RECONSTRUCTION OF NON-UNIFORMLY SAMPLED SIGNALS USING GERCHBERG-PAPOULIS METHOD

Ondrej Mihálik

Doctoral Degree Programme (1), FEEC BUT

E-mail: xmihal06@stud.feec.vutbr.cz

Supervised by: Petr Fiedler

E-mail: fiedlerp@feec.vutbr.cz

Abstract: Analysis of non-uniformly sampled signals is often severely limited, since most signal processing methods rely on constant sampling period. If we still want to apply these methods, the signal must be resampled. Gerchberg-Papoulis algorithm is a method of signal reconstruction. It is commonly used for band-limited extrapolation of uniformly sampled data. We show that it is suitable for reconstruction of non-uniformly sampled signals as well. Our target application is reconstruction of time series measured by a car driving simulator. To demonstrate the benefits of band-limited reconstruction, we compare it with standard interpolation methods. The main advantage of the proposed algorithm is its ability to deal with noise and sampling jitter.

Keywords: Gerchberg-Papoulis, non-uniform sampling, jitter, band-limited signal, one-step reconstruction.

1 INTRODUCTION

Signals cannot be sampled perfectly. During sampling, we are restricted by time, physical or financial constraints. Sometimes the acquired signal is not sampled as uniformly, or densely, as we require. If the quality of sampling cannot be improved, we have to use mathematical post-processing methods to reconstruct the original signal.

An example of such situation occurs in a car-driving simulator [1]. Non-uniform sampling rates are caused by non-real time properties of the operating system. The simulator uses input devices (a steering wheel and pedals) which measure driver's reactions. Sampling of such signals can be greatly improved by adding custom-made digital sensors with built-in microcontroller to assure real-time sampling; there is no need for signal reconstruction. However, many other signals are simulated—e.g., the position of the car. Their samples are generated by a computer. Therefore, their sampling cannot be improved by mounting additional actuators; we are forced to reconstruct the signal by mathematical methods.

The paper presents a solution to the problem of non-uniform sampling. It is divided as follows. In Section 2, we recall some fundamental facts concerning the sampling theory. We focus on the peculiarities of non-uniform sampling and reconstruction. In Section 3 we discuss the computational complexity. Section 4 is devoted to numerical experiments. At first, we reconstruct a theoretical signal to support full understanding of the method. Then we show reconstruction of a signal from the vehicle driving simulator. This demonstrates that the method is fully suitable for practical data processing.

2 THEORY

2.1 UNIFORMLY SAMPLED BAND-LIMITED SIGNALS

A signal $f(t)$ is related to its spectrum $F(\omega)$ by the Fourier transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}t. \tag{1}$$

Here t denotes time and ω stands for angular frequency. Signals which satisfy

$$F(\omega) = 0, \qquad \text{for} \quad |\omega| > \Omega, \tag{2}$$

are called band-limited with band-limit Ω. They possess some very useful properties—engineers are well acquainted with the famous Shannon-Nyquist sampling theorem. It states that band-limited signal $f(t)$ can be fully represented by its samples. There is no loss of information and it can be perfectly reconstructed by means of sinc interpolation:

$$f(t) = \sum_{k=-\infty}^{\infty} f(kT_s) \frac{\sin \pi(t/T_s - k)}{\pi(t/T_s - k)}, \qquad \text{for} \quad 0 < T_s < \frac{\pi}{\Omega}. \tag{3}$$

2.2 Non-uniformly Sampled Band-Limited Signals

It is not so widely known that the sampling density requirement also holds for non-uniformly sampled signals. Henry Landau had shown that non-uniformly sampled band-limited signals can be perfectly reconstructed, if the average sampling frequency is at least two times higher than the maximal frequency in the original signal's spectrum [2].

Of course, the reconstruction can no longer be achieved via the well-known sinc interpolation. Still, it can be solved using Gerchberg-Papoulis (GP) method—a method which is normally used for extrapolation of uniformly sampled data [3]. In [4] GP method was used for reconstruction of non-uniformly sampled data. The authors probably did not realise that they had, in fact, implemented GP method.

