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SELECTED SPECIAL VECTOR FIELDS AND MAPPINGS IN RIEMANNIAN GEOMETRY

VYBRANÁ SPECIÁLNÍ VEKTOROVÁ POLE A ZOBRAZENÍ V RIEMANNOVĚ GEOMETRII

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Abstract

This PhD Thesis is devoted to the study of special vector fields and special mappings of (pseudo-) Riemannian spaces.

We defined and analyzed $\varphi(\boldsymbol{Ric})$ - vector fields on (pseudo-) Riemannian spaces, which are generalized concircular vector fields and characterized by the following condition $\nabla\varphi = \mu \cdot \mathrm{Ric}$, $\mu = \mathrm{const}$.

The metrics of equidistant spaces, which admit special mappings, in particular affine, conformal, geodesic, harmonic, equivolume, and conformally-projective harmonic mappings, are presented.

A counterexample to Petrov's hypothesis on geodesic mappings of Einstein spaces is constructed.

The fundamental equations of conformally-projective harmonic diffeomorphisms were formulated in the form of PDEs of Cauchy type and a new derivation of the fundamental equations of F-planar mappings between manifolds with affine connections was obtained.

Abstrakt

Předložená disertační práce je věnována studiu speciálních vektorových polí a speciálních zobrazení (pseudo-) Riemannových prostorů.

Zavádíme a analyzujeme $\varphi(\boldsymbol{Ric})$ - vektorová pole na (pseudo-) Riemannových prostorech, která zobecňují koncirkulární vektorová pole a jsou charakterizována podmínkou $\nabla\varphi = \mu \cdot \mathrm{Ric}$, $\mu = \mathrm{const}$.

Nalezli jsme dále metriky ekvidistantních prostorů, které připouštějí speciální zobrazení, a to afinní, konformní, geodetická, harmonická, konformně-projektivní harmonická zobrazení a zobrazení zachovávající objemy. V práci byl zkonstruován protipříklad hypotézy Petrova o geodetických zobrazeních Einsteinových prostorů.

Odvodili jsme fundamentální rovnice konformně-projektivních harmonických difeomorfismů ve tvaru parciálních diferenciálních rovnic Cauchyho typu.

Dále práce obsahuje nové efektivní odvození fundamentálních rovnic F-planárních zobrazení mezi varietami s afinní konexí.

Key words: special vector field, geodesic mapping, harmonic mapping, conformally-projective harmonic mapping, F-planar mapping, Riemannian space, equidistant space, Einstein space.

Klíčová slova: speciální vektorové pole, geodetické zobrazení, harmonické zobrazení, konformně-projektivní harmonické zobrazení, F-planární zobrazení, Riemannův prostor, ekvidistantní prostor, Einsteinův prostor.

I declare that I have worked out this PhD thesis by myself under the supervision of doc. RNDr. Miroslav Doupovec, CSc. and that I have quoted all used literature. Computations were partially carried out with the help of Maple. DesignCAD and Maple were applied for graphics.

Brno, February 6, 2009 Mgr. Irena Hinterleitner

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1 Introduction

The geometry of curved spaces, introduced in the 19^{th} century by B. Riemann, generalises Euclidian geometry in a natural way. Today its application in physics is very wide. Above all it provides the mathematical foundation of General Relativity, more recently it is applied for example in gauge field theory and σ models, popular in string theory.

Nowadays Riemannian geometry is usually formulated in terms of manifolds introduced by E. Cartan. Differential geometry on manifolds facilitates the formulation of mappings and can be written in coordinate free form, widely used in contemporary mathematics. In physics, nevertheless, calculations are mostly carried out in local coordinates.

Interesting actual parts of differential geometry are studies of diffeomorphisms and automorphisms of different types of geometric structures on smooth manifolds. In geometry the term morphism denotes a mapping between manifolds which preserves some characteristic properties. Important structures in differential geometry are affine and special Riemannian connections, the latter ones expressed by Christoffel symbols. Connections are very important and useful in physics. A generalisation of Riemannian geometry is Finsler geometry with Berwald connection.

In Riemannian space the natural generalisation of straight lines are geodesics. This is illustrated by their role in General Relativity: geodesics are trajectories of freely falling particles in curved space-time, replacing the rectilinear motion of free particles in flat space (Euclidian). Today the theory of geodesics has reached the stadium of technical application in GPS, but their physical and mathematical significance is well known since the time of Bernulli, Euler, and Lagrange. Already in Beltrami's lifetime geodesic-preserving morphisms were studied - *geodesic mappings* [8]. T. Levi-Civita [58], who laid the foundations of this theory in tensor form, studied it from the point of view of modelling dynamical processes in mechanics. Presently, for example Ferapontov [23] and Hall and Lonie [28, 29] continue working on this subject.

Much work is spent further on isometric, homothetic and conformal mappings, also on various generalisations of geodesic mappings, among them for example holomorphic-projective, quasi-geodesic, semi-geodesic, F-planar mappings, transformations and deformations. The theory of modelling physical-mechanical processes by quasi-geodesic mappings was generalised by A.Z. Petrov [84].

Results on related question can be found in many monographs, reports and theses: [1, 3, 4, 5, 6, 16, 17, 18, 19, 20, 21, 22, 26, 43, 44, 45, 48, 49, 51, 55, 58, 63, 64, 65, 80, 85, 82, 83, 86, 89, 90, 93, 98, 100, 103, 104, 105, 106, 117, 124, 125].

The mathematical apparatus employed here is tensor calculus which is used for global and local relations on n-dimensonal manifolds with affine connection, denoted in the following by A_n and Riemannian manifolds, denoted by V_n. The signature of the metric of V_n can be indefinite, so under the notion of Riemannian manifolds we understand also pseudo-Riemannian manifolds, irrespective of the signature of their metric, as for example in the books [64, 65, 82, 83, 86, 103, 125].

The subject of this thesis are selected examples of Riemannian spaces with special symmetry properties, namely equidistant spaces and generalisations thereof and several kinds of diffeomorphisms which preserve certain geometric structures.

A major part is devoted to mappings between Riemannian spaces of a special kind, so-called equidistant spaces. Equidistant spaces are characterised by the existence of certain vector fields, called concircular (see 2.8). In physics these spaces are represented by the spatially homogenous and isotropic cosmological models (Friedmann-Robertson-Walker-Lemaître models).

The original results in this work are

- the definition and analyzis of $\varphi(\boldsymbol{Ric})$ - vector fields on (pseudo-) Riemannian spaces, characterized by the condition $\nabla\varphi = \mu \cdot \mathrm{Ric}$, $\mu = \mathrm{const}$ in section three,
- metric tensors of equidistant spaces, which admit special mappings, in particular affine, conformal, geodesic, harmonic, equivolume, and conformally-projective harmonic mappings in section four,
- and an effective counterexample to Petrov's hypothesis on geodesic mappings of Einstein spaces in section four,
- fundamental equations of conformally-projective harmonic diffeomorphisms in the form of PDEs of Cauchy type in section five,
- a new derivation of the fundamental equations of F-planar mappings between manifolds with affine connections in section six.

These results were published in the papers [32, 33, 34, 35, 36, 37, 67].

2 Basic Objects of Differential Geometry

In this section we present a brief owerview of basic notations. A profound explanation can be founfd in the books by L. P. Eisenhart [19, 20], I. Kolář [50], O. Kowalský [52], A. Z. Petrov [83], A. P. Norden [80], E. G. Poznyak [85], P. K. Raševsky [89], I. A. Schouten and D. J. Struik [93], N. S. Sinyukov [103, 104], P. A. Shirokov [100], K. Yano and S. Bochner [125], S. W. Hawking and G. F. R. Ellis [31].

2.1 Manifolds

Manifolds are generalisations of our imagination of curves and surfaces to arbitrary dimensions. To begin, assume a topological Hausdorff space M with countable basis (a separable topological space). A topological space is a ***Hausdorff space***, if for every pair of points x, $y \in M$, $x \neq y$, there are open neighbourhoods U, V with $x \in U$, $y \in V$, such that $U \cap V = \emptyset$. A ***homeomorphism*** $f : M \to N$ of topological spaces is a bijection with f and $f^{-1} : N \to M$ being both continuous.

In the next step we introduce local coordinates. A ***chart*** of a neighbourhood $U \subset M$ is a homeomorphism φ to an open set in R^n, it maps each point $x \in U$ to an n-tuple of real numbers, the coordinates of x. Given two neighbourhoods U and V, $U \cap V \neq \emptyset$, with mappings $\varphi : U \to R^n$ and $\psi : V \to R^n$, we can define the transition function $\psi \circ \varphi^{-1} : R^n \to R^n$ from the coordinates defined by φ in U to those defined by ψ in V in the overlap $U \cap V$. A set of neighbourhoods covering M, each with a chart, whose transition functions in all the overlaps are r times differentiable, is called a C^r ***atlas***. An n***-dimensional*** C^r ***manifold*** M is a topological Hausdorff space together with a C^r atlas of local homeomorphisms to R^n, covering it with coordinate patches. In this sense a manifold is said to be locally Euclidean.

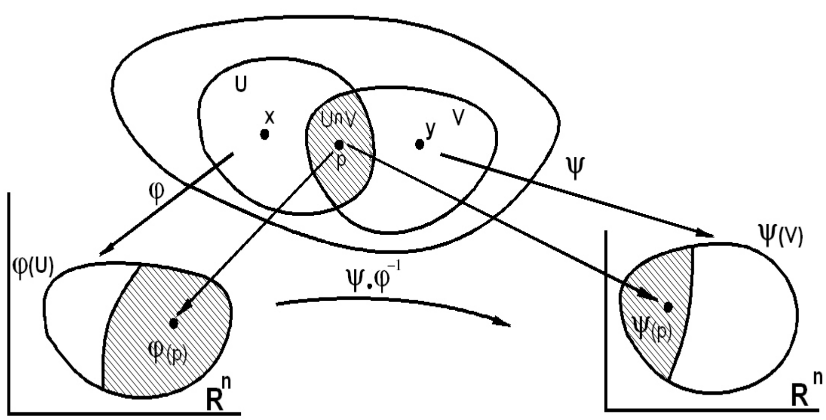


Figure 1: Two overlapping charts of a manifold with transition function.

2.2 Vectors and Tensors

Tangent vectors on manifolds are introduced as tangent vectors to a curve. A curve $\ell(t)$ is a map of an interval of $I \subset R$ into M. The (contravariant) vector $(\partial/\partial t)_\ell|_{t_0}$ tangent to $\ell(t)$ at the point $\ell(t_0)$ is the operator which maps each differantiable function f at $\ell(t_0)$ to the number $(\partial f/\partial t)_\ell|_{t_0}$, the derivative of f in the direction of ℓ with respect to the parameter t. A vector field consists of a vector defined at every point of a manifold. The vector space spanned by n linear independent vectors at a point x is called the ***tangent space*** T_xM. A special basis of the tangent space T_xM is provided by the n tangent vectors of the coordinate lines at x. These vectors are denoted by $\partial/\partial x^i$ or ∂_i and the basis formed by them is called the "coordinate basis" or "natural basis". In this basis an arbitrary vector v is written in the form $v = v^i\partial_i$, v^i are the components with respect to the basis $\{\partial_i\}$.

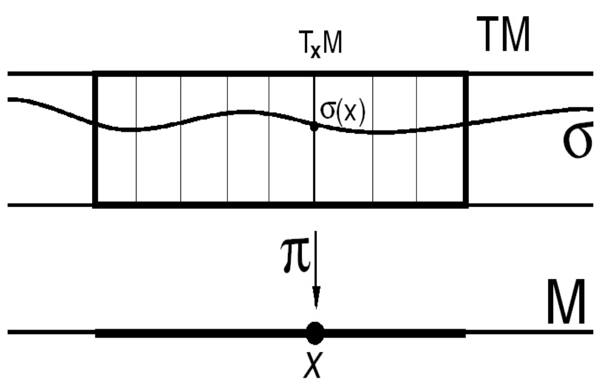


Figure 2: Tangent bundle with canonical projection π and section σ.

The ***tangent bundle*** TM of M is formed by the manifold with the tangent spaces attached to all its points. In a local neighbourhood of the tangent bundle the coordinates (x^i, v^i) of a point are made up from the coordinates x^i of the basis point in $U \in M$ and the coordinates v^i in T_xM. The map $\pi : TM \to M$, which maps a point in TM to its basis point, is called the ***canonical projection***. Its inverse π^{-1} maps each point $x \in M$ to the whole space T_xM, the ***fibre*** of TM at x.

A ***section*** σ ***of a bundle*** is a map $M \to TM$, associating to each point $x \in M$ an element of T_xM, $\sigma \circ \pi = \mathrm{id}$. In other words, a section determines a vector field on M.

A covariant vector (covector) or one-form $\omega(x)$ at a point x is a linear mapping from T_xM to R, written as $\langle\omega, v\rangle$. To each differentiable function f a covector, called the differential $\mathrm{d}f$ of f, is associated in the following way: It maps the vector v to the number $\frac{\partial f(x)}{\partial x^i}v^i$, the derivative of f in the direction of v. Here and throughout this work we use the Einstein convention, implying summation over repeated indices. The differentials of the coordinate functions $\mathrm{d}x^i$ at the point x form a set of n linearly independent one-forms, which span the dual space of T_xM, called the ***cotangent space*** T_x^*M. The basis $\{\mathrm{d}x^i\}$ is dual to the natural basis of vectors $\{\partial_i\}$ in the sense $\langle \mathrm{d}x^i, \partial_j\rangle = \frac{\partial x^i}{\partial x^j} = \delta^i_j$, where δ^i_j is the ***Kronecker symbol***

$$\delta^h_i = \begin{cases} 1, & h = i, \\ 0, & h \neq i. \end{cases}$$

In this basis a one-form $\omega(x)$ is given by $\omega = \omega_i\,\mathrm{d}x^i$. In analogy to the tangent bundle the cotangent bundle T^*M consists of a manifold and the cotangent spaces of all its points.

The Cartesian product

$$\Pi^s_r = \underbrace{T_x^*M \times T_x^*M \times \ldots \times T_x^*M}_{r\text{ factors}} \times \underbrace{T_xM \times T_xM \times \ldots \times T_xM}_{s\text{ factors}}$$

of the space of vectors and the space of one-forms at x is the ordered set of vectors and one-forms $(\omega^1, \ldots, \omega^r, v_1, \ldots, v_s)$. A tensor of type (r, s) at x is a function on Π^s_r which is linear in each argument and maps $(\omega^1, \ldots, \omega^r, v_1, \ldots, v_s)$ into the number $T(\omega^1, \ldots, \omega^r, v_1, \ldots, v_s)$. The space of all such tensors is called the tensor product

$$T^r_s(x) = \underbrace{T_xM \otimes T_xM \otimes \ldots \otimes T_xM}_{r\text{ factors}} \otimes \underbrace{T_x^*M \otimes T_x^*M \otimes \ldots \otimes T_x^*M}_{s\text{ factors}}.$$

With respect to a coordinate system $\{x^i\}$ in a neighbourhood in M the above tensor product has a basis $\{\partial_{i_1} \otimes \ldots \otimes \partial_{i_r} \otimes \mathrm{d}x^{i_1} \otimes \ldots \otimes \mathrm{d}x^{i_s}\}$. A C^r tensor field T of type (r, s) on M is an assignment of an element of $T^r_s(x)$ to each point of M such that the components with respect to any coordinate basis are C^r functions. A particularly important kind of tensors are totally antisymmetric tensors of type $(0, s)$, called s-forms. The space of s-forms in the point x is denoted by $T_x^*M \wedge \ldots \wedge T_x^*M$, the coordinate basis by $\{\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_s}\}$. The symbol $\wedge$ denotes the antisymmetrised tensor product, the so-called exterior product. A further special type of tensor fields, known as ***affinors***, are fields of type $\binom{1}{1}$.

All calculations with tensors in this work will be carried out in expressions with indices referring to local coordinates in a neighbourhood. As these calculations can be repeated in all the charts of an atlas, they can be extended to the whole manifolds. Under a change of coordinates $x^i \to \overline{x}^i$ the tensor components change according to the transformation rule

$$\overline{T}^{i_1\ldots i_r}_{j_1\ldots j_s}(\overline{x}) = \frac{\partial \overline{x}^{i_1}}{\partial x^{p_1}} \cdots \frac{\partial \overline{x}^{i_r}}{\partial x^{p_r}} \frac{\partial x^{q_1}}{\partial \overline{x}^{j_1}} \cdots \frac{\partial x^{q_s}}{\partial \overline{x}^{j_s}} T^{p_1\ldots p_r}_{q_1\ldots q_s}(x). \tag{2.1}$$

2.3 Affine Connections

Vectors in different points of a manifold belong to different vector spaces, namely the tangent spaces at these points, which are so far unrelated. The partial derivative of vector fields depends on the particular choice of coordinates and therefore has no coordinate-independent geometrical character. In consequence, without introducing an additional structure, there is no notion of parallelism at a distance or of parallel transport. Such a structure, which relates neighbouring tangent spaces and thus allows to compare elements of them, is an affine connection.

Definition 1 An ***Affine connection*** on a manifold M is a mapping ∇, which associates to an ordered pair of vector fields X, Y on M a vector field $\nabla_X Y$ on M, satisfying the following conditions:

1. The mapping ∇ is linear with respect to both arguments (in the vector space of all vector fields).
2. $\nabla_{fX} Y = f(\nabla_X Y)$ and
3. $\nabla_X (fY) = (Xf)Y + f(\nabla_X Y)$ for each function f on M.

For basis vectors in the local coordinates $x = (x^1, x^2, \ldots, x^n)$ an affine connection is expressed in the following way:

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k \,, \tag{2.2}$$

where $\Gamma^k_{ij}(x)$ are called the ***connection coefficients***. The connection coefficients specify how the basis vectors change from point to point.

Affine connections satisfying conditons 1 to 3 in definition 1 are no tensors, their components do not transform according to (2.1), but the antisymmetrisation form a tensor of type $\binom{1}{2}$, the ***Torsion tensor*** S with the components

$$S^k_{ij} = \Gamma^k_{ij} - \Gamma^k_{ji} \equiv \Gamma^k_{[ij]}.$$

The affine connection ∇ is ***symmetric***, if for all vector fields X, Y on M: $\nabla_X Y = \nabla_Y X$; the coefficients of symmetric affine connections are symmetric, i.e. $\Gamma^k_{ij} = \Gamma^k_{ji}$. In this case the torsion tensor vanishes.

Definition 2 A manifold M, on which an affine connection ∇ is defined, is called ***a space with affine connection*** and will be denoted by A_n.

An affine connection ∇ makes it possible to define a derivative, which has tensor character, in contrast to the partial derivative. The ***covariant derivative*** ∇T of a tensor field T of the type $\binom{p}{q}$ is a tensor field of type $\binom{p}{q+1}$. In local coordinates the covariant derivative of a tensor $T^{h_1 h_2 \ldots h_p}_{i_1 i_2 \ldots i_q}(x)$ can be defined by the formula

$$\nabla_k T^{h_1 h_2 \dots h_p}_{i_1 i_2 \dots i_q} \stackrel{\text{def}}{\equiv} \partial_k T^{h_1 h_2 \dots h_p}_{i_1 i_2 \dots i_q} +$$

$$+ \sum_{s=1}^{p} \Gamma^{h_s}_{k\alpha} \cdot T^{h_1 \dots h_{s-1} \alpha h_{s+1} \dots h_p}_{i_1 i_2 \dots i_q} - \sum_{s=1}^{q} \Gamma^{\alpha}_{k i_s} \cdot T^{h_1 h_2 \dots h_p}_{i_1 \dots i_{s-1} \alpha i_{s+1} \dots i_q}. \tag{2.3}$$

Covariant derivatives will be denoted by a comma, i.e. $T^{h_1 h_2 \dots h_p}_{i_1 i_2 \dots i_q, k} \equiv \nabla_k T^{h_1 h_2 \dots h_p}_{i_1 i_2 \dots i_q}$.

A special affinor is the ***identical mapping*** $\mathrm{id}\colon TM \to TM$, whose components in any local coordinates are given by the Kronecker delta δ^h_i. For arbitrary connections and for every X, Y from the space of tangent fields $\nabla_X \,\mathrm{id}(Y) = 0$, in local coordinates: $\delta^h_{i,j} = 0$. Tensors, whose covariant derivative is zero, are called ***covariantly constant***.

With the help of the covariant derivative we can define parallel transport of vectors and tensors along a curve $\ell : (a, b) \to M$. Let X be a vector field defined (at least) along $\ell(t)$. If X satisfies the condition $\nabla_V X = 0$, for any t, X is said to be parallel transported along $\ell(t)$, where $V(t)$ is the tangent vector to $\ell(t)$. In components the condition for parallel transport is written as

$$\frac{\mathrm{d}X^k}{\mathrm{d}t} + \Gamma^k_{ij} \frac{\mathrm{d}x^i(\ell(t))}{\mathrm{d}t} X^j = 0. \tag{2.4}$$

In this way parallel transport is defined by the vanishing of the covariant derivative along the curve. Given a vector at a point x, we can define a parallel vector field along a curve going through x according to the prescription (2.4). On a maniold, nevertheless, the result of a parallel transport from a point x to a point y depends on the path from x to y.

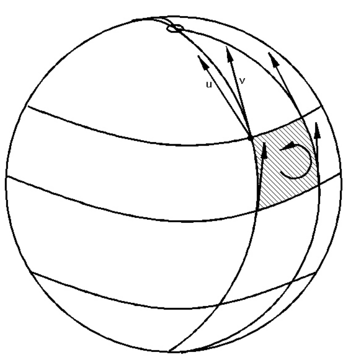


Figure 3: The parallel transport of the vector v tangent to a great circle along a closed path on a sphere results in the vector $u \neq v$.

On an infinitesimal level this is connected with the fact that the covariant derivatives of a vector field in different directions do not commute. As a measure of this noncommutativity in a space with affine connection A_n we define the ***curvature tensor*** or ***Riemann tensor***:

$$R(X,Y)Z \stackrel{\text{def}}{=} \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.5}$$

where X, Y, Z are arbitrary tangent vector fields on A_n.

In local coordinates the components of the Riemann tensor are given by

$$R^h_{ijk} \stackrel{\text{def}}{=} \partial_j \Gamma^h_{ik} - \partial_k \Gamma^h_{ij} + \Gamma^\alpha_{ik} \Gamma^h_{\alpha j} - \Gamma^\alpha_{ij} \Gamma^h_{\alpha k}. \tag{2.6}$$

In spaces with symmetric affine connection the Riemann tensor has the following properties:

$$\begin{aligned} &\text{(a)} \qquad R^h_{ijk} + R^h_{ikj} = 0, \\ &\text{(b)} \qquad R^h_{ijk} + R^h_{jki} + R^h_{kij} = 0, \\ &\text{(c)} \qquad R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0. \end{aligned} \tag{2.7}$$

Formula (c) is called the ***Bianchi identity***.

In spaces with affine connection we further define the ***Ricci tensor*** $Ric(X,Y)$, with the coordinate components

$$R_{ij} \stackrel{\text{def}}{=} R^\alpha_{ij\alpha}. \tag{2.8}$$

Definition 3 A space with affine connection, whose Ricci tensor is symmetric, is called ***equiaffine***.

This means that equiaffine spaces A_n are characterised by the condition $R_{ij} = R_{ji}$. This condition is equivalent to the local existence of a function $f(x)$ such that for the components of the affine connection the relation $\Gamma^\alpha_{\alpha i} = \partial_i f(x)$ holds.

