1 \le k < p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1)$

From (1) and (5), we get

$$p_{(k-m)} = \left\lfloor \frac{k-m+m}{m+1} \right\rfloor \le \frac{k}{m+1} < p_{(k)},$$
$$p_{(k-m)} = \left\lfloor \frac{k-m+m}{m+1} \right\rfloor > \frac{k}{m+1} - 1 = \frac{k-m-1}{m+1} > p_{(k)} - 2.$$

Therefore, $p_{(k-m)} = p_{(k)} - 1$ and, by Definition 2,

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i} \binom{k-m-mi-nj}{i+j+1}. \tag{11}$$

Similarly, omitting details, we get (using (1), and (5)) $q_{(k-n)} = q_{(k)} - 1$ and

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-n-mi-nj}{i+j+1}. \tag{12}$$

Let $q_{(k-m)} \ge 1$. We show that

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if } i \ge 0, j \ge q_{(k-m)}. \tag{13}$$

In accordance with (1),

$$q_{(k-m)} = \left\lfloor \frac{k-m+n}{n+1} \right\rfloor > \frac{k-m+n}{n+1} - 1 = \frac{k-m-1}{n+1}$$

or

$$k - m < (n+1)q_{(k-m)} + 1 \le (m+1)i + (n+1)j + 1 \quad \text{if } i \ge 0, j \ge q_{(k-m)}.$$

From the last inequality, we get

$$k - m - mi - nj < i + j + 1 \quad \text{if } i \ge 0, j \ge q_{(k-m)}$$

and (13) holds by (2). For that reason and since $q_{(k-m)} \le q_{(k)}$, we can replace $q_{(k-m)}$ by $q_{(k)}$ in (11). Thus, we have

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1}. \tag{14}$$

It is easy to see that, due to (3), formula (14) can be used instead of (11) if $q_{(k-m)} < 1$ also.

Let $p_{(k-n)} \ge 1$. Similarly, we can show that

$$\binom{k-n-mi-nj}{i+j+1} = 0 \quad \text{if } i \ge p_{(k-n)}, j \ge 0$$

and, since $p_{(k-n)} \le p_{(k)}$, we can replace $p_{(k-n)}$ by $p_{(k)}$ in (12). Thus, we have

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}. \tag{15}$$

It is easy to see that, due to (3), formula (15) can be used instead of (12) if $p_{(k-n)} < 1$, too.

Due to (1), we also conclude that

$$p_{(k+1)} = p_{(k)}, \qquad q_{(k+1)} = q_{(k)} \tag{16}$$

because

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \le \frac{k}{m+1} + 1 < p_{(k)} + 1$$

and

$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 = \frac{k}{m+1} \ge p_{(k)} - 1 + \frac{1}{m+1}.$$

The second formula can be proved similarly.

Now we are able to prove that

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \\
&= B\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1} \right] \\
&\quad + C\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1} \right].
\end{aligned} \tag{17}$$

With the aid of (7), (8), (9) and (16), we get

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk} \\
&= I + (B+C) \sum_{i=0,j=0}^{p_{(k+1)}-1,q_{(k+1)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\
&\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\
&= I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\
&\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\
&= (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-mi-nj}{i+j+1} - \binom{k-mi-nj}{i+j+1} \right]
\end{aligned}$$

$$= (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j}$$

$$= (B+C)\Bigg[I + \sum_{i=1}^{p_{(k)}-1} B^iC^0\binom{i}{i}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0C^j\binom{j}{0}\binom{k-nj}{j}$$

$$+ \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j}\Bigg].$$

By (10), we have

$$\Delta \mathrm{e}_{mn}^{BCk} = (B+C)\Bigg[I + \sum_{i=1}^{p_{(k)}-1} B^iC^0\binom{i-1}{i-1}\binom{k-mi}{i}$$

$$+ \sum_{j=1}^{q_{(k)}-1} B^0C^j\binom{j-1}{0}\binom{k-nj}{j}$$

$$+ \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}$$

$$+ \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-mi-nj}{i+j}\Bigg]$$

$$= (B+C)\Bigg[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}$$

$$+ \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j-1}{i}\binom{k-mi-nj}{i+j}\Bigg].$$