2.3 Gerchberg-Papoulis Method

Let D denote a time-limiting operator

$$Df(t) = \begin{cases} f(t) & \text{for} \quad |t| \leq 1, \\ 0 & \text{for} \quad |t| > 1. \end{cases} \tag{4}$$

Similarly, let B denote a band-limiting operator

$$Bf(t) = \frac{1}{2\pi} \int_{-\Omega}^{\Omega} F(\omega) e^{j\omega t} d\omega. \tag{5}$$

If we measure a signal $f(t)$, and the measurement is of finite length, we get a time-limited signal $g_0(t) = Df(t)$. If the original signal $f(t)$ is band-limited—i.e., $f(t) = Bf(t)$—then it can be reconstructed using GP method. The method consists of time-limiting and band-limiting operations. The result of m'th iteration $g_m(t)$ is given by the equation

$$g_m(t) = \bar{D} B g_{m-1}(t) + g_0(t), \tag{6}$$

where $\bar{D}$ is the complement of D, so that for any signal

$$f(t) = Df(t) + \bar{D}f(t). \tag{7}$$

The following figure illustrates the first GP iteration.

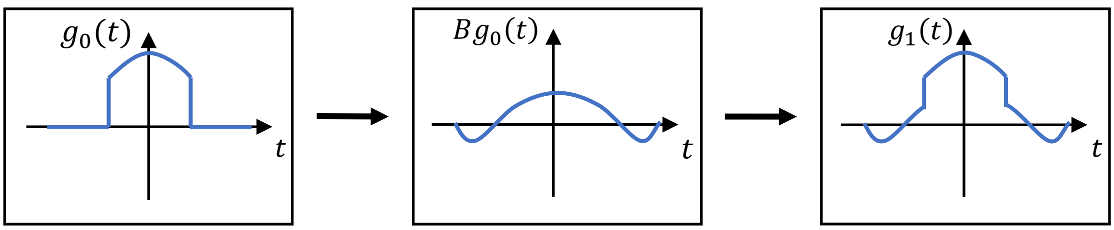

Figure 1: The first iteration of GP method.

We start with the known signal $g_0(t)$. The operator B smoothens the signal so that is spreads out (Figure 1, middle). But is also changes its values in the observed interval, so the following step is time-limiting by $\overline{D}$ and adding $g_0(t)$. This way we correct all known values (Figure 1, right). Then another iteration may follow.

Now we can take the last step towards the reconstruction of non-uniformly sampled signals. We make a generalization, that sole sample may be regarded as an infinitely small observation interval. Then Figure 1 shows what happens to individual sample. It widens, but its central point stays unaltered. So, at non-uniform sampling, D is not a rectangular window; it is a union of many infinitely small intervals.

3 COMPUTATIONAL EFFICIENCY

The iterative algorithm requires large computational efforts. In each cycle, it uses Fast Fourier Transform (FFT) and inverse FFT for long sequences of data. As we get closer to the solution, the convergence slows down. The problem can be bypassed via accelerated GP algorithm [5] or one-step GP algorithm [6]. We choose the latter. Then, the reconstruction can be computed using single matrix inverse and four matrix multiplications.

The last problem is that the calculations become numerically instable as the band-limit and the length of the data increase. To make the inverse computationally feasible, we had to apply Tikhonov regularization. The approach is similar to [7]; however, we do not use Prolate Spheroidal Wave Functions. They would pose a heavy computational burden, which can only be bypassed by storing a large number of their samples to cover all possible positions of signal's non-uniform samples.

4 NUMERICAL EXPERIMENTS

4.1 RECONSTRUCTION OF A SINE WAVE

To demonstrate the importance of band-limit, we show reconstruction of a sine wave with angular frequency of 16π rad/s. The chosen signal is intentionally as simple as possible. This way we can fully concentrate on the proposed method.