The notation ***equiaffine*** is related to the fact that in such a space the volume of an n dimensional parallelepiped is invariant under parallel transport along an arbitrary curve, see [89].

The ***Ricci identity*** for tensors of type $\binom{p}{q}$ expresses the antisymmetrisation of the second covariant derivative. Thanks to the symmetry of the second partial derivatives, $\partial_l \partial_m T^{h_1 \ldots h_p}_{i_1 \ldots i_q} = \partial_m \partial_l T^{h_1 \ldots h_p}_{i_1 \ldots i_q}$, of C^2 tensor fields, it is expressed by the Riemann and the torsion tensor:

$$\begin{aligned} T^{h_1 \ldots h_p}_{i_1 \ldots i_q,[lm]} &= -T^{\alpha h_2 \ldots h_p}_{i_1 \ldots i_q} R^{h_1}_{\alpha lm} - T^{h_1 \alpha \ldots h_p}_{i_1 \ldots i_q} R^{h_2}_{\alpha lm} - \cdots - T^{h_1 \ldots h_{p-1}\alpha}_{i_1 \ldots i_q} R^{h_p}_{\alpha lm} + \\ &+ T^{h_1 \ldots h_p}_{\alpha i_2 \ldots i_q} R^\alpha_{i_1 lm} + T^{h_1 \ldots h_p}_{i_1 \alpha \ldots i_q} R^\alpha_{i_2 lm} + \cdots + T^{h_1 \ldots h_p}_{i_1 \ldots i_{q-1}\alpha} R^\alpha_{i_q lm} + T^{h_1 \ldots h_p}_{i_1 \ldots i_q,\alpha} S^\alpha_{lm}. \end{aligned} \tag{2.9}$$

2.4 Geodesic and F-planar Curves

In the introduction we shortly mentioned geodesics, now we present the exact definition of geodesic curves in spaces with affine connection A_n. Geodesics, that is straight lines, in Euclidian spaces can be characterised in different ways as the shortest line between two points, as curves whose tangent vectors in all of its points are parallel, and others. Here we generalise the second definition to spaces with affine connection, which is possible without the notion of a length.

Definition 4 A curve $\ell \subset A_n$ represented by the equation $x = x(t)$, is a ***geodesic***, when its tangent vector field $\lambda = dx(t)/dt$ and $\lambda \neq 0$ remains tangent to ℓ after parallel transport along the this curve.

Equivalently to this definition, a curve $\ell \subset A_n$ is geodesic if the covariant derivative of its tangent vector is proportional to the tangent vector itself,

$$\nabla_\lambda \lambda = \rho(t)\lambda\,, \tag{2.10}$$

where ρ is some function of the parameter t of the curve ℓ.

We make the remark that when the parameter t on the geodesic curve is chosen so that $\rho(t) \equiv 0$, then this parameter is called ***natural*** or ***affine***. In the literature geodesics are frequently defined in this narrower sense, see [19, 50, 52, 64, 103, 124]. We do not adopt this definition, but prefer the traditional notion based on (2.10), which is independent of the parametrisation.

A generalisation are F-planar curves in A_n, see [65, 77].

Assume an affinor structure F (i.e. a tensor field of type $\binom{1}{1}$) defined in addition to an affine connection ∇ in a space A_n. At every point $x(t)$ of ℓ the tangent vector $\lambda(t) = \mathrm{d}x(t)/\mathrm{d}t$ and the vector $F(x(t))\lambda(t)$ span a plane $\{\lambda(t), F(x(t))\lambda(t)\}$ in the tangent space of this point.

Definition 5 A curve $\ell \subset A_n$ defined by the equation $x = x(t)$, is called ***F-planar*** if its tangent vector $\lambda \neq 0$ at every arbitrary point $x(t')$ after parallel transport along ℓ to $x(t)$ lies in the plane $\{\lambda(t), F(x(t))\lambda(t)\}$.

A curve is F-planar if the equations hold:

$$\nabla_\lambda \lambda = \rho_1(t)\lambda + \rho_2(t)F\lambda, \tag{2.11}$$

where ρ_1 and ρ_2 are some functions.

If $\rho_2(t) \equiv 0$, then from equation (2.11) follows that the F-planar curve is geodesic. By a suitable choice of the parameter t we can make the function $\rho_1(t)$ vanish. In this case for any a priori selected function $\rho_2(t)$ through any chosen point $x_0 \in A_n$ and in any chosen direction there goes exactely one F-planar curve.

2.5 Introduction to Riemannian Geometry

In the foregoing subsection we have introduced the notion of parallel transport on a manifold with the aid of an affine connection. The next geometric element to introduce on a manifold is the distance between infinitesimally separated points.

Definition 6 n-dimensional manifolds M, together with a ***metric tensor*** g, which is regular and symmetric of type $\binom{0}{2}$, are called ***Riemannian spaces*** V_n.

In local coordinates the components of the metric tensor are expressed by the matrix

$$g_{ij} = g(\partial_i, \partial_j), \quad \text{with} \quad \det \|g_{ij}\| \neq 0. \tag{2.12}$$

A metric g is defined by the ***metric form on*** V_n, which associates a quadratic line element to a coordinate differential $\mathrm{d}x^i$ separating two points:

$$\mathrm{d}s^2 = g_{ij}(x)\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{2.13}$$

Unless stated otherwise, we do not specify the signature of the metric. Thus under the notion "Riemannian space" we will understand "classical" Riemannian spaces with a positive definite metric as well as pseudo-Riemannian spaces with an indefinite metric.

The quantity $g(X,Y)$ is called the ***scalar product*** of the vectors X and Y. The vectors X and Y are ***orthogonal***, when $g(X,Y) = 0$. The ***length*** of the vector X is defined as $|X| \stackrel{\text{def}}{=} \sqrt{|g(X,X)|}$. A nonzero vector with vanishing length is called ***isotropic***. Isotropic vectors can exist only in pseudo-Riemannian spaces.

The ***arc length*** of a curve $l\colon\ x^h = x^h(t),\ t \in \langle a,b\rangle$ is calculated according to the formula

$$|l| = \int_a^b \left|\frac{\mathrm{d}x^h(t)}{\mathrm{d}t}\right| \mathrm{d}t.$$

A symmetric affine connection ∇, for which $\nabla g = 0$ holds, is called ***Levi-Civita*** (***Riemann or natural***) connection. In Riemannian spaces V_n there exists a unique Levi-Civita connection, which has in local coordinates the following form:

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k, \tag{2.14}$$

where

$$\Gamma_{ijk} \stackrel{\text{def}}{=} \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \quad \text{and} \quad \Gamma^h_{ij} \stackrel{\text{def}}{=} \Gamma_{ij\alpha}g^{\alpha h} \tag{2.15}$$

are ***Christoffel symbols of first and second type***, g^{ij} are components of the matrix inverse to the matrix $\|g_{ij}\|$, forming a tensor field of type $\binom{2}{0}$. For the contraction of the Christoffel symbols the identity

$$\Gamma^\alpha_{\alpha i} = \partial_i f, \ \text{where}\ f = \frac{1}{2}\ln|\det\|g_{ij}\|| \tag{2.16}$$

holds.

In Riemannian spaces V_n the Riemann tensor defined in (2.5) and (2.6) has further symmetry and antisymmetry properties, (in addition to (2.7a)). They are formulated for the corresponding fields of type $\binom{0}{4}$:

$$R(V, Z, X, Y) \stackrel{\text{def}}{=} g(V, R(X, Y)Z),$$

respectively in coordinate form $R_{hijk} = g_{h\alpha} R^{\alpha}_{ijk}$.

This tensor field is also called the ***Riemann tensor field of first type*** and has the following properties:

$$\begin{aligned}
&\text{(a)} \qquad R_{hijk} + R_{ihjk} = 0, \quad R_{hijk} + R_{hikj} = 0, \quad R_{hijk} = R_{jkhi} \\
&\text{(b)} \qquad R_{hijk} + R_{hjki} + R_{hkij} = 0, \\
&\text{(c)} \qquad R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0.
\end{aligned} \tag{2.17}$$

Like in the general case, in Riemannian spaces the ***Ricci tensor*** (2.8) is expressed by the following formula

$$Ric(X, Y) \stackrel{\text{def}}{=} \sum_{i=1}^{n} g(R(X, e_i)Y, e_i), \tag{2.18}$$

where $\{e_1, \ldots, e_n\}$ is an arbitrary orthonormal basis in the tangent space. In local coordinates formula (2.18) is written in the form (2.8). In Riemannian spaces, where the Riemann tensor is constructed from the Levi-Civita connection, the Ricci tensor is automaticaly symmetric. Thus, according to Definition 3 Riemannian spaces are equiaffine.

We can define the ***Ricci operator*** **Ric** by $g(\boldsymbol{Ric}(X), Y) \stackrel{\text{def}}{=} Ric(X, Y)$, in local coordinates: $R^h_i = g^{h\alpha} R_{\alpha i}$.

The trace of the Ricci operator is called the ***scalar curvature*** R: $R \stackrel{\text{def}}{=} R^{\alpha}_{\alpha}$.

In Riemannian spaces V_n we can introduce the ***sectional curvature*** defined by two noncollinear vectors X and Y in the point $x \in V_n$:

$$K_x(X, Y) = \frac{R(X, Y, X, Y)}{g(X, X) \cdot g(Y, Y) - g(X, Y)^2}. \tag{2.19}$$

Definition 7 A space V_n in which the curvature $K_x(X, Y)$ does not depend on the choice of the two dimensional space spanned by the vectors X, Y and on the point x, is called a ***space with constant curvture***.

Spaces with constant curvature K are characterised by the conditions on the curvature tensor:

$$R(Y,Z)X = K \cdot (g(X,Z) \cdot Y - g(X,Y) \cdot Z) \quad \forall X,Y,Z \in TV_n, \tag{2.20}$$

in the local transcription these conditions are written in the form $R^h_{ijk} = K(g_{ik}\delta^h_j - g_{ij}\delta^h_k)$.

By contraction of this formula with respect to the indices h and k we obtain $R_{ij} = \varrho\, g_{ij}$, $\varrho = K(1-n)$, i.e.

$$Ric(X,Y) = \varrho\, g(X,Y) \quad \forall X,Y \in TV_n. \tag{2.21}$$

2.6 Hypersurfaces

In many situations $n-1$-dimensional submanifolds S imbedded into n-dimensional manifolds M are of interest. Such submanifolds are called hypersurfaces. The most common examples are three-dimensional spatial hypersurfaces of four-dimensional space-time. The tangent space of any point of M can be decomposed into an $n-1$-dimensional subspace tangent to S and one vector n orthogonal to it, unique up to a sign and a normalising factor. In a chart we can choose *adapted coordinates*, such that $n = c\mathrm{d}x^1$, where c is a constant, and $(x^2, \ldots, x^n)$ are coordinates on S.

If g is a metric on M, the imbedding induces a metric h on S, the so-called *first fundamental form* of S. In dependence on the sign of $g_{ij}n^in^j = \pm 1$ the components of the induced metric are

$$h_{ij} = g_{ij} \mp n_i n_j. \tag{2.22}$$

Obviously h_{ij} is a projection operator on the hypersurfaces, because $h_{ij}n^in^j = 0$ and $h_{ik}h_{lj}g^{kl} = h_{ij}$. This is an interior quantity of S. The normal vector n, on the contrary, characterises the imbedding of S in M.

When n is extended in an arbitrary smooth way to a small neighbourhood of S in M, this extension $\overline{n}$ can be covariantly differentiated. The projection of the covariant derivative of $\overline{n}$ to S,

$$K_{ij} = h^k_i\, h^l_j\ \overline{n}_{k,l}, \tag{2.23}$$

is called the *extrinsic curvature* or the *second fundamental form* of S.

In analogy to the manifold M, on the hypersurface S one can construct the Levi-Civita connection from the induced metric and, in the sequel, the Riemann tensor ${R'}^h_{ijk}$, the Ricci tensor ${R'}_{ij}$, and the scalar curvature R', describing the intrinsic curvature of S. Between the curvature tensors of M, those of S, and the extrinsic curvature of S the following relations hold

$$R'^{h}_{\ ijk} = R^{p}_{qrs}\, h^{h}_{p}\, h^{q}_{i}\, h^{r}_{j}\, h^{s}_{k} \pm K^{h}_{j}\, K_{ik} \mp K^{h}_{k}\, K_{ij} \tag{2.24}$$

(*Gauss theorem*) and

$$R' = R \mp 2R_{ij}n^{i}n^{j} \pm (K^{i}_{i})^{2} \mp K^{ij}K_{ij} \tag{2.25}$$

with the same sign dependence as (2.22). Another relation between the extrinsic curvature and the Ricci tensor of M is

$$K^{h}_{i;h} - K^{h}_{h;i} = R_{kl}\, n^{l} h^{k}_{i}, \tag{2.26}$$

where the semicolon denotes the covariant derivative on the hypersurface S. This relation is known as *Codazzi's equation*. For more details on this subject, see [31].

2.7 Einstein Spaces

Riemannian spaces with the Ricci tensor proportional to the metric tensor, as in equation (2.21), are denoted as ***Einstein spaces***.

Spaces with constant sectional curvature are Einstein spaces by definition. In Einstein spaces $\rho = \frac{R}{n}$ holds and moreover the scalar curvature R is constant. They provide simple, highly symmetric cosmological models. From the basic relation (2.21) we obtain in an n dimensional space the form of the scalar curvature

$$R = R^{i}_{i} = \rho\delta^{i}_{i} = n\rho. \tag{2.27}$$

In general relativity the Einstein equations relate the curvature of space-time to the energy and momentum of all the matter present in space in the following way

$$R_{ik} - \frac{1}{2}\, R\, g_{ik} + \Lambda\, g_{ik} = G\, T_{ik}. \tag{2.28}$$

The first two expressions oh the left-hand form the so-called Einstein tensor, representing space-time curvature, Λ is the cosmological constant, on the right-hand side the energy-momentum tensor of matter T_{ik}, multiplied by the gravitational constant G, describes the source of the gravitational field.

After insertion of (2.21) and (2.27) we obtain

$$\left[\left(1 - \frac{n}{2}\right)\rho + \Lambda\right] g_{ik} = G\, T_{ik}. \tag{2.29}$$

For $T_{ik} = 0$ this describes vacuum solutions with a nonzero cosmological constant. In this case, for $n > 2$, the factor of proportionality ρ is determined by

$$\rho = \frac{2\Lambda}{n-2}.$$

Here we make a remark about the cosmological constant: representing a repulsive vacuum energy density of the universe, it plays a crucial role in contemporary cosmology, for its value seems to determine the fate of our universe - eternal expansion or recollaps. Present observations indicate an accelerated expansion that would be in accordance with $\Lambda \approx +10^{-29} g/cm^3$.

From the covariant derivative of formula (2.21) obviously follows $\nabla_X R = 0$. Riemannian spaces in which this condition holds are called ***symmetric*** (in the sense of Cartan). These spaces were studed earlier, beginning with P.A. Shirokov [100]. The monograph [9] deals with the generalisation of these spaces.

2.8 Equidistant Spaces

Geometric properties of Riemannian spaces are frequently studied with respect to the existence of certain vector fields. Vector fields can be related to automorphisms on manifolds and used in this way to classify them according to some symmetry properties. For the sake of completeness here we begin with a rather general class of vector fields and specify it in the sequel to a subclass, which is related to so-called equidistant spaces, a class of Riemannian spaces with cosmological models among them. The general class of vector fields to begin with are the so-called torse-forming vector fields, with the special subclasses of recurrent and concircular fields.

Riemannian spaces in which these vector fields exist have a special form of the metric, called warped product form. Torse-forming vector fields were introduced by K. Yano [123] in 1944. Special types of these spaces were studied before: covariantely constant by T. Levi-Civita [58], convergent by P.A. Shirokov [100] and concircular by H.W. Brinkmann [11], K. Yano [122]. The spaces in which concircular vector fields exist are called ***equidistant*** spaces, this term was introduced by N.S. Sinjukov [103], see [64]. In a number of other papers these spaces are denoted as ***almost warped product spaces*** [13], see also [44, 64, 65, 74, 86, 103, 110, 118, 122].

Firstly we define torse-forming fields and introduce some of their propereties, which are used to define recurrent and concircular vector fields.

Definition 8 Vector fields $\boldsymbol{\xi}$ in A_n are called ***torse-forming***, when:

$$\nabla_X \boldsymbol{\xi} = \rho \cdot X + a(X) \cdot \boldsymbol{\xi}, \tag{2.30}$$

where X is an arbitrary vector field in TA_n), ρ is some function on A_n, a is a linear form on A_n.

In local coordinates (2.30) is written as: $\xi^h_{,i} = \rho\,\delta^h_i + a_i\xi^h$, where ξ^h are components of the torse-forming field $\boldsymbol{\xi}$, δ^h_i is the Kronecker delta, a_i are components of the form a, which generate a covector in A_n, "," is the covariant derivative on A_n.

We suppose here and further, that $\boldsymbol{\xi}$ is a non-vanishing vector field, e.i. $\boldsymbol{\xi}(x) \neq 0$ for each point $x \in A_n$. Each vector field, which is collinear with a torse-forming vector field, is also torse-forming.

In a Riemannian space we can normalize a non isotropic torse-forming field $\boldsymbol{\xi}$ to unity, i.e. $g(\boldsymbol{\xi}, \boldsymbol{\xi}) = g_{ij}\xi^i\xi^j = e = \pm 1$. For a unitary torse-forming vector field $\boldsymbol{\xi}$ equation (2.30) can be written in the form: $\nabla_X \boldsymbol{\xi} = \rho\,(X - e\,\xi(X)\,\boldsymbol{\xi})$, $\forall X \in TV_n$, where $\xi(X) \equiv g(X, \boldsymbol{\xi})$ is a linear form generated by the vector field $\boldsymbol{\xi}$.

Definition 9 A torse-forming vector field $\boldsymbol{\xi}$ which is defined by (2.30) is called:

1. ***recurrent*** if $\rho \equiv 0$,
2. ***concircular*** if the form a is a gradient (or a local gradient), i.e. (locally) there exists a function $\varphi(x)$ that $a = \partial_i\varphi(x)dx^i$,
3. ***specially concircular***, if $a = 0$.

A recurrent vector field $\boldsymbol{\xi}$ is characterized by the equation $\nabla_X \boldsymbol{\xi} = a(X)\boldsymbol{\xi}$.

The fundamental equation of a specially concircular vector field $\boldsymbol{\xi}$ can be written in the form:

$$\nabla_X \boldsymbol{\xi} = \rho \cdot X, \quad \forall X \in TV_n, \tag{2.31}$$

or locally

$$\xi^h_{,i} = \rho \cdot \delta^h_i. \tag{2.32}$$

From this equation some important properties follow immediately.
1) With the index h lowered this becomes

$$\xi_{h,i} = \rho \cdot g_{hi}, \tag{2.33}$$

showing the symmetry of the first covariant derivative of the covector field ξ_i, $\xi_{i,h} = \xi_{h,i}$. In other words, the one-form ξ is closed and so locally there is a function φ whose exterior derivative is ξ. In a coordinate neigbourhood the equations $\varphi(x^i)$ =const. define $(n-1)$ dimensional hypersurfaces in an n-dimensional manifold with the normal form $\xi = \mathrm{d}\varphi$.
2) Contracting (2.33) with ξ^i gives

$$\xi^h_{,i}\xi^i = \rho\xi^h,$$

showing that the vector field ξ^h is geodesic. As long as it is not isotropic, its integral curves form a congruence of geodesics transversal to the family of hypersurfaces given by $\varphi =$const.
3) Differentiating the length of the vector field ξ,

$$(\xi_i\xi^i)_{,k} = 2\xi_i\xi^i_{,k} = 2\xi_i\rho\delta^i_k = 2\rho\xi_k,$$

we find that it varies only along ξ, i.e. it is constant on a hypersurface. In this sense the distance between two hypersurfaces $\varphi = c_1$ and $\varphi = c_2$ is everywhere the same, therefore the notion *equidistant*.

If $\rho \equiv$ const then a concircular field is convergent.

* ***Convergent*** vector fields, characterised by $\rho(x) = \text{const}\cdot\exp(\varphi(x))$, were first studied by P.A. Shirokov [100] and later by other autors.

* ***Concircular*** vector fields were first studied by H.W. Brinkmann [11] (see [67, 79, 83, 99]), in the course of studying conformal mappings of Einstein spaces. His results contain general properties of the metrics of Riemannian and pseudo Riemannian spaces with concircular vector fields.
H.W. Brinkmann [11] in fact found metrics of Riemannian and also pseudo-Riemannian spaces, in which non-isotropic concircular vector fields exist, in the following form

$$\mathrm{d}s^2 = \frac{1}{f(x^1)}(\mathrm{d}x^1)^2 + f(x^1)\mathrm{d}\tilde{s}^2, \tag{2.34}$$

where $f \in C^1$ $(f \neq 0)$ is a function, $\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2, \cdots, x^n)\mathrm{d}x^a\mathrm{d}x^b$ $(a, b = 2, \cdots, n)$ is the metric form of a certain Riemannian space $\tilde{V}_{n-1}$.

Moreover [11], V_n is an Einstein space $\mathcal{E}_n$ (respectively a space $\mathcal{S}_n$ with constant curvature K) if and only if $\tilde{V}_{n-1}$ is an Einstein space $\tilde{\mathcal{E}}_{n-1}$ (respectively a space $\tilde{S}_{n-1}$ with constant curvature $\tilde{K}$) and the condition $f = K\,(x^1)^2 + 2a\,x^1 + b$ holds, where K, a and b are constants and $\mathrm{d}\tilde{s}^2$ is a metric of the Einstein space $\tilde{\mathcal{E}}_{n-1}$ (respectively the space $\tilde{\mathcal{S}}_{n-1}$ with constant curvature $\tilde{K}$). The constants K and $\tilde{K}$ are related to the scalar curvatures R and $\tilde{R}$ of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_{n-1}$ (respectively $\mathcal{S}_n$ and $\tilde{\mathcal{S}}_{n-1}$)

$$\tilde{K} = \frac{\tilde{R}}{(n-1)(n-2)} = b\,K - a^2, \quad \text{where} \quad K = \frac{R}{n(n-1)}. \tag{2.35}$$

These fields were - under the above denotation - introduced by K. Yano [122] and, independently by Ya.L. Shapiro [96] as ***geodesic fields***, and N.S. Sinyukov [102, 103] as ***equidistant fields***.

By a transformation of the coordinate $x^1 \to \overline{x}^1$ the metric (2.34) can be written in the form

$$\mathrm{d}s^2 = a(\overline{x}^1)(\mathrm{d}\overline{x}^1)^2 + f(\overline{x}^1)\mathrm{d}\tilde{s}^2,$$

were $a(\overline{x}^1) \neq 0$ is a function. In particular, we can choose such a transformation that $a(\overline{x}^1) = \pm 1$.

Among equidistant Riemannian spaces there are some cosmological models, see for example [31, 39]. Modern cosmological models are based on the so-called Copernican principle (Bondi 1960), stating that no position in the universe is distinguished in any way. A suitable concrete realisation of this principle is the assumption of spatial homogeneity. At a sufficiently large scale the structure of the universe is everywhere essentially the same. Such a space is invariant under translations, the latter ones form an isometry group, i.e. a group of transformations that leaves the metric invariant. A further interpretation of the Copernican principle is isotropy - the universe appears in every direction approximately the same. The associated isometry group is the group of rotations. Isotropy at every point ascertains homogeneity, the reverse is not true.