Now in the first sum we replace the summation index i by $i+1$ and in the second sum we replace the summation index j by $j+1$. Then

$$\Delta \mathrm{e}_{mn}^{BCk} = (B+C)\Bigg[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1}C^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}$$

$$+ \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^{j+1}\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\Bigg]$$

$$= B + B(B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}$$

$$+ C + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}$$

$$= B\Bigg[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\Bigg]$$

$$+ C\left[I + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right]$$
$$= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)}.$$

Due to (14) and (15), we conclude that formula (17) is valid.

II. $k = p_{(k)}(m+1) \wedge (q_{(k)}-1)(n+1)+1 \le k < q_{(k)}(n+1)$

In this case,

$$p_{(k-m)} = \left\lfloor \frac{k-m+m}{m+1} \right\rfloor = \left\lfloor \frac{k}{m+1} \right\rfloor = p_{(k)},$$
$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor \le \frac{k+1+m}{m+1} = \frac{k}{m+1} + 1 = p_{(k)} + 1,$$
$$p_{(k+1)} = \left\lfloor \frac{k+1+m}{m+1} \right\rfloor > \frac{k+1+m}{m+1} - 1 = \frac{k}{m+1} = p_{(k)}$$

and $p_{(k+1)} = p_{(k)} + 1$. In addition to this (see relevant computations performed in case I), we have $q_{(k-n)} = q_{(k)} - 1$ and $q_{(k+1)} = q_{(k)}$.

Then

$$\mathrm{e}_{mn}^{BCk} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j+1},$$
$$\mathrm{e}_{mn}^{BC(k+1)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k+1-mi-nj}{i+j+1}$$

and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-mi-nj}{i+j+1}, \tag{18}$$
$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-2} B^iC^j\binom{i+j}{i}\binom{k-n-mi-nj}{i+j+1}. \tag{19}$$

Like with the computations performed in the previous part of the proof, we get

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if } i \ge 0, j \ge q_{(k-m)}$$

and

$$\binom{k-n-mi-nj}{i+j+1} = 0 \quad \text{if } i \ge p_{(k-n)}, j \ge 0.$$

So, we can substitute $q_{(k-m)}$ by $q_{(k)}$ in (18) and $p_{(k-n)}$ by $p_{(k)}$ in (19).

Accordingly, we have

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}, \tag{20}$$

$$e_{mn}^{BC(k-n)} = I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1}. \tag{21}$$

It is easy to see that, due to (3), formula (20) can also be used instead of (18) if $q_{(k-m)} < 1$ and formula (21) can also be used instead of (19) if $p_{(k-n)} < 1$.

We have to prove

$$\begin{aligned}
\Delta e_{mn}^{BCk} &= B e_{mn}^{BC(k-m)} + C e_{mn}^{BC(k-n)} \\
&= B\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1} \right] \\
&\quad + C\left[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1} \right].
\end{aligned} \tag{22}$$

Therefore,

$$\begin{aligned}
\Delta e_{mn}^{BCk} &= e_{mn}^{BC(k+1)} - e_{mn}^{BCk} \\
&= I + (B+C) \sum_{i=0,j=0}^{p_{(k)},q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\
&\quad - I - (B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\
&= (B+C)\left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-mi-nj}{i+j+1} - \binom{k-mi-nj}{i+j+1} \right] \right. \\
&\quad \left. + \sum_{j=0}^{q_{(k)}-1} B^{p_{(k)}} C^j \binom{p_{(k)}+j}{p_{(k)}} \binom{k+1-mp_{(k)}-nj}{p_{(k)}+j+1} \right].
\end{aligned}$$