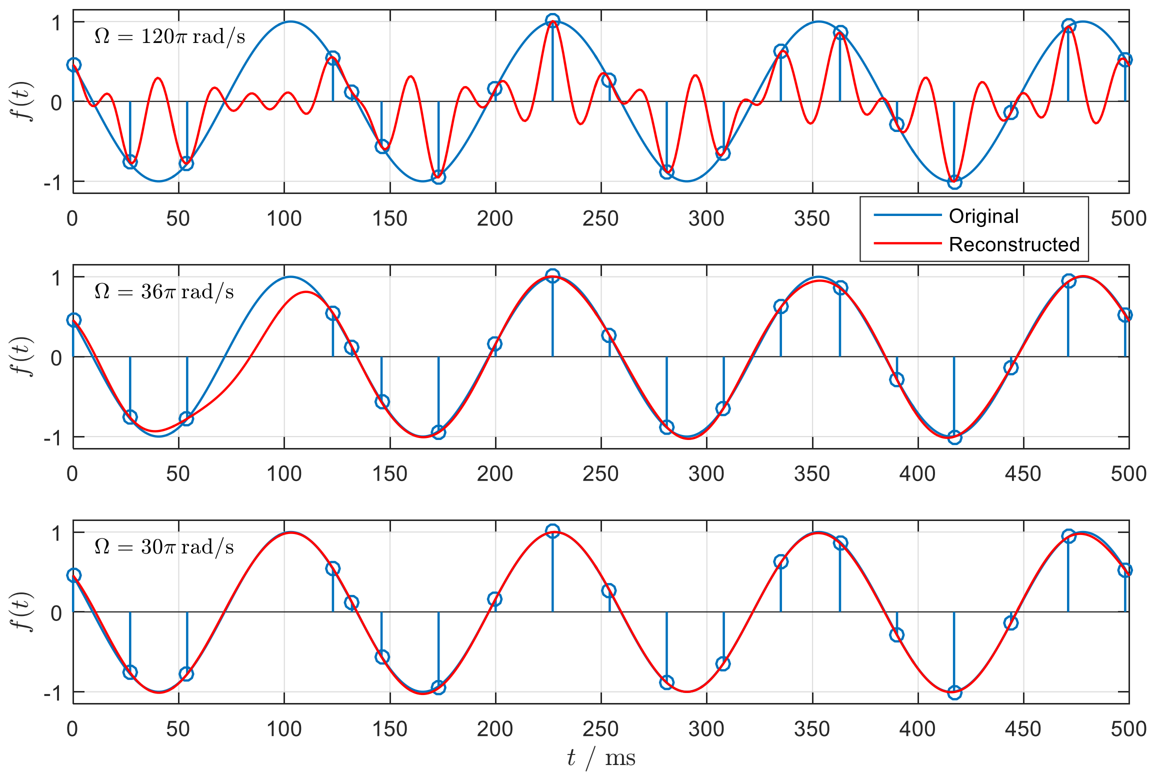

Figure 2: Sampled sine signal and its reconstruction for different band-limiting values Ω.

At first, the sine signal is sampled. We use real non-uniform time stamps as they were recorded by the car driving simulator. Then, in order to make the signal more realistic, we add Gaussian noise with SNR of 40 dB. This way we simulate sampling of a real noisy signal. Continuous sine signal $f(t)$ with its noisy samples $f_s(t)$ is shown in Figure 2 (blue).

Sampled signal $f_s(t)$ can be understood as a sequence of Dirac pulses multiplied by the original signal's amplitude.

$$f_s(t) = \sum_{k=-\infty}^{\infty} f(t_k) \cdot \delta(t - t_k). \tag{8}$$

For large band-limit Ω, the reconstructed signal still resembles closely to the sequence of Dirac pulses (Figure 2, top, red).

By decreasing the band-limit Ω, pulses are widened and smoothened. We set $\Omega = \pi/T$, where T is the average sampling period. This yields relatively good reconstruction (Figure 2, middle, red).

Finally, we reduce the band-limit, so that $\Omega = \pi/(1.2 \cdot T)$. When the Ω is small enough, we get the desired reconstruction (Figure 2, bottom, red). A severe sampling non-uniformity is visible in the time interval 60 ms $< t <$ 120 ms. Clearly, the method is capable of solving the problem. The reconstructed signal is nearly indistinguishable from the original. The small discrepancies are mainly due to the added Gaussian noise.

4.2 Reconstruction of Measured Data

The car driving simulator collects data from a simulated vehicle. An example of frequently used scenario is a long motorway drive. The simulator records car's distance from the centre of the lane (measured in metres) and stores it for further analysis. Obtained samples can be seen in Figure 3 (blue).