Homogenous and isotropic models are the simplest cosmological models characterised by constant spatial curvature. They were introduced and studied by Friedmann, Lemaître, Robertson and Walker. The metric has the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - F^2(t)\left[\frac{\mathrm{d}r^2}{1-\tilde{K}r^2} + r^2\mathrm{d}\vartheta + r^2\sin^2\vartheta\mathrm{d}\varphi^2\right]. \tag{2.36}$$

The parameter $\tilde{K}$ represents the constant (negative, zero, or positive) spatial curvature and can be normalised to -1, 0, or $+1$, the function $a(t)$ describes the time evolution of the size of the universe. Spatial homogeneity is expressed by the dependence of the metric components only on t. Given two values t_1 and t_2 of t, the distance between two corresponding points (r, ϑ, φ) on the t_1 and the t_2 hypersurface, repectively, is independent on the location of the points.

To illustrate (2.34) and the statement below it, we choose a Lorentzian signature to keep contact with physical applications and transform it to the form

$$\mathrm{d}s^2 = \mathrm{d}t^2 - F^2(t)\mathrm{d}\tilde{s}^2.$$

This is achieved by setting

$$\mathrm{d}t^2 = \frac{(\mathrm{d}x^1)^2}{f(x^1)},$$

leading to

$$t = \int_{x_0^1}^{x^1} \frac{\mathrm{d}x}{\sqrt{Kx^2 + 2ax + b}}.$$

Having calculated this integral we express x^1 in terms of t and isert into f to obtain $F^2(t) = f(x^1(t))$.

For the explicit form of the above integral we have to distinguish four cases
1) $K > 0,\ Kb > a^2$

$$x^1 = \frac{\sqrt{Kb - a^2}}{K}\ \sinh\left(\sqrt{K}(t - t_0)\right) - \frac{a}{K},$$

2) $K > 0,\ Kb = a^2,\ Kx^1 + a > 0$

$$x^1 = \frac{1}{2K}\mathrm{e}^{\sqrt{K}(t-t_0)} - \frac{a}{K},$$

3) $K > 0,\ Kb = a^2,\ Kx^1 + a < 0$

$$x^1 = -\frac{1}{2K}\mathrm{e}^{-\sqrt{K}(t-t_0)} - \frac{a}{K},$$

4)$K < 0,\ Kb < a^2,\ |Kx^1 + a| < \sqrt{a^2 - Kb}$

$$x^1 = -\frac{\sqrt{a^2 - Kb}}{K}\ \sin\left(\sqrt{-K}(t - t_0)\right) - \frac{a}{K}.$$

For $F^2(t) = K(x^1(t))^2 + 2ax^1(t) + b$ we obtain
in case 1)

$$F^2(t) = \frac{Kb - a^2}{K}\cosh^2\left(\sqrt{K}(t - t_0)\right),$$

in cases 2) and 3)

$$F^2(t) = \frac{1}{4K}\mathrm{e}^{\pm 2\sqrt{K}(t-t_0)},$$

and in case 4)

$$F^2(t) = \frac{Kb - a^2}{K}\cos^2\left(\sqrt{-K}(t - t_0)\right).$$

Taking $\mathrm{d}\tilde{s}^2$ from (2.36) we obtain the scalar curvature

$$R = -3\frac{(F^2(t))'' + 2\tilde{K}}{F^2(t)},$$

which equals

$$R = -6K\frac{2(Kb - a^2)\cos^2\left(\sqrt{K}(t - t_0)\right) - Kb + a^2 + \tilde{K}}{(Kb - a^2)\cos^2\left(\sqrt{K}(t - t_0)\right)}$$

in case 4), for example. For $\tilde{K} = bK - a^2$ this reduces to the constant $R = -12K$ in all cases[1]. Case 1) is the de Sitter, case 4) is the anti-de Sitter model of the universe with constant space-time curvature. Cases 2) and 3) describe flat spatial hypersurfaces ($\tilde{K} = 0$) imbedded with an exponentially increasing / decreasing scale factor into a space time manifold with constant curvature. Adding one dimension and interchanging one space coordinate with time yields a five dimensional manifold with a metric of this kind. In this manifold the flat space-like hypersurface are replaced by four dimensional Minkowski space. Such manifolds occur in string cosmology as the Randall-Sundrum model of type two [88].

[1] In the original paper [11] there is a typographic error: $\tilde{K} = bK^2 - a^2$.

3 $\varphi(\boldsymbol{Ric})$ - Vector Fields in Riemannian Spaces

In this section we study a class of special curvature-determined vector fields in Riemannian spaces, which are in some sense modifications of concircular vector fields and defined by $\nabla\xi = \mu \cdot \boldsymbol{Ric}$. Results about these vector fields were published in a joint paper, together with a coauthor [35].

Definition 10 A *φ(**Ric**)- vector field* is a vector field φ on an n dimensional Riemannian space V_n with metric g and Levi-Civita connection ∇, which satisfies the condition

$$\nabla\varphi = \mu\, \boldsymbol{Ric}, \tag{3.1}$$

where μ is some constant and $\boldsymbol{Ric}$ is the Ricci tensor.

Obviously, when V_n is an Einstein space, then $\varphi_{i,j} = \mu R_{ij} = \mu\rho g_{ij} = \tilde{\rho} g_{ij}$, where $\rho = \text{const}$, i.e. the vector field φ is concircular. When $\mu = 0$, the vector field φ is covariantly constant.

In the following we suppose that $\mu \neq 0$ and V_n is neither an Einstein space nor a vacuum solution of the Einstein equations, because a vacuum solution is charcterised by $R_{ij} = 0$.

In a local coordinate neighbourhood $U(x)$ equation (3.1) is written as

$$\varphi^h_{,i} = \mu\, R^h_i, \tag{3.2}$$

where φ^i and R^h_i are components of φ and $\boldsymbol{Ric}$, respectively. After lowering indices, (3.2) has the form

$$\varphi_{i,j} = \mu\, R_{ij}, \tag{3.3}$$

where $\varphi_i = \varphi^\alpha\, g_{i\alpha}$ and $R_{ij} = g_{i\alpha}\, R^\alpha_j$.

The Ricci identities, introduced in their general form by (2.9) for a vector field φ, can be written in the form $\varphi^h_{,jk} - \varphi^h_{,kj} = -\varphi^\alpha R^h_{\alpha jk}$.

Using them we obtain from the antisymmetric covariant derivative of (3.3)

$$\varphi_\alpha R^\alpha_{ijk} = \mu \cdot (R_{ij,k} - R_{ik,j}),$$

and applying the Bianchi identity to the right hand side yields the integrability condition of (3.3) in the form:

$$\varphi_\alpha R^\alpha_{ijk} = \mu\, R^\alpha_{ijk,\alpha}, \tag{3.4}$$

where R^h_{ijk} is the Riemann tensor on V_n.

This equation is contracted by g^{ij}, and from the contracted Bianchi identity $R^{\alpha}_{k,\alpha} = \frac{1}{2} R_{,k}$, we can see

$$\varphi_\alpha R^\alpha_k = \frac{\mu}{2} R_{,k}, \tag{3.5}$$

where $R = R_{\alpha\beta} g^{\alpha\beta}$ is the scalar curvature.

Now we consider $\varphi(\boldsymbol{Ric})$ - vector fields of constant length $|\varphi| = \sqrt{|\varphi^\alpha \varphi^\beta g_{\alpha\beta}|}$ or

$$\varphi^\alpha \varphi^\beta g_{\alpha\beta} = \text{const.} \tag{3.6}$$

Differentiating (3.6) and making use of (3.2) we obtain $\mu \cdot \varphi_\alpha R^\alpha_i = 0$. Due to the assumption $\mu \neq 0$ we have $\varphi_\alpha R^\alpha_i = 0$. With the condition (3.5) this leads to

$$R_{,k} = 0,$$

this means that the scalar curvature on V_n is necessarily constant. In this way we have proved the following theorem.

Theorem 1 *Riemannian or pseudo-Riemannian spaces (M^n, g) with a $\varphi(\mathbf{Ric})$-vector field of constant length have constant scalar curvature.*

3.1 $\varphi(\boldsymbol{Ric})$ - and Concircular Vector Fields

We study the question whether a $\varphi(\boldsymbol{Ric})$ - vector field and a concircular field $\boldsymbol{\xi}$ (2.33) can exist simultaneously.

We prove the following theorem

Theorem 2 *If a $\varphi(\mathbf{Ric})$ - vector field exists together with a non-collinear concircular vector field on a Riemannian manifold, then the latter one is necessarily covariantly constant.*

Proof. Assume a $\varphi(\boldsymbol{Ric})$ - vector field φ and a concircular vector field ξ, characterised by the respective equations (3.3) and (2.33). After some easy calculations, the integrability conditions of (2.33) can be written in the form

$$\xi_\alpha R^\alpha_{ijk} = c(\xi)(g_{ij}\xi_k - g_{ik}\xi_j), \tag{3.7}$$

where $c(\xi)$ is some function, depending on the function ξ, which in turn locally generates the covector ξ_i, i. e. $\xi_i = \partial \xi(x)/\partial x^i$.

By differentiating the last formula and by using (2.33) and certain properties of the Riemann tensor we obtain

$$\rho\, R_{lijk} + \xi_\alpha R^\alpha_{ijk,l} = c'(\xi)\,(g_{ij}\xi_k\xi_l - g_{ik}\xi_j\xi_l) + c(\xi)\,\rho\,(g_{ij}g_{kl} - g_{ik}g_{jl}),$$

where $R_{lijk} = g_{\alpha l} R^{\alpha}_{ijk}$, and $c'(\xi)$ is the derivative of the function $c(\xi)$.

After contracting the indices k and l, taking into account (3.4) and (3.7), we have

$$\mu\,\rho\,R_{ij} = (\mu\,c\,\xi_\alpha \xi^\alpha + (n-1)\,\rho\,\mu\,c + c\,\xi_\alpha \varphi^\alpha)\,g_{ij} + c\,\xi_j \varphi_i - \mu\,c'\,\xi_i \xi_j. \tag{3.8}$$

Alternating (3.8) we see that

$$c\,(\xi_j \varphi_i - \xi_i \varphi_j) = 0.$$

Because the vanishing of the expression in parenthesis would mean collinearity of the vectors φ and ξ, which is excluded, c must be equal to zero.

But then (3.8) acquires the form $\mu\rho R_{ij} = 0$. Because $\mu R_{ij} \neq 0$, ρ must vanish. We now conclude from (2.33) that $\nabla\xi = 0$, i.e. ξ is a covariantly constant vector field, which establishes the theorem. ■

3.2 $\varphi(\boldsymbol{Ric})$ - Vector Fields in Symmetric Spaces

We consider $\varphi(\boldsymbol{Ric})$ - vector fields in symmetric spaces V_n, characterized by the covariant constance of the Riemann tensor $\nabla R = 0$, written in coordinates as

$$R^h_{ijk,l} = 0. \tag{3.9}$$

We start from equations (3.2) which characterize $\varphi(\boldsymbol{Ric})$ - vector fields. In symmetric spaces the integrability conditions (3.4) of these equations simplify due to (3.9)

$$\varphi_\alpha R^\alpha_{ijk} = 0. \tag{3.10}$$

Application of (3.2), (3.9) and the fact that $\mu \neq 0$, to the differential prolongations of (3.10) yields

$$R_{\alpha l} R^\alpha_{ijk} = 0. \tag{3.11}$$

Formula (3.11) has an intrinsic character in the space V_n. Its index-free form is $\boldsymbol{Ric}(X, R(Y,Z)V) = 0$ for all tangent fields X, Y, Z, V. In symmetric spaces its differential prolongation is satisfied identically.

As we can easily see, it follows from the theory of partial differential equations that in a symmetric space, satisfying condition (3.11), the equations

$$\varphi^h_{,i} = \mu\,R^h_i\,, \quad \mu = \text{const},$$

have locally a unique solution for an arbitrary constant μ and arbitrary initial conditions

$$\varphi^h(x_0) = \overset{0}{\varphi}{}^h,$$

which fulfill the condition

$$\overset{0}{\varphi}_\alpha R^\alpha_{ijk}(x_0) = 0$$

in the point x_0.

From (3.11) it follows that the last conditions have a nontrivial solution $\varphi^h(x_0)$. In his way we can formulate the following theorem.

Theorem 3 *In a symmetric, non-Einstein pseudo-Riemannian space V_n with $Ric(X, R(Y,Z)V) = 0$ for all tangent fields X, Y, Z, V there exists locally a $\varphi(\mathbf{Ric})$ - vector field.*

We remark that in symmetric spaces no concircular vector fields other than covariantly constant ones exist. We can easily convince ourselves of the existence of symmetric spaces satisfying conditions (3.11). An example of a non-Einstein, symmetric pseudo-Riemannian space V_n satisfying the condition (3.11) is a space with a metric of the following form

$$\mathrm{d}s^2 = \exp(2x^1)\{(2\mathrm{d}x^1\mathrm{d}x^2 + e_3(\mathrm{d}x^3)^2 + \ldots + e_n(\mathrm{d}x^n)^2\},$$

where $e_i = \pm 1, i = 3 \ldots n$.

Lemma 1 *Classical Riemannian spaces V_n with positive definite metric and the above properties (i. e. non-Einsteinian symmetric true Riemannian spaces which satisfy conditions (3.11)) do not exist.*

Proof. After contraction of the formulae (3.11) follows $R_{\alpha\beta}R_{\gamma\mu}g^{\alpha\gamma}g^{\beta\mu} = 0$, this means when the metric is positive definite, then $R_{ij} = 0$, in contradiction to the assumption $R_{ij} \neq 0$. ∎

3.3 A Simple Example of a $\varphi(\boldsymbol{Ric})$ - Vector Field

Here we construct an example of a $\varphi(\boldsymbol{Ric})$ - vector field in a non-isotropic generalization of an equidistant space. We assume a diagonal indefinite metric in a three dimensional space with the coefficients g_{22} and g_{33} depending on the coordinate x^1. In contrast to the equidistant case they are given by two different independent C^2 functions f and g of the coordinate x^1

$$\mathrm{d}s^2 = -(\mathrm{d}x^1)^2 + f(x^1)(\mathrm{d}x^2)^2 + g(x^1)(\mathrm{d}x^3)^2. \tag{3.12}$$

We assume the existence of a $\varphi(\boldsymbol{Ric})$ - vector field in the x^1 direction in the form,

$$\varphi^i = (\varphi^1(x^1), 0, 0), \tag{3.13}$$

depending only on the coordinate x^1, too. The non-vanishing Christoffel symbols (2.15) of the metric (3.12) are

$$\Gamma^1_{22} = \frac{f'}{2}, \;\; \Gamma^2_{12} = \Gamma^2_{21} = \frac{f'}{2f}, \;\; \Gamma^1_{33} = \frac{g'}{2}, \;\; \Gamma^3_{13} = \Gamma^3_{31} = \frac{g'}{2g}. \tag{3.14}$$

The Ricci tensor (2.8) is diagonal with the components

$$\begin{aligned} R_{11} &= \partial_1 \Gamma^2_{12} + \partial_1 \Gamma^3_{13} + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{13}\Gamma^3_{13}, \\ R_{22} &= -\partial_1 \Gamma^1_{22} + \Gamma^2_{12}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^3_{13}, \\ R_{33} &= -\partial_1 \Gamma^1_{33} + \Gamma^3_{13}\Gamma^1_{33} - \Gamma^1_{33}\Gamma^2_{12}. \end{aligned} \tag{3.15}$$

Now we postulate the existence of a vector field of the form (3.13) satisfying (3.1) with $\mu = 1$:

$$\varphi_{1,1} = \varphi_1' = R_{11}, \;\; \varphi_{2,2} = -\Gamma^1_{22}\,\varphi_1 = R_{22}, \;\; \varphi_{3,3} = -\Gamma^1_{33}\,\varphi_1 = R_{33}. \tag{3.16}$$

From the second and the third equation from (3.16) we obtain $R_{22}/\Gamma^1_{22} = R_{33}/\Gamma^1_{33}$, in terms of f and g:

$$-\frac{f''}{f'} + \frac{f'}{2f} - \frac{g'}{2g} = -\frac{g''}{g'} + \frac{g'}{2g} - \frac{f'}{2f}. \tag{3.17}$$

With the definitions $\overline{f} = \ln f$ and $\overline{g} = \ln g$ (3.17) becomes $-\overline{f}''/\overline{f}' = -\overline{g}''/\overline{g}'$ and further, with $F = \overline{f}'$ and $G = \overline{g}'$, we obtain $F'/F = G'/G$ with the solution $G = cF$, where c is an integration constant. So $\overline{g} = c\overline{f} + c_1$ and, finally,

$$g(x^1) = k \cdot (f(x^1))^c. \tag{3.18}$$

Ih the first step the supposed existence of a $\varphi(\boldsymbol{Ric})$ - vector field has led to a specialisation of the metric to the form

$$g_{ik} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & f(x^1) & 0 \\ 0 & 0 & k \cdot f(x^1)^c \end{pmatrix}, \tag{3.19}$$

containing only one function. For this metric the components of the Ricci tensor are

$$\begin{aligned} R_{11} &= \frac{c+1}{2}\ \frac{f''}{f} - \frac{1}{4}\left(1+2c-c^2\right)\ \frac{f'^2}{f^2}, \\ R_{22} &= \frac{-f''}{2} + \frac{1-c}{4}\ \frac{f'^2}{f}, \\ R_{33} &= \frac{kc}{4} f^{c-2}\left(-2ff'' + (1-c)f'^2\right). \end{aligned} \tag{3.20}$$

Making once more use of (3.16) we express φ^1 in terms of the function f,

$$\varphi^1 = -\frac{R_{22}}{\Gamma^1_{12}} = \frac{f''}{f'} - \frac{1-c}{2}\frac{f'}{f} \tag{3.21}$$

the derivative of which is equal to R_{11} according to the first equation of (3.16),

$$\frac{f'''f' - f''^2}{f'^2} - \frac{1-c}{2}\ \frac{f''f - f'^2}{f^2} = \frac{c+1}{2}\ \frac{f''}{f} - \frac{1}{4}(1+2c-c^2)\frac{f'^2}{f^2}.$$

After suitable transformations this equation simplifies slightly,

$$\frac{f'''}{f'} - \left(\frac{f''}{f'}\right)^2 = \frac{f''}{f} - \frac{3-c^2}{4}\left(\frac{f'}{f}\right)^2. \tag{3.22}$$

We make an ansatz for the solution in the form $f(x^1) = (x^1)^\alpha$ which leads to the simple relation

$$c^2 = \frac{4-\alpha^2}{\alpha^2}.$$

Insertion into (3.18) yields $g(x^1) = k \cdot (x^1)^{\pm\sqrt{4-\alpha^2}}$ and with a convenient change of parameters, $\alpha = 2\cos\theta$, and the choice $k = 1$ we obtain a one-parameter family of solutions for the metric

$$g_{ik} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & (x^1)^{2\cos\theta} & 0 \\ 0 & 0 & (x^1)^{2\sin\theta} \end{pmatrix} \tag{3.23}$$

with the parameter θ restricted by $\theta \in \langle 0, 2\pi\rangle$. This choice of parameter encompasses all possible signs of the exponents α in f and $\sqrt{4-\alpha^2}$ in g.

The non-vanishing components of the Ricci tensor are

$$\begin{aligned} R_{11} &= \frac{1-\cos\theta-\sin\theta}{(x^1)^2}, \\ R_{22} &= \frac{\cos\theta\,(1-\cos\theta-\sin\theta)}{(x^1)^{2(1-\cos\theta)}}, \\ R_{33} &= \frac{(1-\cos\theta)\,(\sin\theta-1-\cos\theta)}{(x^1)^{2(1-\sin\theta)}} \end{aligned}$$

and the scalar curvature is

$$R = -\frac{2(1-\cos\theta)\,(1-\sin\theta)}{(x^1)^2}. \tag{3.24}$$

In this space the $\varphi(\boldsymbol{Ric})$ - vector field is given by the component

$$\varphi^1 = \frac{1-\cos\theta-\sin\theta}{x^1}. \tag{3.25}$$

The Riemannian space with the above metric (3.12) with components (3.23) provides a nontrivial example of a space with a $\varphi(\boldsymbol{Ric})$ - vector field, which is in general neither equidistant, nor an Einstein space, nor a space of constant curvature.

The same type of metric components as in (3.23) appeares in the cosmological models of Kasner type with the metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + t^{2p}\mathrm{d}x^2 + t^{2q}\mathrm{d}y^2 + t^{2r}\mathrm{d}z^2.$$

This is a vacuum solution of the Einstein equations if the parameters p, q, r satisfy the two conditions

$$p^2 + q^2 + r^2 = 1 \tag{3.26}$$

and

$$p + q + r = 1. \tag{3.27}$$

In order to satisfy these conditions p, q and r cannot all have the same sign, so the model describes a universe, which expands in two directions and contracts in the third, or vice versa. Condition (3.26) describes a sphere in the space of parameters, the so-called Kasner sphere, (3.27) describes a plane. The Kasner vacuum solutions lie on their intersection, a circle. In the present case the exponents of x^1 lie on a two dimensional analog of the Kasner sphere.

The following figures display the metric distances of the points of the unit circle $(x^2)^2 + (x^3)^2 = 1$ from the origin in dependence of the coordinate x^1. With true lengths measured by the metric (3.23), the circle is in fact an ellipse. The metric orthogonal distances of a point (x^2, x^3) from the x^1 axis are $(x^1)^{\cos(\theta)}x^2$ and $(x^1)^{\sin(\theta)}x^3$, so the polar coordintes $(1, \varphi)$ of a point of the unit circle are transformed to

$$r = \sqrt{(x^1)^{2\cos\theta}(x^2)^2 + (x^1)^{2\sin\theta}(x^3)^2} \quad \text{and} \quad \psi = \tan((x^1)^{\cos\theta-\sin\theta}\arctan\varphi).$$

In this sence the surfaces in the figures show the evolution of the unit circle in the course of the “time” x^1.

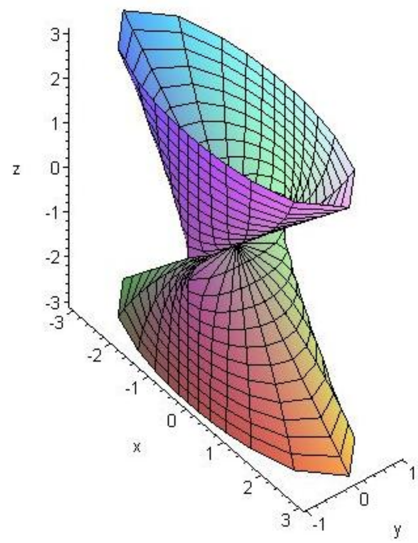


Figure 4: For $\theta = 0$ or $\theta = \dfrac{\pi}{2}$ the vector field φ is equal to zero. In these limiting cases of spaces with a $\varphi(\boldsymbol{Ric})$ - vector field the space is flat, this is the analog to the Kasner vacuum solutions in four dimensions.

