BRNO UNIVERSITY OF TECHNOLOGY
VYSOKÉ UČENÍ TECHNICKÉ V BRNĚ

FACULTY OF ELECTRICAL ENGINEERING AND COMMUNICATION
FAKULTA ELEKTROTECHNIKY
A KOMUNIKAČNÍCH TECHNOLOGIÍ

DEPARTMENT OF MATHEMATICS
ÚSTAV MATEMATIKY

EXISTENCE AND PROPERTIES OF GLOBAL SOLUTIONS OF MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS
EXISTENCE A VLASTNOSTI GLOBÁLNÍCH ŘEŠENÍ FUNKCIONÁLNÍCH DIFERENCIÁLNÍCH ROVNIC SMÍŠENÉHO TYPU

SHORT VERSION OF DOCTORAL THESIS
DIZERTAČNÍ PRÁCE

AUTHOR
AUTOR PRÁCE
Mgr. Gabriela Vážanová

SUPERVISOR
ŠKOLITEL
prof. RNDr. Josef Diblík, DrSc.

BRNO 2020

KEYWORDS

mixed-type functional differential equation, delayed argument, advanced argument, semi-global solution, global solution, monotone iterative method, Schauder-Tychonoff fixed point theorem

KLÍČOVÁ SLOVA

funkcionální diferenciální rovnice smíšeného typu, zpožděný argument, předcházející argument, semi-globální řešení, globální řešení, monotónní iterační metoda, Schauderova-Tychonovova věta o pevném bodu

MÍSTO ULOŽENÍ PRÁCE

Vědecké oddělení, Fakulta elektrotechniky a komunikačních technologií, Vysoké učení technické v Brně, Technická 3058/10, 616 00 Brno

ISBN 978-80-214-XXXX
ISSN 1213-418X

CONTENTS

1 INTRODUCTION . . . 5

1.1 Current State . . . 5

1.2 Aims of the Thesis . . . 5

2 EXISTENCE OF GLOBAL SOLUTIONS TO NONLINEAR MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS . . . 6

2.1 Preliminaries . . . 6

2.2 Problem Description . . . 7

2.3 Existence of Global Solutions to Mixed-Type Functional Differential Equations 8

2.4 A Linear Variant of the Main Result . . . 9

3 LOWER AND UPPER ESTIMATES OF SEMI-GLOBAL AND GLOBAL SOLUTIONS TO MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS . . 10

3.1 Preliminaries . . . 10

3.2 Problem Description . . . 10

3.3 Right Semi-Global Solutions . . . 11

3.3.1 Right Semi-Global Solutions in a Linear Case . . . 12

3.4 Left Semi-Global Solutions . . . 13

3.4.1 Left Semi-Global Solutions in a Linear Case . . . 14

3.5 Semi-Global Solutions in Non-Iterative Case . . . 15

3.6 Global Solutions . . . 16

3.6.1 Global Solutions Found by Monotone Iterative Method . . . 16

3.6.2 Global Solutions Found by Monotone Iterative Method - a Linear Case . 17

3.6.3 Global Solutions in a Non-Iterative Case . . . 18

4 CONCLUSION . . . 18

REFERENCES . . . 21

AUTHOR'S PUBLICATIONS . . . 25

CURRICULUM VITÆ . . . 27

ABSTRACT . . . 28

1 INTRODUCTION

Differential equations are widely used for modeling processes in various areas of the life sciences such as population dynamics, epidemiology, immunology, physiology and neural networks. See, for example, [5,27,28,46,47]. These models usually consist of delayed differential equations. The time delay represents in these models the dependence of the present state on the past history, for example, the time between infection of a cell and the production of new viruses, the duration of the infectious period, the reaction time of market, and so on.

However, using mixed-type rather than delayed differential equations may be beneficial (for additional details see [24]). Adding an advanced term to a differential equation allows us to form predictions for the future. In other words, the advanced term represents the anticipation for the model or the desired outcome. We refer, for example, to [1, 3, 10, 34, 45, 49] and to the references therein, where applications in various fields such as optimal control problems, biology or economics may be found.

The thesis studies the existence of so-called semi-global and global solutions (i.e., solutions defined on $\mathbb{R}$) with graphs staying in a prescribed domain to various classes of mixed-type differential equations including, as particular cases, ordinary, advanced, and delayed differential equations and their combinations. Sufficient conditions under which a global or a semi-global solution exists are derived. Auxiliary computations in the thesis are done by Wolfram Alpha and MATLAB, graphs are constructed in Geogebra.

1.1 CURRENT STATE

There are many papers dealing with either delayed equations or advanced equations. Let us mention at least papers [6–9, 13, 21, 36, 41, 50]. Less literature may be found on the topic of mixed-type differential equations.

Mixed-type functional differential equations are considered, for example, in the books [2, 45] and in the papers [3, 4, 10, 26, 34, 39, 40, 42, 43, 49]. Note that, in the literature, the term "mixed-type equations", is often replaced, for example, by the terms "advanced-delayed equations" or "forward-backward equations". Analysing the relevant literature, we conclude that the results on mixed-type equations can be divided into three groups with respect to the intervals on which solutions are considered. Some of them consider equations on finite intervals only. Most of the results on mixed-type equations on unbounded intervals investigate solutions on half-infinity intervals only, rather then on the entire real axis $\mathbb{R}$, that is, the case of semi-global solutions is considered (we refer, for example to [4, 11, 39, 40, 43, 51] and to the references therein). Not so many papers consider the existence of mixed-type equations on $\mathbb{R}$ (such as [14, 15, 22, 32, 33, 48]).

This has motivated the author to study the existence of such solutions by methods other than the previous ones, deriving a new set of sufficient conditions of their existence. As a co-author, the author of the thesis, has recently achieved new results on this topic, e.g., in [17]- [20].