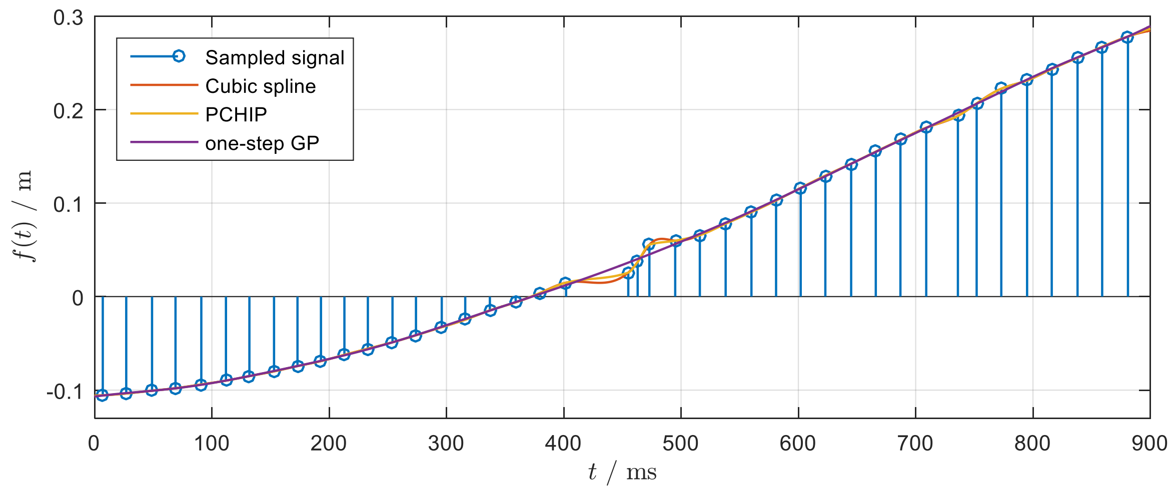

Figure 3: Sampled and reconstructed signal (position of the simulated car).

The figure shows that the car moves from the right edge of the lane to the left. The driver tries to perform rather slow changes, as the car has relatively high speed. Thus, the signal varies slowly. Nevertheless, three samples after $t = 400$ ms are visibly delayed (see Figure 3 in the middle). The anomaly is physically improbable; it should be regarded as a sampling error.

Figure 3 compares one-step GP method with interpolation by cubic spline and Piecewise Cubic Hermite Interpolation Polynomial (PCHIP). If a sample is corrupted, interpolation creates undesirable oscillations. On the other hand, GP method simply ignores all corrupted samples and the signal remains smooth. This is the method's main advantage in practical applications. If the interpolation was used in practice, the spurious changes would complicate further data processing—e.g., the derivative of the signal would be severely distorted.

5 CONCLUSION

We have proposed a method for reconstruction of non-uniformly sampled signals based on GP method. The method has simple interpretation—it yields a band-limited reconstruction of sampled signal. In contrast with interpolation methods, it is capable of dealing with noise and corrupted samples. This is an important advantage, if the processed samples contain noise.

The requirement of iterative computation was bypassed by computing matrix inverse. To avoid the 'blow-up' problem, we have applied Tikhonov regularization. The proposed algorithm was tested with simple sine-wave signal and with the data from vehicle driving simulator. Our results suggest that the reconstruction is fully suitable for practical applications.

We realise that there are other methods, which work for noisy signals—an example is fitting the samples with polynomials [8]—but these methods are rather heuristic. They require more parameters, most of which do not seem to have any meaningful physical interpretation. In contrast, the band-limit Ω is well understood by the engineers and it can be selected easily. It has the same interpretation as for uniformly sampled signals.

The sole disadvantage of the proposed method is the requirement of correctly chosen regularization parameter. We are currently conducting further research to design an algorithm to select it adaptively.

ACKNOWLEDGEMENT

The completion of this paper was made possible by the grant No. FEKT-S-20-6205 – "Research in Automation, Cybernetics and Artificial Intelligence within Industry 4.0" financially supported by the Internal science fund of Brno University of Technology.

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