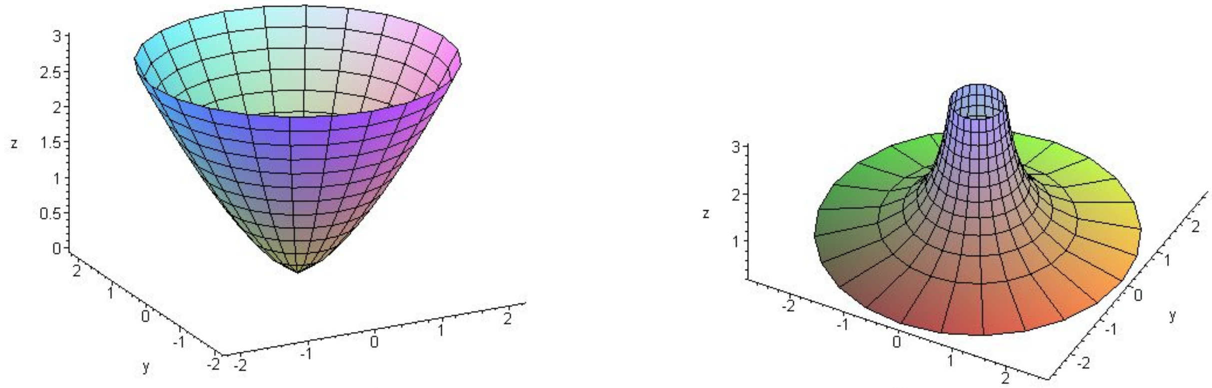


Figure 5: In the cases $\theta = \dfrac{\pi}{4}$ and $\dfrac{5\pi}{4}$ we have $g_{22} = g_{33}$ and the space is obviously equidistant, see (13).

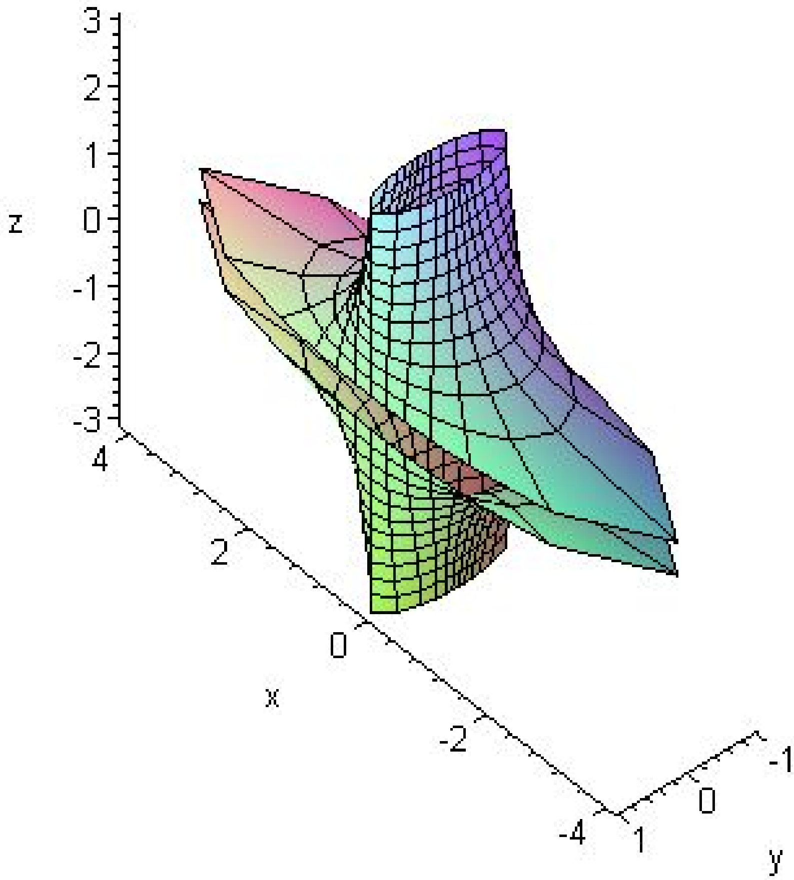


Figure 5: $\theta = \pi, \ \ \dfrac{3\pi}{2}$

When $\theta = \pi$ the metric component g_{33} is equal to one and there is a covariantly constant vector field $v = (0, 0, 1)$ orthogonal to a two-dimensional subspace with metric

$$\mathrm{d}\tilde{s}^2 = -(\mathrm{d}x^1)^2 + \frac{1}{(x^1)^2}\,(\mathrm{d}x^2)^2.$$

For this reason also in this case the space is equidistant; the subspace orthogonal to v does not have constant curvature. The case $\theta = 3\pi/2$ is analogous, with x^2 and x^3 interchanged.

From the expression (3.24) we see that in the generic case the metric (3.12), (3.23) displays a curvature singularity at $x^1 = 0$, like the Kasner metric.

4 Special Mappings Between Riemannian Spaces

4.1 Formalisms Concerning Diffeomorphisms of Manifolds

For the study of affine, conformal, geodesic and other mappings there were established several formalisms, which are generally used for diffeomorphisms between spaces with affine connection. Now we describe the two most frequentely used formalisms that we need for our work.

4.1.1 The Formalism of "Common Coordinate Systems"

Assume a diffeomorphism f: $A_n \to \overline{A}_n$, where A_n and $\overline{A}_n$ are spaces with affine connections ∇ and $\overline{\nabla}$. We restricted oureselves to the study of the respective coordinate neighbourhoods U and $f(U)$ of an arbitrary point $m \in A_n$ and its image $\overline{m} \equiv f(m) \in \overline{A}_n$.

In the neighbourhoods U and $f(U)$ these spaces are related by common coordinate systems x with respect to the map f, e. i. the two corresponding points $m \in U$ and $f(m) \in f(U)$ have the same coordinates $x = (x^1, \dots, x^n)$. In this sense we study the neighbourhood of each point $m \in A_n$.

Corresponding geometric objects on A_n and on $\overline{A}_n$ will be distinguished by a bar. For example $\overline{\Gamma}^h_{ij}$, $\overline{R}^h_{ijk}$ and $\overline{R}_{ij}$ are components of the affine connection, the Riemann and the Ricci tensor of the space $\overline{A}_n$.

4.1.2 The Formalism of the "Common Manifold"

Assume a diffeomorphism f: $A_n \to \overline{A}_n$ as above.

We can specify that A_n is a manifold M with the affine connection ∇ and $\overline{A}_n$ is a manifold $\overline{M}$ with the affine conection $\overline{\nabla}$.

Because f is a diffeomorphism between M and $\overline{M}$, from the point of view of topology we can take $\overline{M} \equiv M$. Then the spaces A_n and $\overline{A}_n$ are defined on the same manifold M with ∇ and $\overline{\nabla}$ determining two different spaces with affine connections.

Similarly we can define on this manifold M other geometric objects, as for example, a metric, a Riemann and a Ricci tensor, and others, which will be distinguished by a bar, as before.

Of particular interest in the following will be the differences between the connection components, which form a tensor, the ***deformation tensor***

$$P^h_{ij} = \overline{\Gamma}^h_{ij} - \Gamma^h_{ij}. \tag{4.1}$$

Of course, if the connections are symmetric, the deformation tensor is symmetric in the indices i and j.

This formalism has a global meaning and is not connected with coordinate neighbourhoods like common coordinates. On the other hand, in view of the formalism discribed in the monograph by A.Z. Petrov [83] and the book by R.W. Penrose [82], it is possible to express not only local, but also global properties with the aid of common coordinates. These methods were used in fact in the work of K. Yano and S. Bochner [125], M. Afwat and A. Švec [1], and many other authors, see [2, 3, 4, 63, 64, 65, 107].

4.2 Special Mapping of Equidistant Spaces

Of particular interest in the following are mappings between equidistant spaces. In this field some of the results of this work were obtained. In equidistant spaces V_n, where the concircular vector fields are nonisotropic (i.e. $g_{ij}\xi^i\xi^j \neq 0$), we can introduce a system of so-called canonical coordinates x, where the metric is of the form

$$\mathrm{d}s^2 = a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2, \tag{4.2}$$

and $a(x^1), b(x^1) \in C^1$ are non-zero functions, $\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\,\mathrm{d}x^a\mathrm{d}x^b$ is the metric form of certain Riemannian space $\tilde{V}_{n-1}$. Here and after the indices $a, b, c, \ldots$ assume values from 2 to n.

Under the regular transformation $\overline{x}^1 = \overline{x}^1(x^1)$; $\overline{x}^a = \overline{x}^a(x^2, \ldots, x^n)$, the principal form of the metric does not change (4.2). Firstly W.H. Brinkmann [11] found a metric in the form (4.2) with $a(x^1) = 1/f(x^1)$ and $b(x^1) = f(x^1)$; often the metric of equidistant spaces is written in a form with $a(x^1) = e = \pm 1$ and $b(x^1) = f(x^1)$, see [103, 64, 122].

It is known that curves $\ell = \{(t, \overset{\circ}{x}{}^2, \ldots, \overset{\circ}{x}{}^n),\ t \in \mathbb{R}\}$ with constant coordinates $(\overset{\circ}{x}{}^2, \ldots, \overset{\circ}{x}{}^n)$ are geodesics and they form a *geodesic congruence*. The hypersurface $S = \{x^1 = \text{const}\}$ is orthogonal to this congruence. We compute the non-zero components of the Christoffel symbols for the metric form (4.2):

$$
\begin{aligned}
&\Gamma_{111} = \frac{1}{2}\, a'(x^1); && \Gamma_{1ab} = \Gamma_{a1b} = \frac{1}{2}\, b'(x^1)\, \tilde{g}_{ab};\\
&\Gamma_{ab1} = -\frac{1}{2}\, b'(x^1)\, \tilde{g}_{ab}; && \Gamma_{abc} = b(x^1)\, \tilde{\Gamma}_{abc};\\
&\Gamma^1_{11} = \frac{1}{2}\, \frac{a'(x^1)}{a(x^1)}; && \Gamma^c_{1a} = \Gamma^c_{a1} = \frac{1}{2}\, \frac{b'(x^1)}{b(x^1)}\, \delta^b_a;\\
&\Gamma^1_{ab} = -\frac{1}{2}\, \frac{b'(x^1)}{a(x^1)}\, \tilde{g}_{ab}; && \Gamma^c_{ab} = \tilde{\Gamma}^c_{ab},
\end{aligned}
\tag{4.3}
$$

where $\tilde{\Gamma}_{abc}$ and $\tilde{\Gamma}^c_{ab}$ are Christoffel symbols of type I and II, respectively, of $\tilde{V}_{n-1}$.

Consider a *special mapping* f between equidistant spaces V_n and $\overline{V}_n$ where the equdistant space V_n has a metric of the form (4.2) and the equdistant space $\overline{V}_n$ has an analogous metric

$$
\mathrm{d}\overline{s}^2 = A(x^1)\,(\mathrm{d}x^1)^2 + B(x^1)\,\mathrm{d}\hat{s}^2, \tag{4.4}
$$

with $A(x^1), B(x^1) \in C^1$ being non-zero functions and $\mathrm{d}\hat{s}^2 = \hat{g}_{ab}(x^2, \ldots, x^n)\,\mathrm{d}x^a\mathrm{d}x^b$ $(a, b = 2, \ldots, n)$ is the metric form of the Riemannian subspace $\hat{V}_{n-1}$.

The non-zero components of the Christoffel symbols for the metric form (4.4) are:

$$
\begin{aligned}
&\overline{\Gamma}_{111} = \frac{1}{2}\, A'(x^1); && \overline{\Gamma}_{1ab} = \overline{\Gamma}_{a1b} = \frac{1}{2}\, B'(x^1)\, \hat{g}_{ab};\\
&\overline{\Gamma}_{ab1} = -\frac{1}{2}\, B'(x^1)\, \hat{g}_{ab}; && \overline{\Gamma}_{abc} = B(x^1)\, \hat{\Gamma}_{abc};\\
&\overline{\Gamma}^1_{11} = \frac{1}{2}\, \frac{A'(x^1)}{A(x^1)}; && \overline{\Gamma}^c_{1a} = \overline{\Gamma}^c_{a1} = \frac{1}{2}\, \frac{B'(x^1)}{B(x^1)}\, \delta^b_a;\\
&\overline{\Gamma}^1_{ab} = -\frac{1}{2}\, \frac{B'(x^1)}{A(x^1)}\, \hat{g}_{ab}; && \overline{\Gamma}^c_{ab} = \hat{\Gamma}^c_{ab},
\end{aligned}
\tag{4.5}
$$

where $\hat{\Gamma}_{abc}$ and $\hat{\Gamma}^c_{ab}$ are Christoffel symbols of type I and II, respectively, of $\hat{V}_{n-1}$. The equdistant space V_n is defined by non-zero differentiable functions $a(x^1), b(x^1)$ and the metric $\mathrm{d}\tilde{s}^2$ of $\tilde{V}_{n-1}$. Analogically, the image of the metric under the special mapping f in $\overline{V}_n$ is defined by non-zero differentiable functions $A(x^1)$, $B(x^1)$, and the metric form $\mathrm{d}\hat{s}^2$ of $\hat{V}_{n-1}$.
Under this map the geodesic curves $\ell = \{(t, \overset{\circ}{x}{}^2, \ldots, \overset{\circ}{x}{}^n),\ t \in \mathbb{R}$ of V_n map into the geodesics of $\overline{V}_n$ and the othogonal surfaces on this geodesic congruence of V_n also map into the othogonal surfaces on the corresponding geodesic congruence of $\overline{V}_n$.

The deformation tensor $P^h_{ij}(x)$ of the mapping $f\colon V_n \to \overline{V}_n$, introduced in (4.1), has in this case the following form

$$\begin{aligned}
&P^1_{11} = \frac{1}{2}\left(\frac{A'(x^1)}{A(x^1)} - \frac{a'(x^1)}{a(x^1)}\right);\\
&P^1_{1a} = P^1_{a1} = P^c_{11} = 0;\\
&P^c_{1a} = P^c_{a1} = \frac{1}{2}\left(\frac{B'(x^1)}{B(x^1)} - \frac{b'(x^1)}{b(x^1)}\right)\delta^c_a;\\
&P^1_{ab} = -\frac{1}{2}\left(\frac{B'(x^1)}{A(x^1)}\hat{g}_{ab} - \frac{b'(x^1)}{a(x^1)}\tilde{g}_{ab}\right);\\
&P^c_{ab} = \hat{\Gamma}^c_{ab} - \tilde{\Gamma}^c_{ab}.
\end{aligned} \tag{4.6}$$

After these general considerations we will go into detail for several types of mappings in the following subsections.

4.3 Isometric and Conformal Mappings

Definition 11 The mapping of Riemannian spaces $f\colon V_n \to \overline{V}_n$ is ***isometric*** if it preserves the length of all arcs.

Isometric mappings are characterised by the condition $\overline{g} = g$. In the common coordinate system x with respect to the mapping f this writes as $\overline{g}_{ij}(x) = g_{ij}(x)$.

Definition 12 The mapping $f\colon V_n \to \overline{V}_n$ is ***conformal*** iff $\overline{g} = \mathrm{e}^{2\sigma(x)}g$, in common coordinates

$$\overline{g}_{ij}(x) = \mathrm{e}^{2\sigma(x)} g_{ij}(x), \tag{4.7}$$

where $\sigma(x)$ is a function on V_n.

Under conformal mappings the angles between curves are preserved. If $\sigma(x) = \text{const}$, these mappings are called ***homothetic***.

Under conformal mappings the following relation holds:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i\sigma_j + \delta^h_j\sigma_i - \sigma^h\, g_{ij}, \tag{4.8}$$

where $\sigma_i = \partial_i\sigma(x)$, $\sigma^h = \sigma_\alpha g^{\alpha h}$.
We can get condition (4.8) in the following way, first we write down the Christoffel

symbols of type I for the metric $\overline{g}$,

$$\overline{\Gamma}_{ijk} = \frac{1}{2}\left(\partial_i \overline{g}_{jk} + \partial_j \overline{g}_{ik} - \partial_k \overline{g}_{ij}\right),$$

now we insert (4.7)

$$\overline{\Gamma}_{ijk} = \frac{1}{2}\left(\partial_i(\mathrm{e}^{2\sigma(x)} g_{jk}) + \partial_j(\mathrm{e}^{2\sigma(x)} g_{ik}) - \partial_k(\mathrm{e}^{2\sigma(x)} g_{ij})\right)$$

and after simple computations we get

$$\overline{\Gamma}_{ijk} = \mathrm{e}^{2\sigma(x)}\left(\sigma_i g_{jk} + \sigma_j g_{ik} - \sigma_k g_{ij} + \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}\right)\right).$$

After using (2.15) this is

$$\overline{\Gamma}_{ijk} = \mathrm{e}^{2\sigma(x)}\left(\sigma_i g_{jk} + \sigma_j g_{ik} - \sigma_k g_{ij} + \Gamma_{ijk}\right),$$

and the Christoffel symbols of type II arise in the form (4.8).

Conformal mappings between equidistant spaces.

For the special case of equidistant spaces the following lemma restricts the functional dependence of the conformal factor ϱ.

Lemma 2 *The special mapping f between equidistant spaces V_n and $\overline{V}_n$ is conformal if and only if there exists a function $\varrho(x^1) \neq 0$ so that the metric of $\overline{V}_n$ has the following form*

$$\mathrm{d}\overline{s}^2 = \varrho(x^1)\,\mathrm{d}s^2 = \varrho(x^1)\left(a(x^1)\,(\mathrm{d}x^1)^2 + b(x^1)\,\mathrm{d}\tilde{s}^2\right).$$

4.4 Harmonic and Equivolume Mappings

Definition 13 A *harmonic map* [120, 121] is a smooth (C^∞) map $\varphi : V_m \to \overline{V}_n$ between Riemannian spaces which extremizes the Dirichlet or energy integral

$$E(\varphi; D) = \frac{1}{2}\int_D |\mathrm{d}\varphi|^2 \sqrt{\det g}\,\mathrm{d}^m x \tag{4.9}$$

with respect to variations of φ on compact domains D.

In local coordinates $(x^1, \ldots, x^m)$, $(y^1, \ldots, y^n)$, with metrics $g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$ in V_m and $\overline{g} = \overline{g}_{\alpha\beta}\mathrm{d}y^\alpha\mathrm{d}y^\beta$ in $\overline{V}_n$, φ maps a point in V_m to a point in $\overline{V}_n$ with coordinates $y^\alpha = \varphi^\alpha(x)$ and we have

$$E(\varphi; D) = \frac{1}{2}\int_D g^{ij}\,\overline{g}_{\alpha\beta}\,\frac{\partial\varphi^\alpha}{\partial x^i}\,\frac{\partial\varphi^\beta}{\partial x^j}\,\sqrt{\det g_{ps}}\;\mathrm{d}x^1\ldots\mathrm{d}x^m. \tag{4.10}$$

Variation with respect to φ leads to the Euler-Lagrange equations for a harmonic map

$$\tau(\varphi)^\gamma = g^{ij}\left(\frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} - \Gamma^k_{ij}\frac{\partial\varphi^\gamma}{\partial x^k} + \overline{\Gamma}{}^\gamma_{\alpha\beta}\frac{\partial\varphi^\alpha}{\partial x^i}\frac{\partial\varphi^\beta}{\partial x^j}\right) = 0. \tag{4.11}$$

$\tau(\varphi)^\gamma$ is called the tension field of φ. These equations are generalisations of the equations of motion of a relativistic point particle: when V_m is one-dimensional with a coordinate s, interpreted as its proper time, and y^α as space-time coordinates of the particle, then Γ^k_{ij} is equal to zero and (4.11) reduces to the geodesic equation. Analogously for a two dimensional V_m (4.10) has the form of the action integral in string theory, whose variation gives the worldsheet of a string, the analog of the worldline of a point particle. More generally, harmonic maps are extremal in relation to the energy functional of sigma models, as described by J. Wood [120, 121].

In the following we specialise to diffeomorphic mappings between spaces of equal dimension, ***harmonic diffeomorphisms***. In common coordinates

$$\frac{\partial\varphi^\alpha}{\partial x^i} = \delta^\alpha_i \qquad \text{and} \qquad \frac{\partial^2\varphi^\gamma}{\partial x^i\partial x^j} = 0$$

and then from (4.11) it follows that a diffeomorphism from V_n onto $\overline{V}_n$ is harmonic if and only if the following conditions hold [111]

$$\left(\overline{\Gamma}{}^h_{ij}(x) - \Gamma^h_{ij}(x)\right)\, g^{ij} = 0. \tag{4.12}$$

Harmonic mappings between equidistant spaces. For harmonic mappings we can rewrite (4.12) in the form $P^h_{\alpha\beta}\,g^{\alpha\beta} = 0$ leading to the following differential equations:

$$\begin{aligned}
&\text{(a)} \quad \frac{1}{a}\left(\frac{A'(x^1)}{A(x^1)} - \frac{a'(x^1)}{a(x^1)}\right) - \frac{1}{b}\left(\frac{B'(x^1)}{A(x^1)}\,\hat{g}_{ab}\tilde{g}^{ab} - (n-1)\,\frac{b'(x^1)}{a(x^1)}\right) = 0,\\
&\text{(b)} \quad \left(\hat{\Gamma}^c_{ab} - \tilde{\Gamma}^c_{ab}\right)\tilde{g}^{ab} = 0.
\end{aligned} \tag{4.13}$$

These equations are more complicated than the corresponding equations for geodesic mappings. We shall investigate special cases. First we look for solutions with $B = \mathrm{const}$. In this case (4.13a) simplifies to

$$\frac{A'(x^1)}{A(x^1)} - \frac{a'(x^1)}{a(x^1)} + (n-1)\frac{b'(x^1)}{b(x^1)} = 0.$$

Straightforward integration leads to

$$\ln\left|\frac{a(x^1)}{A(x^1)}\right| = \ln(b(x^1)^{n-1}C), \quad C = \mathrm{const},\ C \neq 0.$$

Now we can get $A(x^1)$ in the form:

$$A(x^1) = c \cdot a(x^1) b^{1-n}(x^1), \quad c = 1/C.$$

For this special case this prooves the following theorem.

Theorem 4 *The special mapping f between equidistant spaces V_n with a metric of the form (4.2) and $\overline{V}_n$ with the metric*

$$\mathrm{d}\bar{s}^2 = c \cdot a(x^1)\, b^{1-n}(x^1)\, (\mathrm{d}x^1)^2 + B\, \mathrm{d}\hat{s}^2, \ \textit{where} \quad c, B = \mathrm{const} \neq 0$$

is harmonic if and only if the mapping of the subspaces $\tilde{V}_{n-1}$ to $\hat{V}_{n-1}$ is harmonic. Moreover, if $\hat{g}_{ab}\tilde{g}^{ab} = \mathrm{const}$, then for an arbitrary function $B(x^1) \in C^1$ there is a function $A(x^1)$ satisfying the ordinary differential equation (4.13a).

Remark. If $\hat{V}_{n-1} = \tilde{V}_{n-1}$ (i.e. $\hat{g}_{ab} = \tilde{g}_{ab}$) then (4.13b) is satisfied automatically and $\hat{g}_{ab}\tilde{g}^{ab} = n-1$. With this assumption (4.13a) becomes

$$\frac{1}{a(x^1)}\left(\frac{A'(x^1)}{A(x^1)} - \frac{a'(x^1)}{a(x^1)}\right) - (n-1)\frac{1}{b(x^1)}\left(\frac{B'(x^1)}{A(x^1)} - \frac{b'(x^1)}{a(x^1)}\right) = 0.$$

In this case the family of harmonic mappings $f\colon V_n \to \overline{V}_n$ depends on the one function $A(x^1) \neq 0$, because for a concrete function $A(x^1)$ we can find $B(x^1)$ by simple integration of the equation

$$B'(x^1) = \left[\frac{b(x^1)}{(n-1)a(x^1)}\left(\frac{A'(x^1)}{A(x^1)} - \frac{a'(x^1)}{a(x^1)}\right) + \frac{b'(x^1)}{a(x^1)}\right] A(x^1).$$

For $A(x^1) = \frac{a(x^1)}{b(x^1)}$, for example, we obtain $B(x^1) = \frac{n-2}{n-1}\ln|b(x^1)| + C$.