With the aid of the equation $k = p_{(k)}(m+1)$, we get

$$\binom{k+1-mp_{(k)}-nj}{p_{(k)}+j+1} = \binom{p_{(k)}+1-nj}{p_{(k)}+1+j} = 0 \quad \text{if } j > 0$$

and, by (9), we have

$$\begin{aligned}
\Delta e_{mn}^{BCk} &= (B+C)\left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j} + B^{p_{(k)}} \right] \\
&= (B+C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i} \binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0} \binom{k-nj}{j} \right. \\
&\quad \left. + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j} + B^{p_{(k)}} \right].
\end{aligned}$$

By (10), we have

$$\begin{aligned}
\Delta e_{mn}^{BCk} &= (B+C)\Bigg[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1}\binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0}\binom{k-nj}{j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}}\Bigg] \\
&= (B+C)\Bigg[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j} \\
&\quad + \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p_{(k)}}\Bigg].
\end{aligned}$$

Now we replace in the first sum the summation index i by $i+1$ and in the second sum we replace the summation index j by $j+1$. Then

$$\begin{aligned}
\Delta e_{mn}^{BCk} &= (B+C)\Bigg[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1} C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} \\
&\quad + \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^{j+1} \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} + B^{p_{(k)}}\Bigg] \\
&= B + B(B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} + B^{p_{(k)}}(B+C) \\
&\quad + C + C(B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} \\
&= B\Bigg[I + (B+C)\Bigg(\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} + B^{p_{(k)}-1}\Bigg)\Bigg] \\
&\quad + C\Bigg[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\Bigg].
\end{aligned}$$

For $k = p_{(k)}(m+1)$, we have

$$B^{p_{(k)}-1} = \sum_{j=0}^{q_{(k)}-1} B^{p_{(k)}-1} C^j \binom{p_{(k)}-1+j}{p_{(k)}-1}\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1},$$

where

$$\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1} = \binom{k-mp_{(k)}-nj}{p_{(k)}+j} = 0 \quad \text{if } j > 0.$$

Thus,

$$\begin{aligned}\Delta \mathrm{e}_{mn}^{BCk} &= B\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right] \\ &\quad + C\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right] \\ &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)}\end{aligned}$$

and formula (22) is proved.

III. $(p_{(k)} - 1)(m + 1) + 1 \le k < p_{(k)}(m + 1) \wedge k = q_{(k)}(n + 1)$

In this case, we have (see relevant computations in cases I and II)

$$p_{(k-m)} = p_{(k)} - 1, \qquad p_{(k+1)} = p_{(k)}$$

and

$$q_{(k-n)} = q_{(k)}, \qquad q_{(k+1)} = q_{(k)} + 1.$$

Then

$$\begin{aligned}\mathrm{e}_{mn}^{BCk} &= I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-mi-nj}{i+j+1}, \\ \mathrm{e}_{mn}^{BC(k+1)} &= I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^i C^j \binom{i+j}{i}\binom{k+1-mi-nj}{i+j+1}\end{aligned}$$

and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k-m)}-1} B^i C^j \binom{i+j}{i}\binom{k-m-mi-nj}{i+j+1}, \tag{23}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i}\binom{k-n-mi-nj}{i+j+1}. \tag{24}$$

Like with the computations performed in case I, we can get

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if } i \ge 0, j \ge q_{(k-m)}$$

and

$$\binom{k-n-mi-nj}{i+j+1} = 0 \quad \text{if } i \ge p_{(k-n)}, j \ge 0.$$

So, we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (23) and $p_{(k)}$ for $p_{(k-n)}$ in (24).

Thus, we have

$$e_{mn}^{BC(k-m)} = I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - m(i+1) - nj}{i+j+1}, \tag{25}$$

$$e_{mn}^{BC(k-n)} = I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - n(j+1)}{i+j+1}. \tag{26}$$

It is easy to see that, due to (3), formula (25) can also be used instead of (23) if $q_{(k-m)} < 1$ and formula (26) can also be used instead of (24) if $p_{(k-n)} < 1$.

