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SOME NEW SUBSPACES OF AN FK-SPACE AND DEFERRED CESÀRO CONULLITY

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Abstract. In this paper, we construct new important subspaces D_p^qS, D_p^qW, D_p^qF and D_p^qB for a locally convex FK-space X containing ϕ, the space of finite sequences. Then, we show that there is a relation among these subspaces. Also, we study deferred Cesàro conullity of one FK-space with respect to another, and we give some important results. Finally, we examine the deferred Cesàro conullity of the absolute summability domain l_A, and show that if l_A is deferred Cesàro conull, then A cannot be l-replaceable.

1. Introduction

Let w denote the space of all complex valued sequences. It can be topologized with the seminorms $r_n(x) = |x_n|$, $n = 1, 2, \ldots$, and any vector subspace X of w is a sequence space. A sequence space X with a vector space topology τ is a K-space provided that the inclusion map $i : (X, \tau) \to w$, $i(x) = x$, is continuous. If, in addition, τ is complete, matrizable, locally convex, then (X, τ) is called an FK-space. So, an FK-space is a complete, metrizable locally convex topological vector space of sequences for which the coordinate functionals are continuous. An FK-space whose topology is normable is called a BK-space. The basic properties of an FK-space may be found in (see [2, 11, 13]).

By c, c_0, l_∞, we denote the spaces of convergent sequences, null sequences and bounded sequences, respectively. These are FK-spaces under $\|x\| = \sup_n |x_n|$. By cs, l, we denote the spaces of all summable sequences, absolute summable sequences, respectively.

Throughout this paper, e denotes the sequences of ones; δ^j $(j = 1, 2, \ldots)$ the sequence with the one in the j-th position; ϕ the linear span of δ^j's. The linear span of ϕ and e is denoted by ϕ_1. The topological dual of X is denoted by X'. A sequence x in a locally convex sequence space X is said to be the property AK if $x^{(n)} \to x$ in X, where $x^{(n)} = \sum_{k=1}^{n} x_k \delta^k$. The space X is said to have AD if ϕ is dense in X.

We recall (see [2, 11]) that the f, β-duals of a subset X of w are

$$X^f = \left\{ \{f(\delta^k)\} : f \in X' \right\},$$

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$$X^{\beta} = \left\{ x \in w : \sum_{n=1}^{\infty} x_k y_k \text{ is convergent for all } y \in X \right\}.$$

In 1932, Agnew [1] defined the deferred Cesàro mean $D_{p,q}$ of the sequences x by

$$(D_{p,q}x)_n = \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_k,$$

where $\{p(n)\}$ and $\{q(n)\}$ are sequences of nonnegative integers satisfying the conditions $p(n) < q(n)$ and $\lim_{n\to\infty} q(n) = \infty$. We note here that $D_{p,q}$ is clearly regular for any choice of $\{p(n)\}$ and $\{q(n)\}$. We define some new sequence spaces by using a deferred Cesàro mean.

The sequence spaces

$$[\sigma_0]_p^q := \left\{ x \in w : \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_k = 0 \right\},$$

$$[\sigma_c]_p^q := \left\{ x \in w : \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_k \text{ exists} \right\},$$

$$[\sigma_\infty]_p^q := \left\{ x \in w : \sup_n \frac{1}{q(n)-p(n)} \left| \sum_{k=p(n)+1}^{q(n)} x_k \right| < \infty \right\},$$

$$\sigma_p^q[s] := \left\{ x \in w : \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \text{ exists} \right\}$$

and

$$\sigma_p^q[b] := \left\{ x \in w : \sup_n \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \right| < \infty \right\}$$

are BK-spaces with the norms

$$\| x \|_{[\sigma_0]_p^q} = \sup_n \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x_k \right|$$

and

$$\| x \|_{\sigma_p^q[s]} = \sup_n \left| \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \right|.$$

This assertion can be proved along the same lines as in (see [3, 4, 6, 7]), so we omit the details.

A sequence x in a locally convex sequence space X is said to be the property $\sigma_p^q[K]$ if

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \qquad \text{in } X.$$

A sequence x in a locally convex sequence space X is said to be the property $\sigma_p^q[B]$ if $\forall x \in X$

$$\left\{ \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \text{ is bounded in } X$$

Now, we define new d-, $d[b]$-type duals of a sequence space X containing ϕ.

$$\begin{aligned} X^d &= \left\{ x \in w : \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j y_j \text{ exists for all } y \in X \right\} \\ &= \left\{ x \in w : x.y \in \sigma_p^q[s] \text{ for all } y \in X \right\}, \end{aligned}$$

$$\begin{aligned} X^{d[b]} &= \left\{ x \in w : \sup_n \frac{1}{q(n)-p(n)} \left| \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j y_j \right| < \infty \;,\; y \in X \right\} \\ &= \left\{ x \in w : x.y \in \sigma_p^q[b] \text{ for all } y \in X \right\}, \end{aligned}$$

respectively, where $x.y = (x_n y_n)$.