Finally we consider *equivolume mappings*, which were defined and studied by T.V. Zudina and S.E. Stepanov [126]. These mappings $f\colon V_n \to \overline{V}_n$ are characterized by the following condition

$$\overline{\Gamma}^{\alpha}_{i\alpha}(x) = \Gamma^{\alpha}_{i\alpha}(x). \tag{4.14}$$

Equivolume mappings between equidistant spaces.

From the rewritten form $P^{\alpha}_{i\,\alpha} = 0$, of the necessary and sufficient condition (4.14) of equivolume mappings $V_n \to \overline{V}_n$, and with the help of (4.6) we obtain the following formulas

$$\frac{A'(x^1)}{A(x^1)} - \frac{a'(x^1)}{a(x^1)} + (n-1)\left(\frac{B'(x^1)}{B(x^1)} - \frac{b'(x^1)}{b(x^1)}\right) = 0; \quad \hat{\Gamma}^c_{ac} = \tilde{\Gamma}^c_{ac}\,. \tag{4.15}$$

By integration of the first equation we obtain

$$(n-1)\left(\ln|B(x^1)| - \ln|b(x^1)|\right) = \ln|a(x^1)| - \ln|A(x^1)| + \ln|\alpha|, \quad \alpha = \text{const.}$$

After exponentiating we get

$$\left|\frac{B(x^1)}{b(x^1)}\right|^{n-1} = \frac{\alpha a(x^1)}{A(x^1)}.$$

Now we can easily see that $A(x^1)$ is expressed by

$$A(x^1) = \alpha a(x^1)\left|\frac{b(x^1)}{B(x^1)}\right|^{n-1}.$$

After this analysis we have the following theorem:

Theorem 5 *The special non-affine mapping f between equidistant spaces V_n and $\overline{V}_n$ is equivolume if and only if $\tilde{V}_{n-1}$ admits an equivolume mapping on $\hat{V}_{n-1}$, and the metric of $\overline{V}_n$ has the following form*

$$\mathrm{d}\bar{s}^2 = \alpha\cdot a(x^1)\left(\frac{b(x^1)}{B(x^1)}\right)^{n-1}(\mathrm{d}x^1)^2 + B(x^1)\,\mathrm{d}\hat{s}^2, \tag{4.16}$$

where α is a non-zero constant, and $B(x^1)$ is a non-zero differentiable function.

4.5 Affine and Geodesic Mappings

First we introduce affine and geodesic mappings of spaces with affine connection.

Definition 14 Let A_n and $\overline{A}_n$ be two spaces with affine connections ∇ and $\overline{\nabla}$. The diffeomorphism f from A_n to $\overline{A}_n$ is called a ***geodesic mappings*** if it maps all geodesic curves of A_n onto geodesic curves in $\overline{A}_n$.

Moreover, if this mapping preserves the natural parameters of geodesic curves, then f is called ***affine***.

We remark that the foundations of geodesic mappings were laid by J. Lagrange [53] and E. Beltrami [8]. Of particular importance in the theory of geodesic mappings are three fundamental directions:
1. the study of general dependences of geodesic mappings,
2. the integration of fundamental equations,
3. the study of geodesic mappings of special manifolds.

The largest contribution to the study of general dependences of the theory of geodesic mappings was the work of T. Levi-Civita [58], T. Thomas [114], H. Weyl [119], A.Z. Petrov [83], A.S. Solodovnikov [109, 110], N.S. Sinyukov [103], J. Mikeš [64], V.S. Matveev [10], see [19, 20, 49, 64, 83, 80, 86, 90, 103, 124, 125].

N.S. Sinyukov [103] found invariant (with respect to the choice of the coordinate system), necessary and sufficient conditions of intrinsic character for the existence of geodesic mappings between Riemannian spaces. J. Mikeš and V.E. Berezovskij [66] solved these problems in the case of geodesic mappings of spaces with affine connections to Riemannian spaces. These conditions, nevertheless, have an implicit form. With respect to geodesic mappings the notion of the *degree of freedom* of Riemannian manifolds was defined. Practically this is the number of numeric parameters, on which the number of Riemannian manifolds $\overline{V}_n$, depends which map geodesically onto the given Riemannian manifold V_n, respectively A_n. A relation between the degree of freedom with respect to geodesic mappings and some isometric, homothetic and projective transformations was found [64].

The *second* direction in the theory of geodesic mappings, as said before, is the integration of fundamental equations. These questions are connected with the names U. Dini [14], T. Levi-Civita [58], P.A. Širokov [100], A.Z. Petrov, V.I. Golikov [83], A.V. Aminova, see [3, 4].

The *third* important direction is the study of geodesic mappings of special spaces, for example spaces with constant curvature, Einstein spaces, equidistant, symmetric, recurrent, semisymmetric spaces and their generalisations, see the monographs and referential books [3, 4, 19, 20, 49, 58, 64, 80, 83, 86, 103, 104, 109, 110, 117, 124].

For the sake of completeness we point out that the bibliografy of the theory of geodesic mappings and projective transformations contains more than thousand different papers and around 20 monographs and textbooks.

Fundamental equations of geodesic mappings.

It was proved that a necessary and sufficient condition for the existence of a geodesic mapping from A_n to $\overline{A}_n$, is:

$$(\overline{\nabla} - \nabla)_X X = 2\,\psi(X)\,X, \quad \forall X \in TA_n, \tag{4.17}$$

where ψ is a linear form on A_n.

In local coordinates the "coordinate-free" formula (4.17) introduced above is equivalent to the following equation, named after ***Levi-Civita***, who effectively proved it [58], see [19, 64, 83, 103, 124]:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \psi_i \delta^h_j + \psi_j \delta^h_i\,, \tag{4.18}$$

here $\psi_i(x)$ are the components of the linear form ψ.

To derive the relation (4.18) we assume that a given curve with tangent vector λ is a geodesic with respect to both the connections ∇ and $\overline{\nabla}$. The explicit forms of the geodesic equation (2.10) in terms of connection components Γ and $\overline{\Gamma}$ are

$$\frac{\mathrm{d}\lambda^h}{\mathrm{d}t} + \Gamma^h_{ij}(x(t))\,\lambda^i\lambda^j = \rho(t)\,\lambda^h \quad \text{and} \quad \frac{\mathrm{d}\lambda^h}{\mathrm{d}t} + \overline{\Gamma}^h_{ij}(x(t))\,\lambda^i\lambda^j = \overline{\rho}(t)\,\lambda^h.$$

Subtracting these two equations gives

$$\left(\overline{\Gamma}^h_{ij}(x(t)) - \Gamma^h_{ij}(x(t))\right)\ \lambda^i\lambda^j = \left(\overline{\rho}(t) - \rho(t)\right)\ \lambda^h.$$

After defining $\tilde{\rho}(t) = \overline{\rho}(t) - \rho(t)$ and multiplying by λ^l we obtain

$$\left(\overline{\Gamma}^h_{ij}(x(t)) - \Gamma^h_{ij}(x(t))\right)\ \lambda^i\lambda^j\lambda^l = \tilde{\rho}(t)\ \lambda^h\lambda^l.$$

Now we antisymmetrise with respect to h and l:

$$\left(\overline{\Gamma}^h_{ij}(x(t)) - \Gamma^h_{ij}(x(t))\right)\ \lambda^i\lambda^j\lambda^l - \left(\overline{\Gamma}^l_{ij}(x(t)) - \Gamma^l_{ij}(x(t))\right)\ \lambda^i\lambda^j\lambda^h = 0.$$

In terms of the deformation tensor (4.1) this becomes

$$\left(P^h_{ij}\lambda^l - P^l_{ij}\lambda^h\right)\ \lambda^i\lambda^j = 0$$

and making use of $\lambda^l = \lambda^r\delta^l_r$ and $\lambda^h = \lambda^r\delta^h_r$ we arrive at

$$\left(P^h_{ij}\delta^l_r - P^l_{ij}\delta^h_r\right)\ \lambda^i\lambda^j\lambda^r = 0.$$

Becauce $\lambda^i\lambda^j\lambda^r$ is totally symmetric in the indices, the symmetrisation of the expression in paranthesis, denoted in the usual way by round brackets around the indices, is equal to zero,

$$P^h_{(ij}\delta^l_{r)} - P^l_{(ij}\delta^h_{r)} = 0.$$

For $l = r$ we have

$$P^h_{ij}\delta^l_l - P^l_{ij}\delta^h_l + P^h_{jl}\delta^l_i - P^l_{jl}\delta^h_i + P^h_{li}\delta^l_j - P^l_{li}\delta^h_j = 0,$$

and after some calculations

$$P^h_{ij} = \frac{P^l_{jl}}{n+1}\delta^h_i + \frac{P^l_{li}}{n+1}\delta^h_j.$$

Introducing the covector $\quad \psi_i = \dfrac{P^l_{li}}{n+1} \quad$ we obtain the relation (4.18) between the connections.

A geodesic mapping is affine iff $\psi \equiv 0$. This means that under affine mappings the components of the affine connection are preserved:

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x). \tag{4.19}$$

We point out that in the case of geodesic mappings of the space A_n to the Riemannian space $\overline{V}_n$ with the metric tensor $\overline{g}_{ij}$ the equations (4.18) are equivalent to the following equations, which also bear the name of Levi-Civita:

$$\overline{g}_{ij,k} = 2\psi_k\overline{g}_{ij} + \psi_i\overline{g}_{jk} + \psi_j\overline{g}_{ik}, \tag{4.20}$$

here “ , ” denotes the covariant derivative in the space A_n.

Remark about possible applications of geodesic mappings: Recall that geodesics are the trajectories of test particles in a gravitational field. For this reason one can argue that the geometric quantities of two spaces V_n and $\overline{V}_n$, which are related by a geodesic mapping, describe the same gravitational field. In other words, geodesic mappings could be mere gauge transformations between different geometric realisations of one and the same physical situation. General relativity, on the other hand, is not invariant under geodesic mappings, but there are recent attempts for geodisically invariant modifications [116].

Affine mappings between equidistant spaces.

An affine mapping $f\colon\ V_n \to \overline{V}_n$ is characterized by the condition $P^h_{ij} = 0$. From

(4.6) follows

$$\frac{1}{2}\left(\frac{A'(x^1)}{A(x^1)}-\frac{a'(x^1)}{a(x^1)}\right)=0 \;\Rightarrow\; \frac{A'(x^1)}{A(x^1)}=\frac{a'(x^1)}{a(x^1)} \;\Rightarrow$$
$$\Rightarrow\; \ln|A(x^1)|=\ln|a(x^1)|+\ln|\alpha| \;\Rightarrow\; A(x^1)=\alpha\cdot a(x^1)$$

and

$$\frac{1}{2}\left(\frac{B'(x^1)}{B(x^1)}-\frac{b'(x^1)}{b(x^1)}\right)\delta^c_a=0 \;\Rightarrow\; \ln|B(x^1)|=\ln|b(x^1)|+\ln|\beta| \;\Rightarrow\; B=\beta\cdot b(x^1),$$

where $\alpha,\beta=\mathrm{const}\neq 0$. After the detailed analysis we obtain following lemma.

Lemma 3 *The special mapping f between the equidistant spaces V_n and $\overline{V}_n$ is affine if and only if*

1) the metric of $\overline{V}_n$ has the following form $\mathrm{d}\overline{s}^2=\mathrm{const}\cdot\mathrm{d}s^2$, i.e. f is homothetic, and

2) in the case $b(x^1)=\mathrm{const}$, the metric of $\overline{V}_n$ has the following form

$$\mathrm{d}\overline{s}^2=\alpha\, a(x^1)\,(\mathrm{d}x^1)^2+\mathrm{d}\hat{s}^2,$$

and the space $\hat{V}_{n-1}$ with metric $\mathrm{d}\hat{s}^2$ is affine to $\tilde{V}_{n-1}$.

Geodesic mappings between equidistant spaces.

Rewriting the necessary and sufficient condition (4.18) of the geodesic mapping $V_n \to \overline{V}_n$ in terms of the deformation tensor in the form $P^h_{ij}=\psi_i\delta^h_j+\psi_j\delta^h_i$, we obtain

$$\begin{array}{lllll}
P^1_{11} &=& \frac{1}{2}\left(\frac{A'(x^1)}{A(x^1)}-\frac{a'(x^1)}{a(x^1)}\right) &=& \psi_1\delta^1_1+\psi_1\delta^1_1=2\psi_1; \;\Rightarrow\; \frac{A'(x^1)}{A(x^1)}-\frac{a'(x^1)}{a(x^1)}=4\psi_1\\
P^c_{1a} &=& \frac{1}{2}\left(\frac{B'(x^1)}{B(x^1)}-\frac{b'(x^1)}{b(x^1)}\right)\delta^c_a &=& \psi_a\delta^c_1+\psi_1\delta^c_a=\psi_1\delta^c_a; \;\Rightarrow\; \frac{B'(x^1)}{B(x^1)}-\frac{b'(x^1)}{b(x^1)}=2\psi_1\\
P^1_{ab} &=& -\frac{1}{2}\left(\frac{B'(x^1)}{A(x^1)}\hat{g}_{ab}-\frac{b'(x^1)}{a(x^1)}\tilde{g}_{ab}\right) &=& \psi_a\delta^1_b+\psi_b\delta^1_a=0; \;\Rightarrow\; \frac{B'(x^1)}{A(x^1)}-\frac{b'(x^1)}{a(x^1)}=0\\
P^c_{ab} &=& \hat{\Gamma}^c_{ab}-\tilde{\Gamma}^c_{ab} &=& \psi_a\delta^c_b+\psi_b\delta^c_a=0; \;\Rightarrow\; \psi_a=0.
\end{array} \tag{4.21}$$

Now we can analyse these equations. After integration of the first and the second one we get

$$\psi=\frac{1}{4}\ln\left|\frac{C_1A(x^1)}{a(x^1)}\right| \quad \text{and} \quad \psi=\frac{1}{2}\ln\left|\frac{C_2B(x^1)}{b(x^1)}\right|,$$

respectively, where C_1 and C_2 are integration constants. Comparing these two relations we obtain

$$A(x^1) = \frac{C_2^2 B(x^1)^2 a(x^1)}{C_1 b^2(x^1)}$$

and inserting this into the third equation of (4.21) results in

$$\frac{C_1 B'(x^1)}{C_2^2 B^2(x^1)} = \frac{b'(x^1)}{b^2(x^1)},$$

which can easily be integrated to give

$$-\frac{C_1}{C_2^2 B(x^1)} = -\frac{1}{b(x^1)} + C_3.$$

From this we can express

$$B(x^1) = \frac{p \cdot b(x^1)}{(1 + b(x^1) \cdot q)},$$

where we have introduced the constants $p = C_1/C_2^2$ and $q = C_3$. Now we can express also $A(x^1)$ in terms of $a(x^1)$ and $b(x^1)$:

$$A(x^1) = \frac{p \cdot a(x^1)}{(1 + q \cdot b(x^1))^2}.$$

Finally we also find the function ψ in the form:

$$\psi = -\frac{1}{2} \ln \left| \frac{C_2}{C_1}(1 + q \cdot b(x^1)) \right|.$$

Because the function ψ is defined only up to an additive constant, we can write:

$$\psi = -\frac{1}{2} \ln \left| 1 + q \cdot b(x^1) \right|.$$

Now we are in the position to formulate the following theorem.

Theorem 6 *The special mapping f between equidistant spaces V_n and $\overline{V}_n$ is non-trivially geodesic if and only if $\hat{V}_{n-1}$ is homothetic to $\tilde{V}_{n-1}$, and the metric of $\overline{V}_n$ has the following form*

$$d\bar{s}^2 = \frac{p\, a(x^1)}{(1 + q\, b(x^1))^2} (dx^1)^2 + \frac{p\, b(x^1)}{1 + q\, b(x^1)} d\tilde{s}^2, \tag{4.22}$$

where p, q are some constants such that $p \neq 0$, $1 + q\, b(x^1) \neq 0$, and $q\, b'(x^1) \not\equiv 0$. From this follows that the function, which defines the geodesic mapping, has the form $\psi = -\frac{1}{2} \ln |1 + q\, b(x^1)|$.

4.6 Petrov's Hypothesis on Geodesic Mappings of Einstein Spaces

First we formulate some known results concerning geodesic mappings of Einstein spaces. Let us begin with geodesic mappings of spaces of constant curvature, which represent a special case of Einstein spaces and which, in 1865 [8], were the initial objects which the history of geodesic mappings began with.

A theorem by E. Beltrami in modern fomulation states that a Riemannian space V_n admitting a geodesic mapping onto a Euclidean spaces, is a space with constant curvature. The proofs of this theorem (see in the papers [19, 58, 86, 103]) are given under the condition $V_n \in C^2$, i.e. $g_{ij}(x) \in C^2$.

Locally it holds that between two spaces with constant curvatures K and $\overline{K}$, respectively, there exists a nontrivial geodesic mapping, where for the tensor $\psi_{ij} = \psi_{i,j} - \psi_i \psi_j$ the formula $\psi_{ij} = K g_{ij} - \overline{K}\overline{g}_{ij}$ holds [86].

Remark: Compact flat spaces do not admit global nontrivial geodesic mappings, the same applies for compact properly Riemannian spaces with constant negative curvature. There are no global geodesic mappings between compact spaces with constant curvature and with different signatures of the metrics.

Geodesic mappings of Einstein spaces.

The studies of geodesic mappings of Lorentzian four-dimensional Einstein spaces were initiated in 1961 by A.Z. Petrov, see [83]. A space V_n is called *Lorentzian*, if it has a metric with Minkowski signature. The following holds

- V.I. Golikov and A.Z. Petrov, see [83]: Lorentzian four-dimensional Einstein spaces with nonconstant curvature do not admit nontrivial geodesic mappings onto Lorentzian Riemannian spaces.
- J. Mikeš and V.A. Kiosak [69]: Four-dimensional Einstein spaces with non-constant curvature do not admit nontrivial geodesic mappings to Riemannian spaces.
- J. Mikeš [61]: If the Einstein space $\mathcal{E}_n$ admits a nontrivial geodesic mapping onto the Riemannian space $\overline{V}_n$, then $\overline{V}_n$ is an Einstein space.

A.Z. Petrov's hypothesis.

A.Z. Petrov extended methods of studying geodesic mappings of four-dimensional Lorentzian Einstein spaces to Einstein spaces of higher dimensions $n > 4$, and conjectured that *Lorentzian Einstein spaces $\mathcal{E}_n$ $(n > 4)$ which do not have* *constant curvature, do not admit nontrivial geodesic mappings onto Lorentzian-Einstein spaces* ([83], pp. 355, 461).

Counterexamples to this hypothesis were found by Kiosak and Mikeš, see [46, 69]. The construction of these examples is complicated.

A simple counterexample, found by myself [67], is presented in the following. Recall that Theorem 6 describes a method to construct nontrivial geodesic mappings between equidistant spaces. Choose an equidistant Einstein space $\mathcal{E}_n$ with the metric (2.34)

$$\mathrm{d}s^2 = \frac{1}{f(x^1)}\,(\mathrm{d}x^1)^2 + f(x^1)\,\mathrm{d}\tilde{s}^2.$$

According to the mentioned theorem this space can be geodesically mapped to a space $\overline{\mathcal{E}}_n$ with metric

$$\mathrm{d}\overline{s}^2 = \frac{p}{f(x^1)\,\big(1+q\,f(x^1)\big)^2}\,(\mathrm{d}x^1)^2 + \frac{p\,f(x^1)}{1+q\,f(x^1)}\,\mathrm{d}\tilde{s}^2,$$

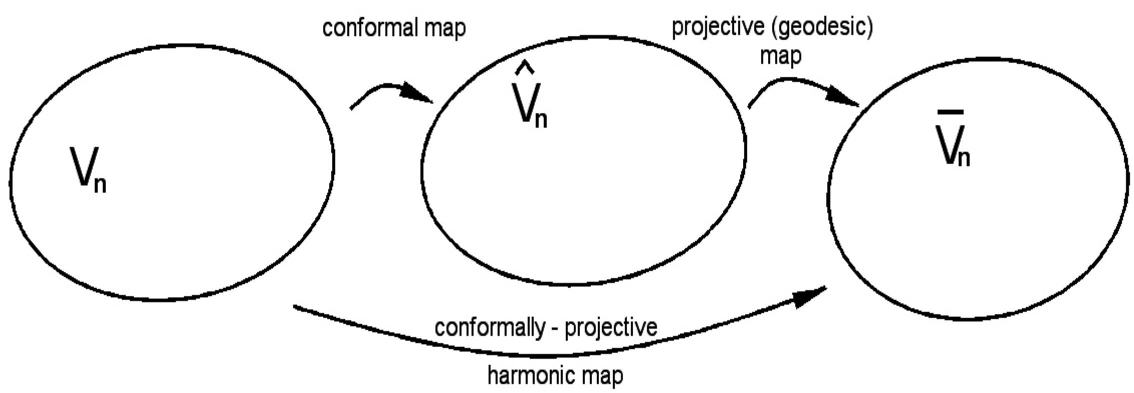
where $f(x^1) = K(x^1)^2 + 2ax^1 + b$. $\mathrm{d}\tilde{s}^2$ is a metric of an Einstein space $\tilde{\mathcal{E}}_{n-1}$, assumed with nonconstant curvature, $\frac{\tilde{R}}{(n-1)(n-2)} = b \cdot \frac{R}{n(n-1)} - a^2$, R and $\tilde{R}$ are the scalar curvatures of $\mathcal{E}_n$ and $\tilde{\mathcal{E}}_{n-1}$. The constants p and q satisfy the conditions $p > 0$, $q > 0$, and $1 + q\,f(x^1) > 0$. Furthermore we assume that the metric $\mathrm{d}\tilde{s}^2$ is negative definite.

Evidently both the above metrics have Minkowski signature and, on the other hand, they have common geodesic curves with respect to the common coordinate system $(x^1, \ldots, x^n)$. In other words, $\mathcal{E}_n$ maps geodesically to $\overline{\mathcal{E}}_n$. This mapping is nontrivial, which is in contradiction with Petrov's hypothesis.

Remark: When $1 + q\,f(x^1) < 0$, the metrics of $\mathcal{E}_n$ and $\overline{\mathcal{E}}_n$ have different signatures; for $n = 4$ they have constant curvature.

5 Conformally-Projective Harmonic Diffeomorphisms

This chapter is devoted to a further kind of diffeomorphisms, which is of special interest. In the study of their properties we obtained results published in the papers [32, 33, 34] and which we are going to present in detail in the sequel. The interest was motivated by the article by S.E. Stepanov and I.G. Shandra [111], in which they studied harmonic diffeomorfisms. In their paper they also considered compositions of conformal and geodesic mappings, which are at the same time harmonic.



Here in the first part we will apply this type of mapping to equidistant manifolds and show again the Friedmann cosmological models as an explicit example. In the second part we will investigate the conditions of their existence in form of differential equations and find their solutions.