Now we have to prove

$$\begin{aligned}\Delta e_{mn}^{BCk} &= B e_{mn}^{BC(k-m)} + C e_{mn}^{BC(k-n)} \\ &= B\left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - m(i+1) - nj}{i+j+1}\right] \\ &\quad + C\left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - n(j+1)}{i+j+1}\right].\end{aligned} \tag{27}$$

Considering the difference by its definition, we get

$$\begin{aligned}\Delta e_{mn}^{BCk} &= e_{mn}^{BC(k+1)} - e_{mn}^{BCk} \\ &= I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1-mi-nj}{i+j+1} \\ &\quad - I - (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j+1} \\ &= (B + C)\left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1-mi-nj}{i+j+1} - \binom{k-mi-nj}{i+j+1}\right]\right. \\ &\quad \left. + \sum_{i=0}^{p_{(k)}-1} B^i C^{q_{(k)}} \binom{i+q_{(k)}}{i} \binom{k+1-mi-nq_{(k)}}{i+q_{(k)}+1}\right].\end{aligned}$$

With the aid of relation $k = q_{(k)}(n+1)$, we get

$$\binom{k+1-mi-nq_{(k)}}{i+q_{(k)}+1} = \binom{q_{(k)}+1-mi}{q_{(k)}+1+i} = 0 \quad \text{if } i > 0$$

and

$$\begin{aligned}\Delta e_{mn}^{BCk} &= (B + C)\left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j} + C^{q_{(k)}}\right] \\ &= (B + C)\left[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i}{i} \binom{k-mi}{i}\right.\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j}{0} \binom{k-nj}{j} \\
&+ \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-mi-nj}{i+j} + C^{q_{(k)}} \Bigg].
\end{aligned}
$$

By (10), we have

$$
\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\Bigg[I + \sum_{i=1}^{p_{(k)}-1} B^i C^0 \binom{i-1}{i-1} \binom{k-mi}{i} + \sum_{j=1}^{q_{(k)}-1} B^0 C^j \binom{j-1}{0} \binom{k-nj}{j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-mi-nj}{i+j} \\
&\quad + \sum_{i=1,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i} \binom{k-mi-nj}{i+j} + C^{q_{(k)}} \Bigg] \\
&= (B+C)\Bigg[I + \sum_{i=1,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i-1} \binom{k-mi-nj}{i+j} \\
&\quad + \sum_{i=0,j=1}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j-1}{i} \binom{k-mi-nj}{i+j} + C^{q_{(k)}} \Bigg].
\end{aligned}
$$

Now we replace in the first sum the summation index i by $i+1$ and in the second sum we replace the summation index j by $j+1$. Then

$$
\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\Bigg[I + \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^{i+1} C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1} \\
&\quad + \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^{j+1} \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1} + C^{q_{(k)}} \Bigg] \\
&= B + B(B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1} \\
&\quad + C + C(B+C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1} + C^{q_{(k)}}(B+C) \\
&= B\Bigg[I + (B+C) \sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k-m(i+1)-nj}{i+j+1} \Bigg] \\
&\quad + C\Bigg[I + (B+C)\Bigg(\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-2} B^i C^j \binom{i+j}{i} \binom{k-mi-n(j+1)}{i+j+1} + C^{q_{(k)}-1} \Bigg) \Bigg].
\end{aligned}
$$

For $k = q_{(k)}(n+1)$, we have

$$
C^{q_{(k)}-1} = \sum_{i=0}^{p_{(k)}-1} B^i C^{q_{(k)}-1} \binom{i+q_{(k)}-1}{i} \binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1},
$$

where

$$\binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1} = \binom{k-mi-nq_{(k)}}{i+q_{(k)}} = 0 \quad \text{if } i > 0.$$

Thus,

$$\begin{aligned}\Delta \mathrm{e}_{mn}^{BCk} &= B\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-2,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right] \\ &\quad + C\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right] \\ &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)}\end{aligned}$$

and formula (27) is proved.