Let X, Y be sets of sequences. Then, for $\nu = f, \beta, b, d[b]$,

(i) $X \subset X^{\nu\nu}$,
(ii) $X^{\nu\nu\nu} = X^{\nu}$,
(iii) If $X \subset Y$, then $Y^{\nu} \subset X^{\nu}$ holds.

Theorem 1.1. *Let X be an FK-space $\supset \phi$, and*

$$\lim_{n\to\infty} \frac{q(n)-i+1}{q(n)-p(n)} = 1 \qquad (i \leq q(n)).$$

Then,

(i) $X^{\beta} \subset X^d \subset X^{d[b]} \subset X^f$,
(ii) *if X is a $\sigma_p^q[K]$-space, then $X^f = X^d$*

and

(iii) *if X is an AD-space, then $X^{d[b]} = X^d$.*

Proof. (ii) Suppose that $u \in X^d$ and

$$f(x) = \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j u_j$$

for $x \in X$. Then, $f \in X'$ by the Banach–Steinhaus theorem. Now, we get

$$f(\delta^i) = \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} u_j \delta^i = \lim_{n\to\infty} \frac{q_n - i + 1}{q(n)-p(n)} u_i = u_i,$$

so $u \in X^f$. Thus, $X^d \subset X^f$.

Let $u \in X^f$. Since X is a $\sigma_p^q[K]$-space, we have

$$f(x) = \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^j)$$
$$= \lim_{n\to\infty} \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j u_j$$

for all $x \in X$. Then, $u \in X^d$. That means $X^f \subset X^d$. This proves (ii).

(iii) Let $u \in X^{d[b]}$. Assume that

$$f_n(x) = \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} u_j x_j$$

for all $x \in X$. Then, $\{f_n\}$ is pointwise bounded and hence equicontinuous by [11]. Since

$$\lim_{n\to\infty} f_n(\delta^i) = u_i, \qquad i < q(n),$$

$\phi \subset \{x : \lim_n f_n(x) \text{ exists}\}$ is a closed subspace of X by the Convergence lemma (see [11]). Also, since X is an AD-space, $X = \{x : \lim_n f_n(x) \text{ exists}\} = \overline{\phi}$, hence $\lim_n f_n(x)$ exists for all $x \in X$. So, $u \in X^d$. The opposite inclusion is trivial.

(i) By the hypothesis, $\overline{\phi} \subset X$. Since $\overline{\phi}$ is an AD-space, by (ii), (iii) and [11, (7.2.4)], we get

$$X^{d[b]} \subset (\overline{\phi})^{d[b]} = (\overline{\phi})^d \subset (\overline{\phi})^f = X^f.$$

□

2. Main results

We shall define some new subspaces of a locally convex FK-space X containing ϕ, the space of finite sequences, which are of importance for each one in the topological sequence spaces theory.

Definition 2.1. Let X be an FK-space $\supset \phi$. Then,

$$D_p^q W := D_p^q W(X) = \left\{ x \in X : \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \ \text{ (weakly) in } X \right\}$$
$$= \left\{ x \in X : f(x) = \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^j) \text{ for all } f \in X \right\},$$

$$D_p^q S := D_p^q S(X) = \left\{ x \in X : \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \right\}$$
$$= \left\{ x \in X : x \text{ has } \sigma_p^q[K] \text{ in} X \right\}$$

$$= \left\{ x \in X : x = \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j \delta^j \right\}.$$

Thus, X is a $\sigma_p^q[K]$-space if and only if $D_p^q S = X$.

$$D_p^q F^+ := D_p^q F^+(X)$$

$$= \left\{ x \in w : \lim_n \left\{ \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \text{ is weakly Cauchy in } X \right\}$$

$$= \left\{ x \in w : \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^j) \text{ exists for all } f \in X' \right\}$$

$$= \left\{ x \in w : \left\{ x_n f(\delta^n) \right\} \in \sigma_p^q[s] \text{ for all } f \in X' \right\} = (X^f)^d.$$

$$D_p^q B^+ := D_p^q B^+(X) = \left\{ x \in w : \left\{ \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\} \text{ is bounded in } X \right\}$$

$$= \left\{ x \in w : (x_n f(\delta^n)) \in \sigma_p^q[b] \text{ for all } f \in X' \right\}$$

also $D_p^q F = D_p^q F^+ \cap X$ and $D_p^q B = D_p^q B^+ \cap X$.