5.1 Definitions and General Properties

First we give the basic notions concerning these maps.

Definition 15 We call a composition of a conformal and a geodesic (projective) diffeomorphism between Riemannian spaces, which is harmonic, *conformally-projective harmonic.*

Necessary and sufficient conditions for the existence of these mappings are given in [111]. A diffeomorphism from an n-dimensional Riemannian space V_n onto a Riemannian space $\overline{V}_n$ is conformally-projective harmonic if and only if the following conditions hold

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \varphi_i\delta^h_j + \varphi_j\delta^h_i - \frac{2}{n}\,\varphi^h g_{ij}, \tag{5.1}$$

where $\varphi_i = \partial_i\varphi(x)$ is a gradient-like vector, $\varphi^h = g^{h\alpha}\varphi_\alpha$.

Proof. The conditions (5.1) are derived in the following way. The solutions in common coordinates, which are characterizing conformal mappings from V_n onto $\hat{V}_n$ and geodesic mappings between $\hat{V}_n$ and $\overline{V}_n$ have the form

$$\hat{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \delta^h_i\sigma_j + \delta^h_j\sigma_i - \sigma^h\, g_{ij},$$

$$\overline{\Gamma}^h_{ij}(x) = \hat{\Gamma}^h_{ij}(x) + \psi_i\delta^h_j + \psi_j\delta^h_i\,,$$

where ψ_i, σ_j are covectors and $\sigma^h = \sigma_\alpha g^{\alpha h}$. These equations are the analogs of (4.8) and (4.18). After subtracting them we get

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + (\psi_j + \sigma_j)\delta^h_i + (\psi_i + \sigma_i)\delta^h_j - \sigma^h g_{ij}.$$

We denote $\psi_i + \sigma_i = \varphi_i$ and can write

$$\overline{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + \varphi_j\delta^h_i + \varphi_i\delta^h_j - \sigma^h g_{ij}. \tag{5.2}$$

If we assume that the considered combination of a conformal and a projective mapping is harmonic, too, then from (4.12) we have

$$g^{ij}\,\left(\overline{\Gamma}^h_{ij} - \Gamma^h_{ij}\right) = g^{ij}\left(\varphi_j\delta^h_i + \varphi_i\delta^h_j - \sigma^h g_{ij}\right) = g^{hj}\varphi_j + \varphi_i g^{ih} - \sigma^h n = 0.$$

From this we can express $\sigma^h = \frac{2}{n}\varphi_k g^{hk}$, insert back into (5.2) and obtain (5.1). ■

For the Riemann tensors of V_n and $\overline{V}_n$ the formula

$$\overline{R}^h_{ijk} = R^h_{ijk} + P^h_{ik,j} - P^h_{ij,k} + P^\alpha_{ik}P^h_{\alpha j} - P^\alpha_{ij}P^h_{\alpha k}$$

holds [103], where $P^h_{ij}(x)$ is the deformation tensor of the diffeomorphism $f\colon V_n \to \overline{V}_n$ defined in (4.1). Then from formula (5.1) it follows that the Riemann tensors of V_n and $\overline{V}_n$ are connected by the condition

$$\begin{aligned}\overline{R}^h_{ijk} = R^h_{ijk} + \delta^h_k(\varphi_{i,j} - \varphi_i\varphi_j) - \delta^h_j(\varphi_{i,k} - \varphi_i\varphi_k)\ +\\ +\tfrac{2}{n}(\varphi^h_{,k} - \tfrac{2}{n}\varphi^h\varphi_k + \varphi_\alpha\varphi^\alpha\delta^h_k)\,g_{ij} - \tfrac{2}{n}(\varphi^h_{,j} - \tfrac{2}{n}\varphi^h\varphi_j + \varphi_\alpha\varphi^\alpha\delta^h_j)\,g_{ik}.\end{aligned} \tag{5.3}$$

By analysis of formula (5.1) the following theorem can be shown, giving the conditions of existence of conformally-projective harmonic mappings in a different form.

Theorem 7 *A necessary and sufficient condition for $f\colon V_n \to \overline{V}_n$ to be conformally-projective harmonic is*

$$\overline{g}_{ij,k} = 2\varphi_k \overline{g}_{ij} + \varphi_i \overline{g}_{jk} + \varphi_j \overline{g}_{ik} - \tfrac{2}{n}\,(\overline{\varphi}_i g_{jk} + \overline{\varphi}_j g_{ik}), \tag{5.4}$$

where $\overline{g}_{ij}$ are components of the metric tensor of $\overline{V}_n$, $\overline{\varphi}_i = \varphi^\alpha \overline{g}_{\alpha i}$.

Proof. Let V_n admit conformally-projective harmonic mappings onto $\overline{V}_n$. We compute the covariant derivative of $\overline{g}_{ij,k}$. (The covariant derivative of the metric $\overline{g}$ with respect to the metric connection derived from g is not equal to zero.) Taking into account

$$\overline{g}_{ij,k} = \partial_k \overline{g}_{ij} - \overline{g}_{i\alpha}\Gamma^\alpha_{jk} - \overline{g}_{j\alpha}\Gamma^\alpha_{ik}, \tag{5.5}$$

and using formulas (5.1), after same elementary computations we get

$$\overline{g}_{ij,k} = \partial_k \overline{g}_{ij} - \overline{\Gamma}_{jki} - \overline{\Gamma}_{ikj} + \overline{g}_{ik}\varphi_j + \overline{g}_{jk}\varphi_i + 2\varphi_k \overline{g}_{ij} - \frac{2}{n}\left(\overline{g}_{i\alpha}\varphi^\alpha g_{jk} + \overline{g}_{j\alpha}\varphi^\alpha g_{ik}\right).$$

Then we can express $\overline{\Gamma}_{jki} + \overline{\Gamma}_{ikj} = \partial_k \overline{g}_{ij}$ and $\overline{\varphi}_i = \overline{g}_{i\alpha}\varphi^\alpha$ and we make sure that (5.4) is satisfied.

In the opposite direction we can verify that (5.1) follows from (5.4) by using (5.5) and

$$\partial_k \overline{g}_{ij} = \overline{g}_{i\alpha}\overline{\Gamma}^\alpha_{jk} + \overline{g}_{j\alpha}\overline{\Gamma}^\alpha_{ik},$$

following from the fact that the covariant derivative of $\overline{g}$ in the sense of the connection derived from $\overline{g}$ is identical to zero. Inserting the last relation into (5.5) gives

$$\overline{g}_{ij,k} = \overline{g}_{i\alpha}\overline{\Gamma}^\alpha_{jk} + \overline{g}_{j\alpha}\overline{\Gamma}^\alpha_{ik} - \overline{g}_{i\alpha}\Gamma^\alpha_{jk} - \overline{g}_{j\alpha}\Gamma^\alpha_{ik}.$$

Using the deformation tensor (4.1) we obtain

$$\overline{g}_{ij,k} = P^l_{ik}\overline{g}_{jl} + P^l_{jk}\overline{g}_{il}.$$

This relation is symmetric in i and j. The antisymmetric analog of the right-hand side is given by

$$P^l_{ik}\overline{g}_{jl} - P^l_{jk}\overline{g}_{il} = \overline{g}_{kj,i} - \overline{g}_{ki,j}.$$

Combining these two relations gives

$$P^l_{ik}\overline{g}_{jl} = \frac{1}{2}\left(\overline{g}_{ij,k} + \overline{g}_{kj,i} - \overline{g}_{ki,j}\right).$$

Now we use (5.4) and after some calculations and contraction with $\overline{g}^{hj}$ we end up at (5.1). This leads to the conclusion that (5.4) and (5.1) are equivalent; when one of them is satisfied, the mapping $f\colon V_n \to \overline{V}_n$ is conformally-projective harmonic. ∎

5.2 On the Equations of Conformally-Projective Harmonic Mappings

In this part we study the equations of conformally-projective harmonic mappings. We obtained the fundamental equations of these mappings in form of a system of first order differential equations in covariant derivatives with initial conditions, that is a Cauchy problem. In the following we will make use of the existence and uniqueness of the solutions of Cauchy type problems in covariant derivatives. The corresponding theory, which is analogous to the standard theory of Cauchy problems, is presented in the appendix. In the studied case the solutions depend on at most $\frac{1}{2}(n+1)(n+2)$ independent parameters. Now we derive the equations characterising conformally-projective harmonic diffeomorphisms.

5.2.1 The Integrability Conditions of the Main Equations

First we study the integrability conditions of the main equations (5.4) of the conformally-projective harmonic mapping V_n onto $\overline{V}_n$, which have the following form

$$\overline{g}_{ij,kl} - \overline{g}_{ij,lk} = \overline{g}_{i\alpha} R^{\alpha}_{jkl} + \overline{g}_{j\alpha} R^{\alpha}_{ikl}.$$

This condition is equivalent to the form

$$\overline{g}_{i\alpha} \overline{R}^{\alpha}_{jkl} + \overline{g}_{j\alpha} \overline{R}^{\alpha}_{ikl} = 0.$$

We use formula (5.3) and find these integrability conditions in the following form

$$\overline{g}_{i\alpha} R^{\alpha}_{jkl} + \overline{g}_{j\alpha} R^{\alpha}_{ikl} + \overline{g}_{il}\varphi_{jk} + \overline{g}_{jl}\varphi_{ik} - \overline{g}_{ik}\varphi_{jl} - \overline{g}_{jk}\varphi_{il} + \tfrac{2}{n}(\overline{\varphi}_{il} g_{jk} + \overline{\varphi}_{jl} g_{ik} - \overline{\varphi}_{ik} g_{jl} - \overline{\varphi}_{jk} g_{il}) = 0, \tag{5.6}$$

where

$$\varphi_{ij} = \varphi_{i,j} - \varphi_i\varphi_j + \tfrac{2}{n}\,\varphi^{\alpha}\varphi_{\alpha}\, g_{ij} \quad \text{and} \quad \overline{\varphi}_{ij} = \overline{g}_{i\alpha}(\varphi^{\alpha}_{,j} - \tfrac{2}{n}\,\varphi^{\alpha}\varphi_j). \tag{5.7}$$

By cyclic permutation of the indices j, k and l in (5.6) we obtain, thanks to the symmetry of the tensors g_{ij}, $\overline{g}_{ij}$ and φ_{ij}, on basis of the Ricci identity of the Riemann tensor, the following condition

$$\overline{g}_{j\alpha} R^{\alpha}_{ikl} + \overline{g}_{k\alpha} R^{\alpha}_{ilj} + \overline{g}_{l\alpha} R^{\alpha}_{ijk} + \frac{2}{n}((\overline{\varphi}_{lk} - \overline{\varphi}_{kl}) g_{ij} + (\overline{\varphi}_{jl} - \overline{\varphi}_{lj}) g_{ik} + (\overline{\varphi}_{kj} - \overline{\varphi}_{jk}) g_{il}) = 0. \tag{5.8}$$

The last formula is contracted with g^{il}:

$$\overline{\varphi}_{jk} - \overline{\varphi}_{kj} = \frac{n}{2(n-2)}(\overline{g}_{k\alpha}R^\alpha_j - \overline{g}_{j\alpha}R^\alpha_k),$$

where $R^h_i = g^{h\alpha}R_{\alpha i}$, and by replacing $\overline{\varphi}_{jk} - \overline{\varphi}_{kj}$ in (5.8) we obtain an expression used later:

$$\overline{g}_{j\alpha}Z^\alpha_{ikl} + \overline{g}_{k\alpha}Z^\alpha_{ilj} + \overline{g}_{l\alpha}Z^\alpha_{ijk} = 0, \tag{5.9}$$

where

$$Z^h_{ijk} = R^h_{ijk} - \frac{1}{n-2}(g_{ij}R^h_k - g_{ik}R^h_j). \tag{5.10}$$

Further we reformulate (5.6) in the following way. We alternate (5.6) with respect to the indices i and k. In the obtained expression we exchange the indices j and k. Then we add this expression to (5.6) and after some suitable transformations we obtain

$$\begin{aligned}&\overline{g}_{il}\varphi_{jk} - \overline{g}_{jk}\varphi_{il} + \tfrac{2}{n}(\overline{\varphi}_{il}g_{jk} - \overline{\varphi}_{jk}g_{il}) = \tfrac{1}{2}\{\overline{g}_{j\alpha}R^\alpha_{lik} + \overline{g}_{k\alpha}R^\alpha_{lij} - \overline{g}_{i\alpha}(R^\alpha_{jkl} + R^\alpha_{kjl})\}+\\ &\tfrac{1}{2(n-2)}\{g_{il}(\overline{g}_{j\alpha}R^\alpha_k - \overline{g}_{k\alpha}R^\alpha_j) - g_{jl}(\overline{g}_{i\alpha}R^\alpha_k - \overline{g}_{k\alpha}R^\alpha_i) - g_{kl}(\overline{g}_{i\alpha}R^\alpha_j - \overline{g}_{j\alpha}R^\alpha_i)\}.\end{aligned} \tag{5.11}$$

It can be shown that these conditions are equivalent to the integrability conditions (5.6) of equations (5.4).

5.2.2 The Main Equations in Cauchy-Type Form

We shall further study the integrability conditions of the equations (5.4), which are, as shown above, equivalent to formulae (5.11). First we restrict ourselves to the case

$$\text{Rank}\,\|\overline{g}_{ij} - \alpha g_{ij}\| > 1, \tag{5.12}$$

where α is a function. Based on (5.7), equation (5.11) can be written in the following form

$$\overline{g}_{il}\varphi_{j,k} - \overline{g}_{jk}\varphi_{i,l} + \frac{2}{n}(\overline{g}_{i\alpha}\varphi^\alpha_{,l}g_{jk} - \overline{g}_{j\alpha}\varphi^\alpha_{,k}g_{il}) = \overset{1}{T}_{ijkl}(\overline{g},\varphi), \tag{5.13}$$

where

$$\begin{aligned}\overset{1}{T}_{ijkl}(\overline{g},\varphi) = {}&\tfrac{1}{2}\{\overline{g}_{j\alpha}R^\alpha_{lik} + \overline{g}_{k\alpha}R^\alpha_{lij} - \overline{g}_{i\alpha}(R^\alpha_{jkl} + R^\alpha_{kjl})\}+\\ &\tfrac{1}{2(n-2)}\{g_{il}(\overline{g}_{j\alpha}R^\alpha_k - \overline{g}_{k\alpha}R^\alpha_j) - g_{jl}(\overline{g}_{i\alpha}R^\alpha_k - \overline{g}_{k\alpha}R^\alpha_i)-\\ &g_{kl}(\overline{g}_{i\alpha}R^\alpha_j - \overline{g}_{j\alpha}R^\alpha_i)\} + \overline{g}_{il}(\varphi_j\varphi_k - \tfrac{2}{n}\varphi^\alpha\varphi_\alpha g_{jk})-\\ &\overline{g}_{jk}(\varphi_i\varphi_l - \tfrac{2}{n}\varphi^\alpha\varphi_\alpha g_{il}) + \tfrac{4}{n^2}(\overline{\varphi}_i\varphi_l g_{jk} - \overline{\varphi}_j\varphi_k g_{il}).\end{aligned}$$

Note that the tensor $\overset{1}{T}$ is algebraically composed from components of the tensors $\overline{g}_{ij}$ and φ_i and also from objects of the Riemannian space V_n, determined by its metric tensor. In the sequel, by $\overset{\sigma}{T}(\overline{g},\varphi)$ we shall denote tensors having a similar construction.

From (5.13) the tensor $\varphi_{j,k}$ can be uniquely expressed in the form

$$\varphi_{j,k} = \overline{\mu} g_{jk} + \mu \overline{g}_{jk} + \overset{2}{T}_{jk}(\overline{g},\varphi), \tag{5.14}$$

where μ and $\overline{\mu}$ are functions. This assertion follows from the following lemma:

Lemma 4 *The homogenous linear algebraic equation*

$$\overline{g}_{il}\Phi_{jk} - \overline{g}_{jk}\Phi_{il} + \frac{2}{n}(\overline{g}_{i\alpha}g^{\alpha\beta}\Phi_{\beta l}g_{jk} - \overline{g}_{j\alpha}g^{\alpha\beta}\Phi_{\beta k}g_{il}) = 0 \tag{5.15}$$

has solutions in the form $\Phi_{jk} = \overline{\mu}g_{jk} + \mu\overline{g}_{jk}$. *When* $\overline{g}_{ij} \neq \alpha g_{ij}$, *this solution is unique.*

Proof. When $\overline{g}_{ij} \neq \alpha g_{ij}$, then there is a tensor ω^{ij} with $\omega^{ij}\overline{g}_{ij} = 1$ and $\omega^{ij}g_{ij} = 0$. By convolving (5.15) with ω^{jk} we obtain uniquely

$$\Phi_{jk} = \overline{\mu}g_{jk} + \mu\overline{g}_{jk}.$$

In the case $\overline{g}_{ij} = \alpha g_{ij}$ formula (5.15) simplifies:

$$\alpha\left(1 - \frac{2}{n}\right)(g_{il}\Phi_{jk} - g_{jk}\Phi_{il}) = 0.$$

Because $\alpha \neq 0$ it follows that $\Phi_{jk} = \mu g_{jk}$. ■

With the aid of (5.14) we eliminate $\varphi_{j,k}$ from (5.13):

$$\overline{\mu}\frac{n+2}{n}(\overline{g}_{il}g_{jk} - g_{il}\overline{g}_{jk}) + \frac{2}{n}\mu(\overline{g}_{i\alpha}g^{\alpha\beta}\overline{g}_{\beta l}g_{jk} - \overline{g}_{j\alpha}g^{\alpha\beta}\overline{g}_{\beta k}g_{il}) = \overset{3}{T}_{ijkl}(\overline{g},\varphi).$$

Evidently for $\overline{g}_{ij} \neq \alpha g_{ij}$, from this it follows that

$$\overline{\mu} = \mu T(\overline{g}) + \overset{4}{T}(\overline{g},\varphi), \text{ where } T(\overline{g}) = \frac{2}{n+2}\,\frac{g_{\alpha\beta}\overline{g}^{\alpha\beta}\,\overline{g}_{\gamma\delta}\overline{g}_{\varepsilon\eta}g^{\gamma\eta}g^{\delta\varepsilon} - n\overline{g}_{\alpha\beta}g^{\alpha\beta}}{n^2 - \overline{g}_{\alpha\beta}g^{\alpha\beta}\,\overline{g}^{\gamma\delta}g_{\gamma\delta}}.$$

Then (5.14) has the following form

$$\varphi_{j,k} = \mu(\overline{g}_{jk} + T(\overline{g})\,g_{jk}) + \overset{5}{T}_{jk}(\overline{g},\varphi). \tag{5.16}$$

The integrability condition of (5.16) has the form

$$\varphi_\alpha R^\alpha_{ijk} = \mu_{,k}(\overline{g}_{ij} + T(\overline{g})g_{ij}) - \mu_{,j}(\overline{g}_{ik} + T(\overline{g})g_{ik}) + \overset{6}{T}_{ijk}(\overline{g}, \varphi, \mu). \tag{5.17}$$

In the case (5.12) from (5.17) follows

$$\mu_{,k} = \overset{7}{T}_k(\overline{g}, \varphi, \mu). \tag{5.18}$$

This assertion follows from the following lemma:

Lemma 5 *In the case (5.12) the homogenous linear algebraic equation*

$$M_k(\overline{g}_{ij} + T(\overline{g})g_{ij}) - M_j(\overline{g}_{ik} + T(\overline{g})g_{ik}) = 0$$

has only the trivial solution $M_k = 0$.

Proof. When $M_k \neq 0$, then there is a vector ω^k with $\omega^\alpha M_\alpha = 1$. By convolving this equation with ω^k we obtain

$$\overline{g}_{ij} + T(\overline{g})g_{ij} = M_j(\overline{g}_{i\alpha} + T(\overline{g})g_{i\alpha})\omega^\alpha.$$

Then it follows that $\operatorname{Rank}\|\overline{g}_{ij} - \alpha g_{ij}\| \leq 1$, but this is in contradiction with (5.12). ∎

Obviously, from the statements shown above, the following theorem holds.

Theorem 8 *Let V_n be a Riemannian space with metric tensor g_{ij}. Formulas (5.4), (5.16), and (5.18) form a system of differential equations of Cauchy type in covariant derivatives with respect to the unknown functions $\overline{g}_{ij}$, φ_i, and μ whose solutions, which satisfy the condition (5.12), generate the metrics $\overline{g}_{ij}$ of all spaces $\overline{V}_n$, to which V_n can be mapped conformally-projectively harmonically.*

The system of equations (5.4), (5.16), and (5.18) has at most one solution for the initial conditions

$$\overline{g}_{ij}(x_o) = \overset{o}{\overline{g}}_{ij} \quad (\operatorname{Rank}\| \overset{o}{\overline{g}}_{ij} - \alpha g_{ij}(x_o)\| > 1), \quad \varphi_i(x_o) = \overset{o}{\varphi}_i, \quad \mu(x_o) = \overset{o}{\mu},$$

at a point $x_0 \in V_n$. Hence the initial conditions depend on at most $N = \frac{1}{2}(n+1)(n+2)$ independent parameters.

Lemma 6 *There are not more than $\frac{1}{2}(n+1)(n+2) - (n-2)$ independent parameters.*

Proof. When the tensor $Z^h_{ijk} \neq 0$ it follows from (5.9) that at least $n-2$ components depend linearly on the other components of $\overline{g}_{ij}$. So there are maximally $N-(n-2)$ independent parameters.

When $Z^h_{ijk} = 0$, then from (5.10) we find that $R^h_{ijk} = 0$, i.e. V_n is flat. By a detailed investigation of the integrability conditions of (5.4) in the flat space V_n we find that the vectors $\overline{\varphi}_i = \overline{g}_{i\alpha}\varphi^\alpha$ and φ_i are parallel. Therefore $n-1$ components of $\overline{g}_{ij}$ are linearly dependent on the other components. This means that for the flat space the initial conditions depend on at most $N-(n-1)$ independent parameters. ■

5.2.3 Special Case of the Main Equations

Now we study conformally-projective harmonic mappings $V_n \to \overline{V}_n$ for which

$$\text{Rank}\,\|\overline{g}_{ij} - \alpha g_{ij}\| \leq 1. \tag{5.19}$$

The following theorem holds.

Theorem 9 *Let $f : V_n \to \overline{V}_n$ be a conformally-projective harmonic mapping. If (5.19) holds, then V_n and $\overline{V}_n$ are equidistant and*

$$\overline{g}_{ij} = C\,\mathrm{e}^{2\varphi}\,g_{ij} + \beta(\varphi)\,\varphi_i\varphi_j, \tag{5.20}$$

where C is constant, β is a function of φ, which generates a gradient-like concircular vector φ_i. This solution depends maximally on $n+2$ independent real parameters.

Proof. Let $f : V_n \to \overline{V}_n$ be a conformally-projective harmonic mapping. The condition (5.19) is written in the form

$$\overline{g}_{ij} = \alpha g_{ij} + \beta a_i a_j \tag{5.21}$$

where α and β are functions and a_i is a covector.