IV. $k = p_{(k)}(m+1) \wedge k = q_{(k)}(n+1)$

In this case, we have (see similar combinations in cases II and III)

$$p_{(k-m)} = p_{(k)}, \qquad p_{(k+1)} = p_{(k)} + 1$$

and

$$q_{(k-n)} = q_{(k)}, \qquad q_{(k+1)} = q_{(k)} + 1.$$

Then

$$\begin{aligned}\mathrm{e}_{mn}^{BCk} &= I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-nj}{i+j+1}, \\ \mathrm{e}_{mn}^{BC(k+1)} &= I + (B+C)\sum_{i=0,j=0}^{p_{(k)},q_{(k)}} B^iC^j\binom{i+j}{i}\binom{k+1-mi-nj}{i+j+1}\end{aligned}$$

and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k-m)}-1} B^iC^j\binom{i+j}{i}\binom{k-m-mi-nj}{i+j+1}, \tag{28}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B+C)\sum_{i=0,j=0}^{p_{(k-n)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-n-mi-nj}{i+j+1}. \tag{29}$$

As before,

$$\binom{k-m-mi-nj}{i+j+1} = 0 \quad \text{if } i \geq 0, j \geq q_{(k-m)}$$

and

$$\binom{k-n-mi-nj}{i+j+1} = 0 \quad \text{if } i \geq p_{(k-n)}, j \geq 0.$$

So, we can substitute $q_{(k)}$ for $q_{(k-m)}$ in (28) and $p_{(k)}$ for $p_{(k-n)}$ in (29) and

$$\mathrm{e}_{mn}^{BC(k-m)} = I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - m(i+1) - nj}{i+j+1}, \tag{30}$$

$$\mathrm{e}_{mn}^{BC(k-n)} = I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - n(j+1)}{i+j+1}. \tag{31}$$

It is easy to see that, due to (3), formula (30) can also be used instead of (28) if $q_{(k-m)} < 1$ and formula (31) can also be used instead of (29) if $p_{(k-n)} < 1$.

Now it is possible to prove the formula

$$\begin{aligned} \Delta \mathrm{e}_{mn}^{BCk} &= B\mathrm{e}_{mn}^{BC(k-m)} + C\mathrm{e}_{mn}^{BC(k-n)} \\ &= B\left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - m(i+1) - nj}{i+j+1} \right] \\ &\quad + C\left[I + (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - n(j+1)}{i+j+1} \right]. \end{aligned} \tag{32}$$

By definition, we get

$$\begin{aligned} \Delta \mathrm{e}_{mn}^{BCk} &= \mathrm{e}_{mn}^{BC(k+1)} - \mathrm{e}_{mn}^{BCk} \\ &= I + (B + C) \sum_{i=0,j=0}^{p_{(k)},q_{(k)}} B^i C^j \binom{i+j}{i} \binom{k+1 - mi - nj}{i+j+1} \\ &\quad - I - (B + C) \sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \binom{k - mi - nj}{i+j+1} \\ &= (B + C) \left[\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^i C^j \binom{i+j}{i} \left[\binom{k+1 - mi - nj}{i+j+1} - \binom{k - mi - nj}{i+j+1} \right] \right. \\ &\quad + \sum_{j=0}^{q_{(k)}} B^{p_{(k)}} C^j \binom{p_{(k)} + j}{p_{(k)}} \binom{k+1 - mp_{(k)} - nj}{p_{(k)} + j + 1} \\ &\quad \left. + \sum_{i=0}^{p_{(k)}} B^i C^{q_{(k)}} \binom{i + q_{(k)}}{i} \binom{k+1 - mi - nq_{(k)}}{i + q_{(k)} + 1} \right]. \end{aligned}$$