We now study some inclusions which are analogous to those given in [11, Chapter 10]. Also, we prove some theorems related to the f-, d- and $d[b]$-duality of a sequence space X.

Theorem 2.2. *Let X be an FK-space $\supset \phi$. Then,*

$$\phi \subset D_p^q S \subset D_p^q W \subset D_p^q F \subset D_p^q B \subset X \text{ and } \phi \subset D_p^q S \subset D_p^q W \subset \overline{\phi}.$$

Proof. The only non-trivial part is $D_p^q W \subset \overline{\phi}$. Let $f \in X'$ and $f = 0$ on ϕ. The definition of $D_p^q W$ shows that $f = 0$ on $D_p^q W$. Hence, the Hahn–Banach theorem gives the result. □

Theorem 2.3. *The subspaces $E = D_p^q S$, $D_p^q W$, $D_p^q F$, $D_p^q F^+$, $D_p^q B$ and $D_p^q B^+$ of X FK-spaces are monotone, i.e., if $X \subset Y$, then $E(X) \subset E(Y)$.*

Proof. The inclusion map $i : X \to Y$ is continuous by [11, (4.2.4)], so

$$\frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \to x \qquad \text{in } X$$

implies the same in Y. This proves the assertion for $D_p^q S$. For $D_p^q W$, it follows from the fact that i is weakly continuous by [11, (4.0.11)]. Now, $z \in D_p^q F^+$, $D_p^q B^+$ if and only if $(z_n f(\delta^n)) \in \sigma_p^q[s]$, $\sigma_p^q[b]$, respectively, for all $f \in X'$, hence for all $g \in Y'$ since $g|X \in X'$ by [11]. The result follows for $D_p^q F^+$, $D_p^q B^+$ and, thus, for $D_p^q F$, $D_p^q B$. □

Since $[\sigma_0]_p^q$ is an AK-space, we immediately get the following:

Theorem 2.4. *Let X be an FK-space $\supset [\sigma_0]_p^q$. Then, $[\sigma_0]_p^q \subset D_p^q S \subset D_p^q W$.*

Theorem 2.5. *Let X be an FK-space $\supset \phi$. Then, $D_p^q B^+ = X^{fd[b]}$.*

Proof. By Definition 2.1, $z \in D_p^q B^+$ if and only if $z \cdot u \in \sigma_p^q[b]$ for each $u \in X^f$. This is precisely the assertion. □

Theorem 2.6. *Let X be an FK-space $\supset \phi$. Then, $D_p^q B^+$ is the same for all FK-spaces Y between $\overline{\phi}$ and X; i.e., $\overline{\phi} \subset Y \subset X$ implies $D_p^q B^+(Y) = D_p^q B^+(X)$. Here, the closure of ϕ is calculated in X.*

Proof. By Theorem 2.3, we have $D_p^q B^+(\overline{\phi}) \subset D_p^q B^+(Y) \subset D_p^q B^+(X)$. By Theorem 2.5 and [11, (7.2.4)], the first and the last are equal. □

Theorem 2.7. *Let X be an FK-space such that $D_p^q B \supset \overline{\phi}$. Then, $\overline{\phi}$ has $\sigma_p^q[K]$ and $D_p^q S = D_p^q W = \overline{\phi}$.*

Proof. Suppose first that X has $\sigma_p^q[B]$. Define $f_n : X \to X$ by

$$f_n(x) = x - \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)}.$$

Then, $\{f_n\}$ is pointwise bounded, hence equicontinuous by [11, (7.0.2)]. Since $f_n \to 0$ on ϕ, then also $f_n \to 0$ on $\overline{\phi}$ by [11, (7.0.3)]. This is the desired conclusion. □

Theorem 2.8. *Let X be an FK-space $\supset \phi$. Then, $D_p^q F^+ = X^{fd}$.*

Proof. This may be proved as in Theorem 2.5, with d instead of $d[b]$. □

Theorem 2.9. *Let X be an FK-space $\supset \phi$. Then, $D_p^q F^+$ is the same for all FK-spaces Y between $\overline{\phi}$ and X; i.e., $\overline{\phi} \subset Y \subset X$ implies $D_p^q F^+(Y) = D_p^q F^+(X)$ (the closure of ϕ is calculated in X).*

The proof is similar to that of Theorem 2.6.