The covariant derivative of (5.21) w. r. to x^k is

$$\overline{g}_{ij,k} = \alpha_{,k}g_{ij} + \beta_{,k}a_ia_j + \beta a_{i,k}a_j + \beta a_{j,k}a_i.$$

After inserting (5.4) we have

$$\begin{aligned}&(\alpha_{,k} - 2\alpha\varphi_k)g_{ij} + a_i(\beta a_{j,k} + \tfrac{1}{2}\beta_{,k}a_j - \beta\varphi_j a_k - \beta\varphi_k a_j)+\\ &a_j(\beta a_{i,k} + \tfrac{1}{2}\beta_{,k}a_j - \beta\varphi_i a_k - \beta\varphi_k a_i) - g_{jk}(\tfrac{2}{n}\overline{\varphi}_i - \alpha\varphi_i) - g_{ik}(\tfrac{2}{n}\overline{\varphi}_j - \alpha\varphi_j) = 0.\end{aligned} \tag{5.22}$$

If a_i and $(\frac{2}{n}\overline{\varphi}_i - \alpha\varphi_i)$ are non-parallel, then there exists a vector ε^i so that:

$$a_\alpha \varepsilon^\alpha = 0 \quad \text{and} \quad \left(\frac{2}{n}\overline{\varphi}_\alpha - \alpha\varphi_\alpha\right)\varepsilon^\alpha = 1.$$

By contraction of (5.22) with $\varepsilon^i\varepsilon^j$ we convince ourselves that

$$(\alpha_{,k} - 2\alpha\varphi_k) \quad \text{and} \quad \varepsilon_k (\equiv \varepsilon^\alpha g_{\alpha i})$$

are parallel. Then, contracting (5.22) with ε^i we find that $\operatorname{Rank}\|g_{jk}\| \leq 2$, which is in contradiction to $n > 2$.

Therefore $a_i = \gamma(\frac{2}{n}\overline{\varphi}_i - \alpha\varphi_i)$. Note that $\overline{\varphi}_i = \overline{g}_{i\alpha}\varphi^\alpha$. From insertion of (5.21) it follows that a_i is parallel to φ_i. For this reason we may write (5.21) in the form

$$\overline{g}_{ij} = \alpha\, g_{ij} + \beta\, \varphi_i\varphi_j.$$

Formulae (5.22) simplify to

$$(\alpha_{,k} - 2\alpha\varphi_k)g_{ij} + \varphi_i(\beta\varphi_{j,k} + \tfrac{1}{2}\beta_{,k}\varphi_j - 2\beta\varphi_j\varphi_k - \gamma g_{jk}) +$$
$$\varphi_j(\beta\varphi_{i,k} + \tfrac{1}{2}\beta_{,k}\varphi_i - 2\beta\varphi_i\varphi_k - \gamma g_{ik}) = 0.$$

From this evidently follows $\alpha_{,k} = 2\alpha\varphi_k$, i.e. $\alpha = C\,\mathrm{e}^{2\varphi}$, $C = \text{const}$, and $\varphi_{j,k} = -\frac{\beta_{,k}}{2\beta}\varphi_j + 2\varphi_j\varphi_k + \frac{\gamma}{\beta}g_{jk}$. Because φ_i is a gradient, from the last formula it follows that $\beta_{,k}$ is parallel to $\varphi_{,k}$ and, in consequence, $\beta = \beta(\varphi)$, i.e. (5.20) holds. Finally we obtain

$$\varphi_{j,k} = \left(2 - \frac{\beta'}{2\beta}\right)\varphi_j\varphi_k + \left(C\,\frac{n-2}{\beta n}\mathrm{e}^{2\varphi} - \frac{2}{n}\varphi_\alpha\varphi^\alpha\right) g_{jk}. \tag{5.23}$$

After a suitable normalization of φ_i these equations acquire the form $\hat{\varphi}_{i,j} = \rho g_{ij}$, which shows that $\hat{\varphi}^i = g^{i\alpha}\hat{\varphi}_\alpha$ is concircular and V_n is equidistant, see [64, 103, 122]. Applying (5.1) we deduce that $\overline{V}_n$ is also equidistant.

Studying the integrability condition of (5.23) we find that the general solution depends only on $n+2$ real parameters. ∎

5.3 Conformally-Projective Harmonic Mappings of Equidistant Spaces

In the foregoing subsection 5.2.3 (Theorem 9) we could see that conformally-projective harmonic mappings are closely related to concircular vector fields and consequently to equidistant spaces. Below we introduce three cases of conformally-projective harmonic mappings as special transfotmations of equidistant spaces.

First case:

From the necessary and sufficient condition (5.1) of conformally-projective harmonic mappings between V_n and $\overline{V}_n$, with the equidistant metrics (4.2) and (4.4), rewritten in terms of the deformation tensor

$$P^h_{ij} = \varphi_i \delta^h_j + \varphi_j \delta^h_i - \frac{2}{n}\,\varphi^h g_{ij},$$

and with the help of (4.6) we obtain that $\varphi = \varphi(x^1)$ and

$$\begin{aligned}
&\tfrac{A'(x^1)}{A(x^1)} - \tfrac{a'(x^1)}{a(x^1)} = \tfrac{4}{n}\,(n-1)\varphi' \\
&\tfrac{B'(x^1)}{B(x^1)} - \tfrac{b'(x^1)}{b(x^1)} = 2\varphi' \\
&\tfrac{B'(x^1)}{A(x^1)}\,\hat{g}_{ab} - \tfrac{b'(x^1)}{a(x^1)}\,\tilde{g}_{ab} = \tfrac{4b}{na}\,\varphi'\,\tilde{g}_{ab} \\
&\hat{\Gamma}^c_{ab} = \tilde{\Gamma}^c_{ab}\,.
\end{aligned} \tag{5.24}$$

For analysing these equations we first consider the case $B = \text{const}$. From the second equation follows

$$\varphi = \ln\left|b^{-\frac{1}{2}} C_1\right|,$$

whereas from the third one we get

$$\varphi = \ln\left|b^{-\frac{n}{4}} C_2\right|.$$

These relations can hold simultanously only for constant b, namely $b^{\frac{n-2}{4}} = \frac{C_2}{C_1}$ and for this reason also φ is constant. Now from the first equation it follows that $A = aC_3$, and $f\colon V_n \to \overline{V}_n$ is affine, see 4.2. (C_1, C_2, and C_3 are integration constants.)

So, the case $B \neq \text{const}$ is more interesting for us. From (5.24) it follows that $\hat{g}_{ab} = \text{const.}\,\tilde{g}_{ab}$, i.e. $\hat{V}_{n-1}$ and $\tilde{V}_{n-1}$ are homothetic. From the first equation of (5.24) we find

$$\varphi = \frac{n}{4(n-1)} \ln\left|\frac{C_1 A(x^1)}{a(x^1)}\right| \Rightarrow A(x^1) = \alpha\, a(x^1)\, e^{\frac{4(n-1)\varphi}{n}}, \quad \alpha = \frac{1}{C_1}$$

and from the second

$$\varphi = \frac{1}{2} \ln\left|\frac{C_2 B(x^1)}{b(x^1)}\right| \Rightarrow B(x^1) = \beta\, b(x^1)\, e^{2\varphi}, \quad \beta = \frac{1}{C_2}.$$

Now we can formulate the following theorem.

Theorem 10 *The special non-affine mapping f between equidistant spaces V_n and $\overline{V}_n$ is conformal-projective harmonic if and only if $\tilde{V}_{n-1}$ admits a homothetic mapping on $\hat{V}_{n-1}$, and the metric of $\overline{V}_n$ has the following form*

$$\mathrm{d}\bar{s}^2 = \alpha\cdot a(x^1)\,\mathrm{e}^{4\,\frac{n-1}{n}\varphi(x^1)}\,(\mathrm{d}x^1)^2 + \beta\cdot b(x^1)\;\mathrm{e}^{2\varphi(x^1)}\mathrm{d}\tilde{s}^2, \tag{5.25}$$

where α, β are non-zero constants, and the function $\varphi(x^1)$ satisfies the following ordinary differential equation

$$\beta\, n\,(b'(x^1) + 2b(x^1)\varphi')\cdot \mathrm{e}^{2\,\frac{2-n}{n}\,\varphi} - \alpha n\, b'(x^1) - 4\alpha\, b(x^1)\,\varphi' = 0.$$

At last we consider the case $A = \text{const}$. Here the first equation of (5.24) yields

$$\varphi' = \frac{-n}{4(n-1)}\frac{a'(x^1)}{a(x^1)} \Rightarrow \varphi = \ln\left|C_1\; a^{\frac{n}{4(n-1)}}(x^1)\right|.$$

With φ' inserted into the second equation we get

$$B'(x^1) = B(x^1)\left(\frac{b'(x^1)}{b(x^1)} - \frac{n}{2(n-1)}\frac{a'(x^1)}{a(x^1)}\right) \Rightarrow B(x^1) = C_2\; b(x^1)\, a^{\frac{-n}{2(n-1)}}(x^1).$$

Now the third equation gives a relation between $\hat{g}_{ab}$ and $\tilde{g}_{ab}$:

$$\hat{g}_{ab} = \frac{2A}{C_2} a^{\frac{2-n}{2(n-1)}}(x^1)\frac{(n-1)a(x^1)b'(x^1) - a'(x^1)b(x^1)}{2(n-1)a(x^1)b'(x^1) - na'(x^1)b(x^1)}\tilde{g}_{ab}$$

and the metric of $\overline{V}_n$ becomes

$$\mathrm{d}\bar{s}^2 = A\left[(\mathrm{d}x^1)^2 + 2\;\frac{b(x^1)}{a(x^1)}\;\frac{(n-1)a(x^1)b'(x^1) - a'(x^1)b(x^1)}{2(n-1)a(x^1)b'(x^1) - na'(x^1)b(x^1)}\;\tilde{g}_{ab}\right].$$

This metric is singular in two cases:
(1) $(n-1)a(x^1)b'(x^1) - a'(x^1)b(x^1) = 0 \quad\Rightarrow\quad a(x^1) = \alpha\; b^{n-1}(x^1),\quad \alpha = \text{const}.$
and
(2) $2(n-1)a(x^1)b'(x^1) - na'(x^1)b(x^1) = 0 \Rightarrow\; a(x^1) = \beta\; b^{\frac{2(n-1)}{n}}(x^1),\; \beta = \text{const}.$
This is an indication that the metric may change signature under conformally-projective harmonic mappings, and indeed this happens in the example

$$\mathrm{d}s^2 = x^3\mathrm{d}x^2 + x^2(\mathrm{d}y^2 + \mathrm{d}z^2),$$

with $n = 3$, and $a(x) = x^3$, and $b(x) = x^2$, which is mapped onto $\overline{V}_n$ with metric

$$\mathrm{d}\bar{s}^2 = A\left(\mathrm{d}x^2 - \frac{2}{x}(\mathrm{d}y^2 + \mathrm{d}z^2)\right).$$

In this case we can see that a positive definite metric was mapped to an indefinite one.

A simple 3-dimensional example where $\mathrm{d}\bar{s}^2$ is singular is given by $a(x) = x^2$ and $b(x) = x$.

Second case:

By a suitable transformation of the coordinate x^1 the metric form (4.2) of an equidistant space $\hat{V}_n$ can be transformed to the equivalent form (2.34) with $a(x^1) = 1/f(x^1)$ and $b(x^1) = f(x^1)$ (see [11, 67, 83, 103, 122]):

$$\mathrm{d}\hat{s}^2 = \frac{1}{f(x^1)}(\mathrm{d}x^1)^2 + f(x^1)\,\mathrm{d}\tilde{s}^2, \tag{5.26}$$

where $f(x^1) \neq 0$ and C^1, $\mathrm{d}\tilde{s}^2$ is a metric of $\tilde{V}_{n-1}$. An equidistant manifold $\hat{V}_n$ with metric (5.26) admits geodesic diffeomorphisms onto the Riemannian space $\overline{V}_n$, whose metric form is

$$\mathrm{d}\bar{s}^2 = \frac{p}{f\cdot(1+qf)^2}(\mathrm{d}x^1)^2 + \frac{p\,f}{1+qf}\mathrm{d}\tilde{s}^2, \tag{5.27}$$

where p, q are some constants such that $1 + qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is trivial, and x are common coordinates for V_n and $\overline{V}_n$, see [67]. The function $\psi(x)$, which defines a geodesic mapping, has the following form: $\psi(x) = -\frac{1}{2}\ln|1 + q\,f|$.

The following theorem holds:

Theorem 11 *An equidistant manifold V_n with the metric*

$$\mathrm{d}s^2 = (1 + q\,f(x^1))^{\frac{2}{n-2}}\left(\frac{1}{f(x^1)}(\mathrm{d}x^1)^2 + f(x^1)\,\mathrm{d}\tilde{s}^2\right), \tag{5.28}$$

where $f \in C^1$ $(f \neq 0)$ is a function and $d\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\,dx^a dx^b$ $(a, b = 2, \ldots, n)$ is the metric of some $(n-1)$-dimensional Riemannian space $\tilde{V}_{n-1}$, is mapped conformally-projectively harmonically onto the Riemannian space $\overline{V}_n$ with the metric (5.27).

Proof. Let (5.28) and (5.27) be the metric forms of the Riemannian spaces V_n and $\overline{V}_n$. We calculate the Christoffel symbols Γ^h_{ij} and $\overline{\Gamma}^h_{ij}$ of these spaces. Formula (5.1) holds for $\varphi = -\dfrac{n}{2(n-2)}\ln|1 + q\,f|$.

Analysing the formulas related to conformally-projective harmonic mappings we can convince ourselves that the following holds:

Proposition 1. The equidistant manifold V_n with metric (5.28) is conformally mapped onto a Riemannian space with metric (2.34), which is geodesically mapped onto a Riemannian space $\overline{V}_n$ with metric (5.27).

Proposition 2. By comparison of the metrics (5.28) and (5.27) we find that, dependent on the choice of the parameter q, the signatures of the two metrics can be the same or not.

Proposition 3. There are spaces with a metric of the form (5.28), satisfying condition (2.35), admitting conformally-projectively harmonic mappings onto an Einstein space, resp. a space of constant curvature.

By a detailed analysis we can convince ourselves of the existence of compact Riemannian spaces, for which global non-affine conformally-projective harmonic mappings exist.

Third case:

Equidistant manifolds $\hat{V}_n$ with geodesic coordinate system and Friedmann metric: After another transformation of the coordinate x^1 we can rewrite the metrics (2.34), (5.27) and (5.28) in the form:

$$\mathrm{d}\hat{s}^2 = e(dx^1)^2 + f(x^1)\,\mathrm{d}\tilde{s}^2, \tag{5.29}$$

where $e = \pm 1$, $f \in C^1$ $(f \neq 0)$ is a function, $\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\mathrm{d}x^a\mathrm{d}x^b$ $(a, b = 2, \ldots, n)$ is the metric of a certain Riemannian subspace $\tilde{V}_{n-1}$ (see [64, 103]).

The *Friedmann metric* is a metric of the type (5.29) with $\tilde{V}_{n-1}$ being a space with constant curvature, modeling a spatially homogenous and isotropic universe. The function f describes the evolution in the time coordinate x^1.

An equidistant space V_n with metric (5.29) admits geodesic mappings onto a Riemannian space $\overline{V}_n$, whose metric form is [64]

$$\mathrm{d}\overline{s}^2 = \frac{ep}{(1+qf)^2}\,(\mathrm{d}x^1)^2 + \frac{p\,f}{1+qf}\,\mathrm{d}\tilde{s}^2, \tag{5.30}$$

where p, q are some constants such that $1+qf \neq 0$, $p \neq 0$. If $qf' \not\equiv 0$, the mapping is nontrivial; otherwise it is affine. Like in the second case the function $\psi(x)$ has the form $\psi(x) = -\frac{1}{2}\ln|1+q\,f|$.

Theorem 12 *An equidistant manifold V_n with the metric*

$$\mathrm{d}s^2 = (1+q\,f(x^1))^{\frac{2}{n-2}}\left(e\,(\mathrm{d}x^1)^2 + f(x^1)\,\mathrm{d}\tilde{s}^2\right), \tag{5.31}$$

where $f \in C^1$ $(f \neq 0)$ is a function, $\mathrm{d}\tilde{s}^2 = \tilde{g}_{ab}(x^2, \ldots, x^n)\,\mathrm{d}x^a\mathrm{d}x^b$ $(a, b = 2, \ldots, n)$ is the metric of some $(n-1)$-dimensional Riemannian space $\tilde{V}_{n-1}$, is mapped *conformally-projectively harmonically on the Riemannian space $\overline{V}_n$ with the metric (5.30).*

The proof of Theorem 12 is analogous to that of Theorem 11 for the same function ψ. The manifold V_n with metric (5.31) is conformally mapped onto a Riemannian space with metric (5.29), which is geodesically mapped onto a Riemannian space $\overline{V}_n$ with metric (5.30).

Conformally-projective harmonic mappings of equidistant manifolds V_n were studied in [32, 33]. The results mentioned here are examples of these mappings under the condition (5.19). We note, if the choice of the function $f(x^1)$ and the metric $d\tilde{s}^2$ in (5.31) is special, than the equidistant space V_n will be a flat space, a space of constant curvature, an Einstein space, a semisymmetric or pseudosymmetric space, see [11, 64, 67, 83, 103].

6 F-planar Mappings of Spaces With Affine Connections

Further results, published in [36], were obtained with a type of mapping, which conserves F-planar curves, introduced in subsection (2.4), in analogy to the conservation of geodesics by geodesic mappings, considered in subsection (4.4).

6.1 F-planar Mappings Between two Spaces With Affine Connection

We suppose two spaces A_n and $\overline{A}_n$ with torsion-free affine connections ∇ and $\overline{\nabla}$, respectively. Affine structures F and $\overline{F}$ are defined on A_n, resp. $\overline{A}_n$.

Definition 16 *(J. Mikeš, N.S. Sinyukov [77]) A diffeomorphism $f\colon A_n \to \overline{A}_n$ between two manifolds with affine connections is called F-planar if any F-planar curve in A_n is mapped onto an $\overline{F}$-planar curve in $\overline{A}_n$.*

Important convention. Due to the diffeomorphism f we always suppose that ∇, $\overline{\nabla}$, and the affinors F, $\overline{F}$ are defined on A_n. Moreover, we always identify a given curve $\ell\colon I \to A_n$ and its tangent vector function $\lambda(t)$ with their images $\overline{\ell} = f \circ \ell$ and $\overline{\lambda} = f_*(\lambda(t))$ in $\overline{A}_n$.

Two principially different cases are possible for the investigation:

$$\text{a)}\quad \overline{F} = a\,F + b\,I; \tag{6.1}$$
$$\text{b)}\quad \overline{F} \neq a\,F + b\,I, \tag{6.2}$$

a, b are some functions.

Naturally, case a) characterizes F-planar mappings *which preserve F-structures.* In case b) the structures of F and $\overline{F}$ are essentially distinct. The following holds.

Theorem 13 *An F-planar mapping f from A_n onto $\overline{A}_n$ preserves F-structures and is characterized by the following condition*

$$P(X,Y) = \psi(X)\,Y + \psi(Y)\,X + \varphi(X)\,FY + \varphi(Y)\,FX \tag{6.3}$$

for any vector fields X, Y, where $P \stackrel{\text{def}}{=} \overline{\nabla} - \nabla$ is the deformation tensor field of f, ψ, φ are some linear forms.

Let us recall that on each tangent space T_xA_n, $P(X,Y)$ is a symmetric bilinear mapping $T_xA_n \times T_xA_n \to T_xA_n$ and a tensor field of type $\binom{1}{2}$.

Theorem 13 was proved by J. Mikeš and N.S. Sinyukov [77] for finite dimension $n > 3$. Here we can show a more rational proof of this Theorem for $n > 3$ and also a proof for $n = 3$. We show a counterexample for $n = 2$.

6.2 F-planar Mappings Which Preserve F-structures

First we prove the following theorem

Theorem 14 *An F-planar mapping f from A_n onto $\overline{A}_n$ which preserves F-structures is characterized by condition (6.3).*

In the sequel we shall need the following lemma:

Lemma 7 *Let V be an n-dimensional vector space, $Q\colon V \times V \to V$ be a symmetric bilinear mapping and $F\colon V \to V$ a linear mapping. If, for each vector $\lambda \in V^*$*

$$Q(\lambda, \lambda) = \varrho_1(\lambda)\, \lambda + \varrho_2(\lambda)\, F(\lambda) \tag{6.4}$$

holds, where $\varrho_1(\lambda)$, $\varrho_2(\lambda)$ are functions on $V^* = V/\{0\}$, then there are linear forms ψ and φ such that the condition

$$Q(X, Y) = \psi(X)\, Y + \psi(Y)\, X + \varphi(X)\, F(Y) + \varphi(Y)\, F(X) \tag{6.5}$$

holds for any $X, Y \in V$.

Proof. Formula (6.4) has the following coordinate expression

$$Q^h_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_1(\lambda)\, \lambda^h + \varrho_2(\lambda)\, F^h_\alpha\lambda^\alpha, \tag{6.6}$$

where $\lambda^i, F^h_i, Q^h_{ij}$ are the components of λ, F, Q.

By multiplying (6.6) with $\lambda^i\, F^j_\alpha\lambda^\alpha$ and antisymmetrizing the indices h, i and j we obtain

$$\left\{\, Q^{[h}_{\alpha\beta}\delta^i_\gamma F^{j]}_\delta \,\right\} \lambda^\alpha\lambda^\beta\lambda^\gamma\lambda^\delta = 0, \tag{6.7}$$

where square brackets denote the alternation of indices. The term in curly brackets does not depend on λ and (6.7) holds for any vector $\lambda \in V$, therefore

$$Q^{[h}_{(\alpha\beta}\delta^i_\gamma F^{j]}_{\delta)} = 0 \tag{6.8}$$

holds, where the round brackets denote symmetrization of indices.
It is natural to assume that

$$F^h_i \neq a\, \delta^h_i \quad \text{with} \quad a = \text{const.}$$

By virtue of this there exist some vectors ξ^h such that

$$\xi^\alpha F^h_\alpha \neq b\, \xi^h, \quad b = \text{const.}$$

Introducing
$$P_i^h \overset{\text{def}}{=} P_{i\alpha}^h \xi^\alpha, \quad P^h \overset{\text{def}}{=} P_\alpha^h \xi^\alpha \quad \text{and} \quad F^h \overset{\text{def}}{=} F_\alpha^h \xi^\alpha,$$
we contract (6.8) with $\xi^\alpha \xi^\beta \xi^\gamma \xi^\delta$. Since $F^h \neq b\,\xi^h$, we obtain
$$P^h = 2a\,\xi^h + 2b\,F^h,$$
where a, b are certain constants. Contracting (6.8) with $\xi^\beta \xi^\gamma \xi^\delta$, and taking into account the above formula, we have
$$P_i^h = a\,\delta_i^h + b\,F_i^h + a_i\,\xi^h + b_i\,F^h,$$
where a_i, b_i are some components of linear forms. Analogously, contracting (6.8) with $\xi^\gamma \xi^\delta$, we have
$$Q_{ij}^h = \psi_i \delta_j^h + \psi_j \delta_i^h + \varphi_i F_j^h + \varphi_j F_i^h + \xi^h a_{ij} + F^h b_{ij}, \tag{6.9}$$
where ψ_i, φ_i are components of 1-forms ψ, φ defined on V, and a_{ij}, b_{ij} are components of a symmetric tensors of type $(0,2)$ defined on V.

In the case $a_{ij} = b_{ij} = 0$, evidently from (6.9) we obtain formula (6.5).