With the aid of equations $k = p_{(k)}(m+1)$, $k = q_{(k)}(n+1)$, we get

$$\begin{aligned} \binom{k+1 - mp_{(k)} - nj}{p_{(k)} + j + 1} &= \binom{p_{(k)} + 1 - nj}{p_{(k)} + 1 + j} = 0 \quad \text{if } j > 0, \\ \binom{k+1 - mi - nq_{(k)}}{i + q_{(k)} + 1} &= \binom{q_{(k)} + 1 - mi}{q_{(k)} + 1 + i} = 0 \quad \text{if } i > 0 \end{aligned}$$

and

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[\sum_{i=0,j=0}^{p(k)-1,q(k)-1} B^i C^j \binom{i+j}{i}\binom{k-mi-nj}{i+j} + B^{p(k)} + C^{q(k)}\right] \\
&= (B+C)\left[I + \sum_{i=1}^{p(k)-1} B^i C^0 \binom{i}{i}\binom{k-mi}{i} + \sum_{j=1}^{q(k)-1} B^0 C^j \binom{j}{0}\binom{k-nj}{j}\right. \\
&\quad \left. + \sum_{i=1,j=1}^{p(k)-1,q(k)-1} B^i C^j \binom{i+j}{i}\binom{k-mi-nj}{i+j} + B^{p(k)} + C^{q(k)}\right].
\end{aligned}$$

By (10), we have

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[I + \sum_{i=1}^{p(k)-1} B^i C^0 \binom{i-1}{i-1}\binom{k-mi}{i} + \sum_{j=1}^{q(k)-1} B^0 C^j \binom{j-1}{0}\binom{k-nj}{j}\right. \\
&\quad + \sum_{i=1,j=1}^{p(k)-1,q(k)-1} B^i C^j \binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j} \\
&\quad \left. + \sum_{i=1,j=1}^{p(k)-1,q(k)-1} B^i C^j \binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p(k)} + C^{q(k)}\right] \\
&= (B+C)\left[I + \sum_{i=1,j=0}^{p(k)-1,q(k)-1} B^i C^j \binom{i+j-1}{i-1}\binom{k-mi-nj}{i+j}\right. \\
&\quad \left. + \sum_{i=0,j=1}^{p(k)-1,q(k)-1} B^i C^j \binom{i+j-1}{i}\binom{k-mi-nj}{i+j} + B^{p(k)} + C^{q(k)}\right].
\end{aligned}$$

We replace in the first sum the summation index i by $i+1$ and in the second sum we substitute the summation index j by $j+1$. Then

$$\begin{aligned}
\Delta \mathrm{e}_{mn}^{BCk} &= (B+C)\left[I + \sum_{i=0,j=0}^{p(k)-2,q(k)-1} B^{i+1} C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right. \\
&\quad \left. + \sum_{i=0,j=0}^{p(k)-1,q(k)-2} B^i C^{j+1} \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} + B^{p(k)} + C^{q(k)}\right] \\
&= B + B(B+C)\sum_{i=0,j=0}^{p(k)-2,q(k)-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} + B^{p(k)}(B+C) \\
&\quad + C + C(B+C)\sum_{i=0,j=0}^{p(k)-1,q(k)-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} + C^{q(k)}(B+C) \\
&= B\left[I + (B+C)\left(\sum_{i=0,j=0}^{p(k)-2,q(k)-1} B^i C^j \binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1} + B^{p(k)-1}\right)\right] \\
&\quad + C\left[I + (B+C)\left(\sum_{i=0,j=0}^{p(k)-1,q(k)-2} B^i C^j \binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1} + C^{q(k)-1}\right)\right].
\end{aligned}$$