Lemma 2.10. *Let X be an FK-space in which $\overline{\phi}$ has $\sigma_p^q[K]$. Then, $D_p^q F^+ = (\overline{\phi})^{dd}$.*

Proof. Observe that $D_p^q F^+ = X^{fd}$ by Theorem 2.8. Since $X^f = (\overline{\phi})^f$ by [11, Thm. 7.2.4], we have $X^{fd} = (\overline{\phi})^{fd}$. Hence, by [8, Thm. 1.9], the result follows. □

An FK-space X is said to have $F\sigma_p^q[K]$ (functional $\sigma_p^q[K]$) if $X \subset D_p^q F^+$, i.e., $X = D_p^q F$.

Theorem 2.11. *Let X be an FK-space $\supset \phi$. Then, X has $F\sigma_p^q[K]$ if and only if $\overline{\phi}$ has $\sigma_p^q[K]$ and $X \subset (\overline{\phi})^{dd}$.*

Proof. Necessity. X has $\sigma_p^q[B]$ since $D_p^q F \subset D_p^q B$, so $\overline{\phi}$ has $\sigma_p^q[K]$ by Theorem 2.7. The remainder of the proof follows from Lemma 2.10. Sufficiency is given by Lemma 2.10. □

Theorem 2.12. *Let X be an FK-space $\supset \phi$. The following are equivalent:*

(i) *X has $F\sigma_p^q[K]$,*
(ii) $X \subset (D_p^q S)^{dd}$,
(iii) $X \subset (D_p^q W)^{dd}$,
(iv) $X \subset (D_p^q F)^{dd}$,
(v) $X^d = (D_p^q S)^d$,
(vi) $X^d = (D_p^q F)^d$.

Proof. Observe that (ii) implies (iii) and (iii) implies (iv) and that they are trivial since

$$D_p^q S \subset D_p^q W \subset D_p^q F \ .$$

If (iv) is true, then $X^f \subset (D_p^q F)^d = (X^f)^{dd} \subset X^d$, so (i) is true by [8, Thm. 1.9]. If (i) holds, then Theorem 2.11 implies that $\overline{\phi} = D_p^q S$ and that (ii) holds. The equivalence of (v), (vi) with the others is clear. □

Theorem 2.13. *Let X be an FK-space $\supset \phi$. The following are equivalent:*
(i) *X has $S\sigma_p^q[K]$,*
(ii) *X has $\sigma_p^q[K]$,*
(iii) $X^d = X'$.

Proof. Clearly, (ii) implies (i). Conversely, if X has $S\sigma_p^q[K]$, it must have AD for $D_p^q W \subset \overline{\phi}$ by Theorem 2.2. It also has $\sigma_p^q[B]$ since $D_p^q W \subset D_p^q B$. Thus, X has $\sigma_p^q[K]$ by Theorem 2.7, which proves that (i) and (ii) are equivalent. Assume that (iii) holds. Let $f \in X'$, then there exists $u \in X^d$ such that

$$f(x) = \lim_{n\to\infty} \frac{1}{q(n) - p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} u_j x_j$$

for $x \in X$. Since $f(\delta^j) = u_j$, it follows that each $x \in D_p^q W$, which shows that (iii) implies (i). That (ii) implies (iii) is known (see [6, p. 97]). □

Theorem 2.14. *Let X be an FK-space $\supset \phi$. The following are equivalent:*
(i) *$D_p^q W$ is closed in X,*
(ii) $\overline{\phi} \subset D_p^q B$,
(iii) $\overline{\phi} \subset D_p^q F$,
(iv) $\overline{\phi} = D_p^q W$,
(v) $\overline{\phi} = D_p^q S$,
(vi) *$D_p^q S$ is closed in X.*

Proof. (ii) implies (v): By Theorem 2.7, $\overline{\phi}$ has $\sigma_p^q[K]$, i.e., $\overline{\phi} \subset D_p^q S$. The opposite inclusion is clear from Theorem 2.2. Note that (v) implies (iv), (iv) implies (iii) and (iii) implies (ii) because

$$D_p^q S \subset D_p^q W \subset \overline{\phi} \ , \ D_p^q W \subset D_p^q F \subset D_p^q B \ ;$$

(i) implies (iv) and (vi) implies (v) since $\phi \subset D_p^q S \subset D_p^q W \subset \overline{\phi}$. Finally, (iv) implies (i) and (v) implies (vi). □