Now we will suppose that either $a_{ij} \neq 0$, or $b_{ij} \neq 0$. Since ξ^h and F^h are noncollinear, it is evident that
$$\xi^h a_{ij} + F^h b_{ij} \neq 0. \tag{6.10}$$

By virtue of (6.9) formula (6.8) has the form
$$\Omega_{(\alpha\beta\gamma}^{[hi} F_{\delta)}^{j]} = 0, \tag{6.11}$$
where $\Omega_{\alpha\beta\gamma}^{hi} \overset{\text{def}}{=} (\xi^h a_{\alpha\beta} + F^h b_{\alpha\beta})\delta_\gamma^i - (\xi^i a_{\alpha\beta} + F^i b_{\alpha\beta})\delta_\gamma^h$. It is possible to show that there exists some vector ε^h for which $\Omega_{\alpha\beta\gamma}^{hi} \varepsilon^\alpha \varepsilon^\beta \varepsilon^\gamma \neq 0$, otherwise (6.10) would be violated.

Contracting (3.12) with $\varepsilon^\alpha \varepsilon^\beta \varepsilon^\gamma \varepsilon^\delta$, we have
$$F_\alpha^h \varepsilon^\alpha = a\,\xi^h + b\,F^h + c\,\varepsilon^h,$$
with a, b, c being constants. Analogously, contracting (6.11) with $\varepsilon^\beta \varepsilon^\gamma \varepsilon^\delta$, we obtain that F_i^h is represented in the following manner:
$$F_i^h = a\,\delta_i^h + a_i\,\xi^h + b_i\,F^h + c_i\,\varepsilon^h, \tag{6.12}$$
where a_i, b_i, c_i are components of 1-forms.

By virtue of (6.12) formula (6.11) has the form

$$\omega^{[hi}_{(\alpha\beta\gamma}\delta^{j]}_{\delta)} = 0, \tag{6.13}$$

where

$$\omega^{hi}_{\alpha\beta\gamma} \overset{\text{def}}{=} \xi^{[h}F^{i]}(a_{(\alpha\beta}b_{\gamma)} - b_{(\alpha\beta}a_{\gamma)}) + \xi^{[h}\varepsilon^{i]}a_{(\alpha\beta}c_{\gamma)} + F^{[h}\varepsilon^{i]}b_{(\alpha\beta}c_{\gamma)}.$$

a) If $n > 3$ then $\omega^{hi}_{\alpha\beta\gamma} = 0$ follows from (6.11), and because ξ^h, F^h and ε^h are linearly independent, we obtain $a_{(\alpha\beta}c_{\gamma)} = 0$ and $b_{(\alpha\beta}c_{\gamma)} = 0$. Therefore $c_i = 0$ and

$$F^h_i = a\,\delta^h_i + a_i\,\xi^h + b_i\,F^h. \tag{6.14}$$

b) If $n = 3$ the matrix F^h_i has always the previous form (6.14) while ξ^h, F^h and ε^h are not linearly dependent.

Then formula (6.11) becomes (6.13), whereas $\omega^{hi}_{\alpha\beta\gamma} \overset{\text{def}}{=} \xi^{[h}F^{i]}(a_{(\alpha\beta}b_{\gamma)} - b_{(\alpha\beta}a_{\gamma)})$. For $n > 2$ it follows $\omega^{hi}_{\alpha\beta\gamma} = 0$ and consequently

$$a_{(\alpha\beta}b_{\gamma)} = b_{(\alpha\beta}a_{\gamma)}. \tag{6.15}$$

If a_α and b_α are linearly independent, then from (6.15) we obtain

$$a_{ij} = a_{(i}\omega_{j)} \text{ and } b_{ij} = b_{(i}\omega_{j)},$$

where ω_i are components of a 1-form. Afterwards it is possible to show that on the basis of (6.14) formula (6.9) assumes the following form

$$Q^h_{ij} = (\psi_i - a\omega_i)\delta^h_j + (\psi_j - a\omega_j)\delta^h_i + (\varphi_i + a\omega_i)F^h_j + (\varphi_j + a\omega_j)F^h_i,$$

i.e. formula (6.5) also holds.

Now there remains the case that a_α and b_α are linearly dependent. For example, $b_\alpha = \alpha\,a_\alpha$, $\alpha \neq 0$. Then from (6.15) follows $b_{\alpha\beta} = \alpha\,a_{\alpha\beta}$. We denote

$$\Lambda^h = \xi^h + \alpha\,F^h, \quad \omega_i = \psi_i + \alpha\,\varphi_i, \quad \omega_{ij} = a_{ij} + a_{(i}\varphi_{j)},$$

from (6.9) and (6.14) we obtain that Q^h_{ij} and F^h_i are represented by

$$Q^h_{ij} = \psi_i\delta^h_j + \psi_j\delta^h_i + \Lambda^h\omega_{ij} \text{ and } F^h_i = a\delta^h_i + \Lambda^h a_i.$$

Then formula (6.6) appears in the following way

$$\Lambda^h\,(\omega_{\alpha\beta}\lambda^\alpha\lambda^\beta - \varrho_2(\lambda)\,a_\alpha\lambda^\alpha) = \lambda^h\,(\varrho_1(\lambda) + a\,\varrho_2(\lambda) - 2\psi_\alpha\lambda^\alpha).$$

From this it follows that

$$\omega_{\alpha\beta}\lambda^\alpha\lambda^\beta = \varrho_2(\lambda)\, a_\alpha\lambda^\alpha, \qquad \forall\lambda^h \neq \alpha\,\Lambda^h.$$

By simple analysis we obtain that $\omega_{ij} = a_{(i}\sigma_{j)}$, where σ_i are components of a 1-form. Then due to (6.15) we have

$$Q^h_{ij} = (\psi_i - a\sigma_i)\delta^h_j + (\psi_j - a\sigma_j)\delta^h_i + \sigma_i F^h_j + \sigma_j F^h_i$$

and Lemma 7 is proved. ∎

Now we start the **proof** of Theorem 14. It is obvious that geodesics are a special case of F-planar curves. Let a geodesic with tangent vector λ in A_n, which satisfies $\nabla_\lambda\lambda = 0$, be mapped onto an F-planar curve in $\overline{A}_n$, which satisfies

$$\overline{\nabla}_\lambda\lambda = \overline{\varrho}_1(t)\lambda + \overline{\varrho}_2(t)F\lambda.$$

Here $\overline{\varrho}_1, \overline{\varrho}_2$ are functions of the parameter t.

Because the deformation tensor satisfies $P(\lambda,\lambda) = \overline{\nabla}_\lambda\lambda - \nabla_\lambda\lambda$, we have

$$P(\lambda(t), \lambda(t)) = \overline{\varrho}_1(t)\lambda + \overline{\varrho}_2(t)F\lambda.$$

It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda,\lambda) = \varrho_1(\lambda)\lambda + \varrho_2(\lambda)F\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Based on Lemma 7 it follows that there exist linear forms ψ and φ, for which formula (6.3) holds.

6.3 F-planar Mappings Which do not Preserve F-structures

We now assume that the structures F and $\overline{F}$ are essentially distinct, i.e.

$$\overline{F}^h_i \neq a\delta^h_i + b\,F^h_i.$$

a) It is obvious, that geodesics are a special case of F-planar curves. Let a geodesic in A_n, which satisfies $\nabla_\lambda\lambda = 0$, be mapped onto an $\overline{F}$-planar curve in $\overline{A}_n$, which satisfies

$$\overline{\nabla}_\lambda\lambda = \overline{\varrho}_1(t)\lambda + \overline{\varrho}_2(t)\overline{F}\lambda,$$

here $\overline{\varrho}_1, \overline{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have

$$P(\lambda(t), \lambda(t)) = \overline{\varrho}_1(t)\lambda + \overline{\varrho}_2(t)\overline{F}\lambda.$$

It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda, \lambda) = \varrho_1(\lambda)\lambda + \varrho_2(\lambda)\overline{F}\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Based on Lemma 7 it follows that there exist linear forms ψ and φ, for which formula

$$P(X, Y) = \psi(X)\, Y + \psi(Y)\, X + \varphi(X)\, \overline{F}Y + \varphi(Y)\, \overline{F}X \tag{6.16}$$

holds.

b) Let a special F-planar curve with tangent vector λ in A_n, which satisfies $\nabla_\lambda \lambda = F\lambda$, be mapped onto an $\overline{F}$-planar curve in $\overline{A}_n$, which satisfies

$$\overline{\nabla}_\lambda \lambda = \overline{\varrho}_1(t)\lambda + \overline{\varrho}_2(t)\overline{F}\lambda,$$

here $\overline{\varrho}_1, \overline{\varrho}_2$ are functions of the parameter t.

For the deformation tensor we have

$$P(\lambda(t), \lambda(t)) = F\lambda + \overline{\varrho}_1(t)\lambda + \overline{\varrho}_2(t)\overline{F}\lambda.$$

It follows from the previous formula that in each point $x \in A_n$

$$P(\lambda, \lambda) = F\lambda + \varrho_1(\lambda)\lambda + \varrho_2(\lambda)\overline{F}\lambda.$$

for each tangent vector $\lambda \in T_x$; $\varrho_1(\lambda), \varrho_2(\lambda)$ are functions dependent on λ.

Applying (6.16) we obtain

$$F\lambda = \tilde{\varrho}_1(\lambda)\lambda + \tilde{\varrho}_2(\lambda)\overline{F}\lambda.$$

Analyzing this expression like in Lemma 7 we convince ourselves that formula (6.1) holds. In this way we prove

Theorem 15 *Any F-planar mapping of a space with affine connection A_n onto $\overline{A}_n$ preserves F-structures.*

6.4 F-planar Mappings for Dimension $n = 2$

It is easy to see that for $n = 2$ Theorems 14 and 15 do not hold. If they were valid, the functions ϱ_1 and ϱ_2, appearing in (6.4), would be linear in λ.

In the case

$$F_i^h = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

for example, these functions have the forms

$$\varrho_1(\lambda) = \frac{\lambda^1 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta + \lambda^2 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2} \quad \text{and} \quad \varrho_2(\lambda) = \frac{\lambda^1 P^2_{\alpha\beta}\lambda^\alpha\lambda^\beta - \lambda^2 P^1_{\alpha\beta}\lambda^\alpha\lambda^\beta}{(\lambda^1)^2 + (\lambda^2)^2},$$

which are not linear in general.

On the other hand, an arbitrary diffeomorphism from A_2 onto $\overline{A}_2$ is an F-planar mapping with (6.4) being valid for the above functions ϱ_1 and ϱ_2.

7 Conclusion

The aim of the thesis is the study of some special diffeomorphisms between Riemannian spaces. Essentially it is constructed from published articles, having as subject the one or the other special mapping or a vector field, closely related with these mappings. The considered diffeomorphisms are mainly generalisations of conformal and geodesic mappings: F-planar, harmonic, conformally-projective harmonic mappings. In the thesis the brief text of the published papers was amplified and some explanatory examples were added.

- The work contributes to the theory of geodesic mappings of equidistant and Einstein spaces, continuing results by [3, 4, 11, 13, 19, 46, 61, 64, 83, 102, 103, 109]. Some known facts concerning geodesic mappings of Einstein spaces and a hypothesis by A.Z. Petrov were made more precise. These results were published in the paper [67], quoted by G.S. Hall and D.P. Lonie [29].
- General properties of F-planar mappings of spaces with affine connection were studied. In the paper [36] the derivation of the fundametal equations of F-planar mappings was simplified.

Generally speaking, this work consists in a synopsis of different mappings between manifolds and related vector fields, pointing out common features on the one hand, and the new results in the papers mentioned above on the other hand.

We already published our results in the papers [32, 33, 34, 35, 36, 37, 67].

8 Appendix: On Systems of Partial Differential Equations of Cauchy Type

The conformal-projective harmonic diffeomorphisms in this work are characterized by conditions in the form of first-order partial differential equations, the fundamental equations. Equations of this type are characteristic also for other mappings.

Obviously the existence of a solution of the fundamental equations implies the existence of the mentioned mappings. These fundamental equations were found in several forms, in this work in form of a ***system of partial differential equations of Cauchy type*** see [68]. When they are linear, the question of solvability can be answered by algebraic methods. The investigation of such systems has many aspects concerning existence and uniqueness of a solution, differentiability of the functions under consideration, local and global properties of solutions.

8.1 Systems of PDEs of Cauchy Type in $\mathbb{R}^n$

Now we introduce the basic notions of the theory of systems of differential equations with initial conditions. We restrict ourselves to the local theory.

Assume a convex domain $D \subset \mathbb{R}^n$ with coordinates $x = (x^1, x^2, \ldots, x^n)$ and functions $F_i^A(x,y)$, $i = 1, \ldots, n$; $A = 1, \ldots, N$, on $\tilde{D} \subset D \times \mathbb{R}^N$. Suppose the functions $F_i^A(x,y)$ are continuous with respect to x and differentiable with respect to y in the domain $\tilde{D}$.

The system of *differential equations of Cauchy type* has the following form

$$\frac{\partial y^A(x)}{\partial x^i} = F_i^A(x, y(x)), \qquad A = 1, \ldots, N, \quad i = 1, \ldots, n, \tag{8.1}$$

where $y(x) = (y^1(x), \ldots, y^N(x))$ are unknown functions.

For *initial data* (= *initial Cauchy conditions*)

$$y^A(x_0) \;=\; y_0^A, \quad A = 1, \ldots, N, \tag{8.2}$$

where $x_0 \in D$ and $(x_0, y_0^A) \in \tilde{D}$, the system (8.1) has at most one solution

$$y^A = y^A(x^1, \ldots, x^n) \tag{8.3}$$

in the class C^1 such that $(x, y(x)) \in \tilde{D}$. For this reason the general solution of the system (8.1) depends on $r \leq N$ real parameters.

Let us suppose that $F_i^A(x,y) \in C^1(\tilde{D})$ and that a solution we are looking for satisfies $y^A(x) \in C^2(D)$. Then the integrability conditions of (8.1) read

$$\partial_{jk} y^A(x) = \partial_{kj} y^A(x) \tag{8.4}$$

and according to (8.1) $\partial_k(F_j^A(x,y(x))) = \partial_j(F_k^A(x,y(x)))$ which can be written as

$$\partial_k F_j^A(x,y) + \partial_B F_j^A(x,y)\, \partial_k y^B - \partial_j F_k^A(x,y) - \partial_B F_k^A(x,y)\, \partial_j y^B = 0.$$

If we apply (8.1) we get

$$\partial_k F_j^A + \partial_B F_j^A\, F_k^B - \partial_j F_k^A - \partial_B F_k^A\, F_j^B = 0. \tag{8.5}$$

Here we denoted $\partial_k F_j^A = \dfrac{\partial F_j^A}{\partial x^k}$ and $\partial_B F_j^A = \dfrac{\partial F_j^A}{\partial y^B}$.

For any solution (8.3) of (8.1) the conditions (8.5) are satisfied identically for $x \in D$. Among others, these conditions must be satisfied for the initial data (8.2).

The conditions (8.5) are called ***integrability conditions*** of the system (8.1).

If the conditions (8.5) are identically satisfied for $x \in D$, then the system (8.1) is called ***completely integrable***. In this case, the system has a solution for any initial data (8.2), i.e. the general solution of (8.1) depends on N real parameters.

8.2 On Mixed Systems of PDEs of Cauchy Type in $\mathbb{R}^n$

Suppose that the functions $y^A(x)$ satisfy, besides (8.1), also the additional equations

$$f^p(x^1, \ldots, x^n, y^1, \ldots, y^N) = 0, \quad p = 1, \ldots, m. \tag{8.6}$$

The functions $f^p(x,y)$ are defined in the domain $\tilde{D}$.

The system (8.1) and (8.6) is called ***a mixed system of PDEs of Cauchy type***. If we study such a system, we investigate the integrability conditions (8.5) and (8.6) together, and denote them (B) in short.

Let $F_i^A(x,y) \in C^{r+2}(\tilde{D})$ and $f^p(x,y) \in C^{r+1}(\tilde{D})$. Then differentiating step by step we get a system of differential prolongations (B_1), (B_2), …, (B_r). Denote $(B_0) \equiv (B)$. Now (B_{k+1}) is obtained from the system of conditions (B_k) by means of differentiating all equations by ∂_i, $i = 1, \ldots, n$. The conditions (B_0), (B_1), …, (B_r) must hold for the initial data (8.2).

The following theorem can be proved (Eisenhart [20], Sinyukov [103] – here the theorem is formulated for analytic solutions)

Theorem 16 *A mixed system of PDEs (8.1) and (8.6) of Cauchy type has in a neighborhood of a point $x_0 = (x_0^i)$ a unique solution (8.3) of class C^{r+1} which satisfies the initial conditions (8.2) if and only if the conditions (B_0), $(B_1), \ldots$, (B_r) hold in the point (x_0, y_0) and r is the least integer for which (B_{r+1}) is a consequence of the system of all preceding prolongations.*

The system (8.1) may be written in terms of covariant derivatives. A fundamental investigation of (8.1) consists in checking of the integrability conditions, which are essentially algebraic equations for the unknown variables y^A. In the case when they are identically fulfilled, we have $r = N$.

8.3 Mixed Linear Systems of PDEs of Cauchy Type in $\mathbb{R}^n$

The systems mentioned in the previous subsection are of particular importance in case they are linear. In such a case, the equations (8.1) and the conditions (8.6) read

$$\frac{\partial y^A(x)}{\partial x^i} = F^A_{iB}(x) \cdot y^B(x) + F^A_i(x), \tag{8.7}$$

$$f^p_B(x) \cdot y^B + f^p(x) = 0, \tag{8.8}$$

where $F^A_{iB}(x)$, $F^A_i(x)$, $f^p_B(x)$, $f^p(x)$ are functions on D.

The integrability conditions of (8.7) are linear algebraic equations with respect to y^A and obviously, they have a form analogous to the equations (8.8):

$$\begin{aligned}&(\partial_k F^A_{jB} - \partial_j F^A_{kB} F^A_{jC} F^C_{kB} - F^A_{kC} F^C_{jB})\, y^B + \\ &+ \partial_k F^A_j - \partial_j F^A_k + F^A_{jC} F^C_k - F^A_{kC} F^C_j = 0.\end{aligned} \tag{8.9}$$

When $F^A_{iB}, F^A_i \in C^{r+2}(D)$, and $f^p_B, f^p \in C^{r+1}(D)$ there exist conditions $(B_0) \equiv (B)$ and their differential prolongations (B_1), $\ldots, (B_r)$, (B_{r+1}) that all are linear algebraic equations in unknown functions $y^A(x)$, with coefficients being certain functions of the variable $x \in D$.

Obviously, Theorem 16 is satisfied, and consequently, the problem of solvability of the linear system (8.7) and (8.8) can be decided by analysis of the linear algebraic equations (B_0), (B_1), ...

8.4 Mixed PDEs in Tensor Form

In applications of the theory of PDEs' the equations (8.1) and (8.7) are often written in tensor form.

Let $D \subset \mathbb{R}^n$ be a coordinate domain of $A_n$with a linear connection ∇. A system of PDEs of Cauchy type in covariant derivatives (with respect to the affine connection ∇) of m unknown tensor fields $\underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma}}(x)$, $\sigma = 1, \ldots, m$, of type (p_σ, q_σ) takes the form

$$\begin{aligned} &\underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma}, k}(x) = \underset{\sigma}{F}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma} k}(x, \underset{1}{Y}, \ldots, \underset{m}{Y}), \\ &i_1, i_2, \ldots, i_{p_\sigma}, j_1, j_2, \ldots, j_{q_\sigma}, k = 1, 2, \ldots, n. \end{aligned} \tag{8.10}$$

On the right-hand side of (8.10) there are tensor functions of type $(p_\sigma, q_{\sigma+1})$ constructed in a certain way by means of a finite number of tensor operations from the unknown tensor fields $\underset{\sigma}{Y}$ and also from components of certain known objects, including the linear connection ∇. The integrability conditions of (8.10) follow from the Ricci identities for the tensors $\underset{\sigma}{Y}$:

$$\begin{aligned} \underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_q, [lm]} &= -\underset{\sigma}{Y}{}^{\alpha i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma}} R^{i_1}_{\alpha lm} - \cdots - \underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma - 1} \alpha}_{j_1 j_2 \cdots j_{q_\sigma}} R^{i_{p_\sigma}}_{\alpha lm} \\ &+ \underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{\alpha j_2 \cdots j_{q_\sigma}} R^{\alpha}_{j_1 lm} + \cdots + \underset{\sigma}{Y}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma - 1} \alpha} R^{\alpha}_{j_{q_\sigma} lm} + \underset{\sigma}{Y}{}^{i_1 i_2 \ldots i_{p_\sigma}}_{j_1 j_2 \ldots j_{q_\sigma}, \alpha} S^{\alpha}_{lm} \\ &= \underset{\sigma}{F}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma} l, m} - \underset{\sigma}{F}{}^{i_1 i_2 \cdots i_{p_\sigma}}_{j_1 j_2 \cdots j_{q_\sigma} m, l}. \end{aligned} \tag{8.11}$$

Hence the integrability conditions are written in a tensor form.

8.5 On Systems of PDEs of Cauchy Type on Manifolds

It is a question, which of the results being formulated for PDEs in $\mathbb{R}^n$ remain valid, eventually can be generalized, for an arbitrary n-dimensional manifold M_n.

The theorem on the existence of a solution is of local character, there are examples of equations of type (8.1) that are defined on the whole M_n which admit a local solution on a coordinate neighborhood of each point, but no global soluiton. There exist, for example, counterexamples to global metrizability, which can be found in [12] and [79] where, among others, the aim is to find a global solution of the equation $\nabla g = 0$, i.e.

$$\partial_k g_{ij} = g_{i\alpha} \Gamma^{\alpha}_{jk} + g_{j\alpha} \Gamma^{\alpha}_{ik},$$

where Γ^h_{ij} are componente of a given linear connection and g_{ij} are components of a metric we are searching for.

Of course, uniqueness properties are guaranteed even in this general case.

Now we reformulate the problem on PDEs on a manifold M_n.

Let an n-dimensional manifold M_n be given, and a geometric object $y \in C^1$ defined on it, given by a set of N functions $y^A(x)$, $A = 1, \ldots, N$ in each local chart $(U_\alpha, \varphi_\alpha)$. A first order system of PDEs of Cauchy type on M_n with respect to a geometric object y can be given in the following way. In any coordinate domain, a system of differential equations (8.1) is given, where on the right-hand sides, there are expressions depending on chart coordinates x as well as on coordinates of the geometric object y. Of course, on each chart, the differential system can be completed by additional requirements of type (8.5), and a mixed system arises. By step-by-step integration, accounting uniqueness conditions in each coordinate chart $(U_\alpha, \varphi_\alpha)$ (that is, demanding differentiability with respect to x and y^A), we check that there exists at most one global solution.

There might exist local solutions in each chart $(U_\alpha, \varphi_\alpha)$ while no global solution exists. In principle, on an overlapping $U \cap V$ of coordinate domains, there are two possibilities.

Either a solution existing in U coincides with some solution found in V; then we can extend the solution to $U \cup V$. Of course, a solution which can be step-by step extended (in an obvious way) onto each overlap of coordinate neighborhoods has a global character.

Or else, there exists U such that none of the possible solutions found in U can be "glued" to any solutions being found on overlapping neighborhoods to represent a solution on the corresponding union of domains. Then the solution cannot exist globally.

A global solution y of the linear system has usual properties.

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