Because $k = p_{(k)}(m+1) = q_{(k)}(n+1)$, we can express $B^{p_{(k)}-1}$ and $C^{q_{(k)}-1}$ in the form

$$B^{p_{(k)}-1} = \sum_{j=0}^{q_{(k)}-1} B^{p_{(k)}-1}C^j \binom{p_{(k)}-1+j}{p_{(k)}-1}\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1},$$
$$C^{q_{(k)}-1} = \sum_{i=0}^{p_{(k)}-1} B^i C^{q_{(k)}-1} \binom{i+q_{(k)}-1}{i}\binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1},$$

where

$$\binom{k-m(p_{(k)}-1+1)-nj}{p_{(k)}-1+j+1} = \binom{k-mp_{(k)}-nj}{p_{(k)}+j} = 0 \quad \text{if } j>0,$$
$$\binom{k-mi-n(q_{(k)}-1+1)}{i+q_{(k)}-1+1} = \binom{k-mi-nq_{(k)}}{i+q_{(k)}} = 0 \quad \text{if } i>0.$$

Thus,

$$\begin{aligned}\Delta e_{mn}^{BCk} &= B\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-m(i+1)-nj}{i+j+1}\right]\\ &\quad + C\left[I + (B+C)\sum_{i=0,j=0}^{p_{(k)}-1,q_{(k)}-1} B^iC^j\binom{i+j}{i}\binom{k-mi-n(j+1)}{i+j+1}\right]\\ &= Be_{mn}^{BC(k-m)} + Ce_{mn}^{BC(k-n)}.\end{aligned}$$

Therefore, formula (32) is valid.

We proved that formula (6) holds in each of the considered cases I, II, III and IV for $k \geq 1$. If $k = 0$, the proof can be done directly because $p_{(0)} = q_{(0)} = 0$, $p_{(1)} = q_{(1)} = 1$,

$$\begin{aligned}\Delta e_{mn}^{BC0} &= e_{mn}^{BC1} - e_{mn}^{BC0}\\ &= I + (B+C)\sum_{i=0,j=0}^{0,0} B^iC^j\binom{i+j}{i}\binom{1-mi-nj}{i+j+1}\\ &\quad - I - (B+C)\sum_{i=0,j=0}^{-1,-1} B^iC^j\binom{i+j}{i}\binom{-mi-nj}{i+j+1} = I + B + C - I = B + C\end{aligned}$$

and

$$Be_{mn}^{BC(-m)} + Ce_{mn}^{BC(-n)} = BI + CI = B + C.$$

Formula (6) holds again. Theorem 2 is proved. □

Open problems and concluding remarks

Formula (4) is valid for $k \in \mathbb{Z}_{-m}^{\infty}$. However, formula (6) holds for $k \in \mathbb{Z}_0^{\infty}$ only. Therefore, there is a difference between the definition domains of the formulas, and it is a challenge how to modify Definition 2 of discrete matrix delayed exponential for two delays in such a way that formula (6) will hold for $k \in \mathbb{Z}^{\infty}_{-\max\{m,n\}}$. In [1] formula (4) is used to get a representation of the solution of the problems (both homogeneous and nonhomogeneous)

$$\Delta y(k) = By(k-m) + f(k), \quad k \in \mathbb{Z}^{\infty}_{0},$$
$$y(k) = \varphi(k), \quad k \in \mathbb{Z}^{0}_{-m},$$

where $f : \mathbb{Z}^{\infty}_{0} \to \mathbb{R}^{r}$, $y : \mathbb{Z}^{\infty}_{-m} \to \mathbb{R}^{r}$ and $\varphi : \mathbb{Z}^{0}_{-m} \to \mathbb{R}^{r}$.

It is an open problem how to use formula (6) to get a representation of the solution of the homogeneous and nonhomogeneous problems

$$\Delta y(k) = By(k-m) + Cy(k-n) + f(k), \quad k \in \mathbb{Z}^{\infty}_{0},$$
$$y(k) = \varphi(k), \quad k \in \mathbb{Z}^{0}_{-s}, s = \max\{m,n\}$$

if $BC = CB$.

Let us note that the first concept of matrix delayed exponential was given in [3] and the first concept of discrete matrix delayed exponential was given in [1]. Further development of the delayed matrix exponentials method and its utilization to various problems can be found, *e.g.*, in [4–16].

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors have made the same contribution. Both authors have read and approved the final manuscript.

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