3. Combinations of new type subspaces of an FK-space

Let $A=(a_{nk})$, $n,k=1,2,\ldots$, be an infinite matrix with complex entries and $c_A=\{x: Ax\in c\}$. Then, c_A is an FK-space with seminorms

$$\rho_0(x)=\sup_n\left|\sum_{k=1}^{\infty}a_{nk}x_k\right| \quad (n=1,2,\ldots),$$

$\rho_n(x)=|x_n|$, $(n=1,2,\ldots)$; and $h_n(x)=\sup_m\left|\sum_{k=1}^{m}a_{nk}x_k\right|$ $(n=1,2,\ldots)$. Also, every $f\in c'_A$ if and only if

$$f(x)=\sum_{k=1}^{\infty}\beta_k x_k+\sum_{n=1}^{\infty}t_n\sum_{k=1}^{\infty}a_{nk}x_k+\mu\lim{}_A x\ ,$$

where $t\in l$, $\mu\in\mathbf{C}$, $(\beta_k)\in c_A^{\beta}$, the β-dual of c_A (see [11, 4.4.3]). The representation is not unique; we say that A is μ-unique if all representations for some f have the same μ. If A is μ-unique, $c_A\subset c_D$, D is conull with respect to A if and only if $\mu_A(\lim_D)=0$ in [12].

Let X and Y be FK-spaces, X with paranorm ρ and Y with paranorm s. It is shown that $Z=X+Y$ with the unrestricted inductive limit topology is an FK-space as in [11, Thm. 4.5.1]. The paranorm τ of Z is given by

$$\tau(z)=\inf_{\substack{x+y=z\\ x\in X,\,y\in Y}}(\rho(x)+s(y)).$$

Let $\{X^n\}_{n=1}^{\infty}$ be a sequence of FK-spaces, ρ_n be the paranorm of X^n and $\{s_{nk}\}_{k=1}^{\infty}$ be the seminorms of X^n. Let $Y=\bigcap_n X^n$. It is well known that Y is an FK-space with paranorm $s=\sum_{n=1}^{\infty}\frac{\rho_n}{2^n(1+\rho_n)}$ and seminorms $\{s_{nk}\}_{n,k=1}^{\infty}$.

We now investigate some important subspaces of a locally convex FK-space X containing ϕ which are analogous to those given in [5]. To prove the theorems of this section, we use the same technique by DeVos [5].

Theorem 3.1. *Let X, Y be FK-spaces and $Z=X+Y$. Then, $E(X)+E(Y)\subseteq E(Z)$ for $E=D_p^qS$, D_p^qW, D_p^qF or D_p^qB.*

Proof. Let $E=D_p^qS$. We take $x\in D_p^qS(X)$ and $y\in D_p^qS(Y)$. Then,

$$\rho\left(\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}x^{(k)}-x\right)\to 0$$

and

$$s\left(\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}y^{(k)}-y\right)\to 0$$

as $n\to\infty$. Hence,

$$r\left(\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}(x+y)^{(k)}-(x+y)\right)$$

$$\leq \rho\left(\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} x^{(k)} - x\right) + s\left(\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} y^{(k)} - y\right),$$

which implies that $x+y \in D_p^q S(Z)$.

Let $E = D_p^q W$. We take $x \in D_p^q W(X)$, $y \in D_p^q W(Y)$ and $f \in Z'$. Then, $f|X \in X'$ and $f|Y \in Y'$.

$$\begin{aligned} & f(x+y) = f(x) + f(y) \\ &= \lim_n \frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\sum_{j=1}^{k} f(\delta^j)x_j + \lim_n \frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\sum_{j=1}^{k} f(\delta^j)y_j \\ &= \lim_n \frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\sum_{j=1}^{k} f(\delta^j)(x_j+y_j). \end{aligned}$$

The proofs for $E = D_p^q F$ or $D_p^q B$ are similar, so the details are omitted. □

Theorem 3.2. *Let $\{X^n\}_{n=1}^{\infty}$ be a sequence of FK-spaces and $Y = \bigcap_n X^n$. Then, $E(Y) = \bigcap_n E(X^n)$ for $E = D_p^q S$, $D_p^q W$, $D_p^q F$ or $D_p^q B$.*

Proof. By Theorem 2.3, for each n, $E(Y) \subseteq E(X^n)$, hence $E(Y) \subseteq \bigcap_n E(X^n)$ for $E = D_p^q S$, $D_p^q W$, $D_p^q F$ or $D_p^q B$.

Let $z \in \bigcap_n D_p^q S(X^n)$. Then,

$$s_{nk}\left(\frac{1}{q(r)-p(r)}\sum_{n=p(r)+1}^{q(r)} z^{(n)} - z\right) \to 0, \qquad r \to \infty,$$

for each fixed n and k, but these are the seminorms for Y. Hence,

$$\lim_{r\to\infty} \frac{1}{q(r)-p(r)}\sum_{n=p(r)+1}^{q(r)} z^{(n)} = z \qquad \text{in } Y,$$

which implies that $z \in D_p^q S(Y)$.

Let $z \in \bigcap_n D_p^q W(X^n)$ and $f \in Y'$. Then, we have $f = \sum_{j=1}^{h} f_j$, where $f_j \in (X^j)'$ (see [12]). Since

$$f_j\left(\frac{1}{q(r)-p(r)}\sum_{n=p(r)+1}^{q(r)} z^{(n)}\right) \to f_j(z)$$

for $j = 1, 2, \ldots, h$. Therefore,

$$f\left(\frac{1}{q(r)-p(r)}\sum_{n=p(r)+1}^{q(r)} z^{(n)}\right) \to f(z).$$

Hence, $z \in D_p^q W(Y)$.

The proof for $E = D_p^q F$ is similar to the one of the previous paragraph, so we omit the details.

Let $z \in \bigcap_n D_p^q B(X^n)$. Then, for any fixed l and k,

$$s_{lk}\left(\frac{1}{q(r)-p(r)}\sum_{n=p(r)+1}^{q(r)} z^{(n)}\right) \le H_{lk}$$

for all r. Hence, $z \in D_p^q B(Y)$. □

In [10], let X be an FK-space containing ϕ_1 and

$$\begin{aligned}\zeta^n &:= e - \frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} e^{(k)} \\ &= \left(0,0,\dots,0,\frac{1}{q(n)-p(n)},\frac{2}{q(n)-p(n)},\frac{3}{q(n)-p(n)},\right. \\ &\qquad\left.\dots,\frac{q(n)-p(n)-1}{q(n)-p(n)},1,1,\dots\right), \end{aligned} \tag{3.1}$$

where $e^{(k)} := \sum_{j=1}^{k}\delta^j$. If $\zeta^n \to 0$ in X, then X is called strongly deferred Cesàro conull. If the convergence holds in the weak topology in (3.1), then X is called deferred Cesàro conull. Hence, X is deferred Cesàro conull iff

$$f(e) = \lim_n \frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)}\sum_{j=1}^{k} f(\delta^j)\ , \qquad \forall f \in X'.$$

Now, we define deferred Cesàro conullity of one FK-space with respect to another.

Definition 3.3. Let X be an FK-space with $D_p^q W(X) \neq D_p^q B(X)$ and Y be an FK-space, $X \subseteq Y$. Y is deferred Cesàro conull with respect to X iff $D_p^q B(X) \subseteq D_p^q W(Y)$.

Theorem 3.4. *Let X, Y, Z be FK-spaces with $X \subseteq Y \subseteq Z$. Then,*

(i) *If Y is deferred Cesàro conull with respect to X, then Z is deferred Cesàro conull with respect to X,*
(ii) *If Z is deferred Cesàro conull with respect to X and Y is closed in Z, then Y is deferred Cesàro conull with respect to X.*

The proof of the Theorem is clear by Definition 3.3 and Theorem 2.3.

Theorem 3.5. *Let $\{Y^n\}_{n=1}^{\infty}$ be FK-spaces such that each Y^n is deferred Cesàro conull with respect to X. Then, $\bigcap_n Y^n$ is deferred Cesàro conull with respect to X.*

The proof of the Theorem is obtained by Definition 3.3 and Theorem 3.2.

Let $E(c_A) = E(A)$ for a matrix A and $E = D_p^q W$ or $D_p^q B$ and $\mu_A(\lim_D) = \mu_A(D)$. For many cases, the following theorem gives an equivalence between Wilansky's and our extensions of deferred Cesàro conullty.

Theorem 3.6. *Let A and D be matrices with $D_p^q W(A) \neq D_p^q B(A)$ and $c_A \subset c_D$. $\mu_A(D) = 0$ if and only if c_D is deferred Cesàro conull with respect to c_A.*

Proof. Let c_D be deferred Cesàro conull with respect to c_A. For $x \in D_p^q B(A)$, $\lim_D x = \mu_A(D) \lim_A x + \beta x$.

Now, let $z \in D_p^q B(A) \setminus D_p^q W(A)$. Firstly, by [10],

$$\lim{}_A \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right)$$

is not convergent to zero. Also, since $D_p^q B(A) \subset c_A$ and $\gamma \in c_A^\beta$, we obtained

$$\gamma \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) = \gamma \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} z_j \delta^j \right)$$
$$= \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=k+1}^{\infty} z_j \delta^j .$$

By hypothesis, for each $f \in (c_D)'$,

$$f \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \to 0 \ .$$

In particular, we take $f = \lim_D \in (c_D)'$. Thus,

$$\lim_D \left(z - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} z^{(k)} \right) \to 0.$$

Conversely, let $f \in (c_D)'$. By [12], we have $\mu_A(f) = \mu_D(f) \cdot \mu_A(D) = 0$. Hence, $f(x) = t(Ax) + \beta x$ for $x \in c_A$.

Now, we are able to write

$$f(x) = \mu_A(f) \lim{}_A x + \beta x$$

with $\gamma = tA + \beta$ for $x \in D_p^q B$ by [11]. Therefore, we get $f(x) = \gamma x$ for $x \in D_p^q B(A)$, which implies that $x \in D_p^q W(D)$. So, c_D is deferred Cesàro conull with respect to c_A. □

We establish some relations among the subspaces $D_p^q S$, $D_p^q W$, $D_p^q F$, $D_p^q F^+$.

Remark 3.7. Let X be an FK-space such that weakly convergent sequences are convergent in the FK-topology, A be a matrix such that $X_A \supset \phi$. The subspaces $D_p^q S$, $D_p^q W$ and $D_p^q F$ are calculated in X_A.

Lemma 3.8. *If X is as in Remark 3.7, then, for X itself, we have $D_p^q S = D_p^q W = D_p^q F = D_p^q F^+$.*

Proof. The inclusions $D_p^q S \subset D_p^q W \subset D_p^q F \subset D_p^q F^+$ are trivial by definitions. Conversely, if $x \in D_p^q F^+$, then

$$\left\{ \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right\}$$

is weakly Cauchy, hence by [11, 12.0.1] is Cauchy in the FK-topology of X, so convergent, say

$$\left\{\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} x^{(k)}\right\} \to y \ .$$

Since $x^{(k)} \to x$ in w, we have

$$\left\{\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} x^{(k)}\right\} \to x \qquad \text{in } w.$$

By the contiunity of $i : X \to w$, $y = x$, and $x \in D_p^q S$. □

Now, we note that if X is an FK-space containing ϕ_1, then

$$D_p^q F^+ = X^{fd}. \tag{3.2}$$

To see this, it is enough to take $\sigma_p^q[s]$ instead of cs in [11, Thm. 10.4.2]. If X is also $\sigma_p^q[K]$, we can conclude that $X^{dd} = X$ by (3.2). We have $X^{dd} = D_p^q F = D_p^q F^+ \subset X$, hence the result follows.

Theorem 3.9. *With X, A as in Remark 3.7, for the X_A, we have $D_p^q S = D_p^q W = D_p^q F = D_p^q F^+$.*

Proof. If $x \in D_p^q F^+$, then

$$\left\{\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} x^{(k)}\right\}$$

is weakly Cauchy, hence it is Cauchy in the FK-topology of X, so convergent. Since by Lemma 12.0.2 of [11] the matrix mapping $A : X_A \to X$ is continuous,

$$\left\{\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} Ax^{(k)}\right\}$$

is convergent in X, say

$$\left\{\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} Ax^{(k)}\right\} \to y.$$

On the other hand, by [11, 4.3.8], $(w_A, \rho \cup h)$ is an AK-space. Hence, it is also a $\sigma_p^q[K]$-space. Hence,

$$\left\{\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} x^{(k)}\right\} \to x \ .$$

The matrix mapping $A : w_A \to w$ is continuous, and therefore

$$\left\{\frac{1}{q(n)-p(n)}\sum_{k=p(n)+1}^{q(n)} Ax^{(k)}\right\} \to Ax \qquad \text{in } w.$$

Since $X \subset w$ and X is complete, $Ax = y$. We have

$$\left\{ \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} Ax^{(k)} \right\} \to Ax \qquad \text{in } X.$$

That is,

$$r\left(Ax - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} Ax^{(k)} \right) = (r \circ A)\left(x - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} x^{(k)} \right) \to 0$$

as $n \to \infty$, where r is a typical seminorm of X. Hence, $x \in D_p^q S$, which proves Theorem 3.9. □

3.1. Replaceability, deferred Cesàro conullity of l_A

Recall that a matrix A with $l_A \supset \phi$ is called l-replaceable if there is a matrix $D = (d_{nk})$ with $l_D = l_A$, and $\sum_n d_{nk} = 1$, $k \in \mathbb{N}$ (see [9]). It is easy to see that A is replaceable if and only if there exists $f \in l_A'$ with $f(\delta^k) = 1$ ($k \in \mathbb{N}$), namely, $f = \sum_D$.

Theorem 3.10. *Suppose that $D_p^q F = l_A$. Then, A is l-replaceable if and only if $l_A \subset \sigma_p^q[s]$.*

Proof. Assume that A is l-replaceable. Then, it follows from [9] that A is l-replaceable if and only if there is $f \in l_A'$ such that $f(\delta^j) = 1$ for all $j \in \mathbb{N}$. Since $D_p^q F = l_A$,

$$\lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j f(\delta^j) = \lim_n \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j$$

exists for all $x \in l_A$, hence $x \in \sigma_p^q[s]$.

Conversely, if $l_A \subset \sigma_p^q[s]$, then

$$f(x) = \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} x_j$$

defines an element f of l_A', and we get $f(\delta^\nu) = 1$ ($\nu = 1, 2, \ldots$). □

We now establish a relation between deferred Cesàro conullity and replaceability.

Theorem 3.11. *If the space l_A is deferred Cesàro conull, then A is not l-replaceable.*

Proof. Suppose that A is l-replaceable. Then, it follows from [9] that A is l-replaceable if and only if there is $f \in l_A'$ such that $f(\delta^j) = 1$ for all $j \in \mathbb{N}$. Hence, we have

$$
\begin{aligned}
&f(e) - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} f(\delta^j) \\
&= \left(f(e) - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} \sum_{j=1}^{k} (1) \right) = \left(f(e) - \frac{1}{q(n)-p(n)} \sum_{k=p(n)+1}^{q(n)} (k) \right) \\
&= \left(f(e) - \frac{1}{q(n)-p(n)} \frac{(q(n)-p(n))(q(n)+p(n)+1)}{2} \right) \\
&= \left(f(e) - \frac{q(n)+p(n)+1}{2} \right)
\end{aligned}
$$

is not convergent as $n \to \infty$. So, l_A is not deferred Cesàro conull. □

4. Conclusion

Deferred Cesàro conullity is a generalization of the Cesàro conullity and C_λ conullity. Indeed, if $p(n) = 0$ and $q(n) = n$, we get Cesàro conullity and if $p(n) = 0$ and $q(n) = \lambda(n)$, where $\{\lambda(n)\}$ is a strictly increasing sequence of positive integers, we get C_λ conullity.

References

[1] R. P. Agnew, *On deferred Cesàro means*, Annals of Math. **33** (1932), 413–421.
[2] J. Boos, *Classical and Modern Methods in Summability*, Oxford University Press, New York, 2000.
[3] M. Buntinas, *Convergent and bounded Cesàro sections in FK-spaces*, Math. Z. **121** (1971), 191–200.
[4] I. Dağadur, *On some subspaces of an FK-space*, Math. Commun. **7** (2002), 15–20.
[5] R. DeVos, *Combinations of distinguished subsets and conullity*, Math. Z. **192** (1986), 447–451.
[6] G. Goes and S. Goes, *Sequences of bounded variation and sequences of Fourier coefficients I*, Math. Z. **118** (1970), 93–102.
[7] G. Goes, *Sequences of bounded variation and sequences of Fourier coefficients II*, J. Math. Anal. Appl. **39** (1972), 477–494.
[8] G. Goes, *Summan von FK-Räumen funktionale Abschnıttskonvergenz und Umkehrsatz*, Tôhoku. Math. Journ. **26** (1974), 487–504.
[9] M. S. Macphail and C. Orhan, *Some properties of absolute summability domains*, Analysis **9** (1989), 317–322.
[10] Ş. Sezgek and İ. Dağadur, *Deferred Cesàro conull FK spaces*, submitted.
[11] A. Wilansky, *Summability through Functional Analysis*, North Holland, 1984.
[12] A. Wilansky, *The μ property of FK-spaces*, Comment. Math. **21** (1978), 371–380.
[13] K. Zeller, *Allgemeine Eigenschaften von limitierungsverfahren*, Math. Z. **53** (1951), 463–487.

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