1.2 AIMS OF THE THESIS

The aims of the thesis are to prove the existence of left semi-global solutions, right semi-global solutions as well as the existence of global solutions for systems of advance-delay nonlinear differential equations giving relevant criteria for linear particular cases as well.

Methodically, investigation follows the papers [15] and [16]. The criteria mentioned in these papers are generalized so that they guarantee the existence of solutions for mixed-type differential equationson on half-axes or on $\mathbb{R}$.

The authors of the papers mentioned above have obtained the following results: The paper [16] considers an advanced differential system. In part 2.2 the authors employ the Schauder–Tychonoff fixed-point theorem and give conditions for the existence of solutions to an advanced system on a half infinity interval satisfying lower and upper inequalities. These results are modified in Chapter 2. The details about using the Schauder-Tychonoff fixed point theorem for functional differential equations is described in [12].

In [15] the authors use the monotone iterative method for a delayed differential system. With the aid of "starting" functions and by the means of a suitable operator, they prove the existence of a global solution staying in the area bounded by exponential-type functions. In Chapter 3 a similar method is used to derive the criteria for mixed-type functional differential equations. Details of the monotone iterative method may also be found in [52].

2 EXISTENCE OF GLOBAL SOLUTIONS TO NONLINEAR MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

In this chapter we use the Schauder-Tychonoff fixed point theorem to prove the existence of global solutions.

2.1 PRELIMINARIES

The following two notions of delayed- and advanced-type functions are taken from or motivated by [23, 30, 31, 35, 38]. For a function $y\colon [t_0 - r_1^*, t_0] \to \mathbb{R}^n$ where $t_0 \in \mathbb{R}$, $r_1^* > 0$, define a delayed-type function y_{t_0} by $y_{t_0}(\theta) = y(t_0 + \theta)$ where $\theta \in [-r_1^*, 0]$. Similarly, for a function $y\colon [t_0, t_0 + r_2^*] \to \mathbb{R}^n$ where $t_0 \in \mathbb{R}$, $r_2^* > 0$, define an advanced-type function y^{t_0} by $y^{t_0}(\theta) = y(t_0 + \theta)$ where $\theta \in [0, r_2^*]$. In the thesis, we will work with functions that can be with delayed-type as well as advanced-type argument on a given interval. In the sequel, I stands for any connected set $I \subseteq \mathbb{R}$ of the $(-\infty, \infty)$, $(-\infty, a]$, $[a, b]$ or $[b, \infty)$ types, where $a \leqslant b$, $a, b \in \mathbb{R}$.

Definition 2.1.1. Let I be an interval such that $\{0\} \in I$, $t_0 \in \mathbb{R}$ and $y\colon [t_0 + r_1, t_0 + r_2] \to \mathbb{R}^n$ where

$$r_1 := \inf_{x \in I}\{x\}, \quad r_2 := \sup_{x \in I}\{x\}.$$

Then, the formula $y_{t_0 I}(\theta) := y(t_0 + \theta)$ where $\theta \in [r_1, r_2]$ defines a mixed-type (delayed-advanced-type) function.

Remark 2.1.2. The point t_0 in Definition 2.1.1 plays the role of a “fixed” point defining the “delay” to the left of t_0 and the “advance” to the right of t_0. For example, for $I = [-1, 1]$ and $t_0 = 10$, we have $r_1 = -1$, $r_2 = 1$. If, say, $y\colon [9, 11] \to \mathbb{R}^n$, the function $y_{0I}(\theta) = y(10 + \theta)$, $\theta \in [-1, 1]$ is of a mixed-type. If $\theta \in [-1, 0)$, then $y_{0I}(10 + \theta)$ is a delayed-type function and, if $\theta \in (0, 1])$, $y_{0I}(\theta)$ is an advanced-type function. If $I = \mathbb{R}$ and $y\colon \mathbb{R} \to \mathbb{R}^n$, then $r_1 = -\infty$, $r_2 = \infty$ and, for an arbitrary $t_0 \in \mathbb{R}$, we have $y_{t_0 I}(\theta) = y(t_0 + \theta)$, $\theta \in (-\infty, \infty)$.

Definition 2.1.1 reduces to a delayed-type function y_{t_0} if $I = [-r_1^*, 0]$ (then $r_2 = 0$) and to an advanced-type function y^{t_0} if $I = [0, r_2^*]$ (then $r_1 = 0$). If $r_1 = r_2 = 0$ (that is, $I = \{0\}$), then a mixed-type function reduces to $y_{t_0 I}(\theta) = y_{t_0 I}(0) = y(t_0)$. If $I = (-\infty, \infty)$, then $r_1 = -\infty$, $r_2 = \infty$.

Let $\mathcal{C}(I, \mathbb{R}^n)$ be the set of n-dimensional real continuous vector-functions defined on a fixed set I. Let Ω be a closed bounded subset of $\mathbb{R}^n$. Denote by $\mathcal{C}(I, \Omega)$ the set of n-dimensional real continuous vector-functions mapping I to Ω. Let $f = f(t, y_{tI})\colon \mathbb{R} \times \mathcal{C}(I, \mathbb{R}^n) \to \mathbb{R}^n$ be a functional. We say that $f(t, y_{tI})$ is
(i) continuous if it is continuous with respect to t and y_{tI},
(ii) quasi-bounded if it is bounded on every set of the form $[a^*, b^*] \times \mathcal{C}([r_1, r_2], \Omega^*)$ where $a^* < b^*$, $a^*, b^* \in \mathbb{R}$, $r_1 < r_2$, $r_1, r_2 \in \mathbb{R}$, and Ω^* is an arbitrary fixed bounded subset of $\mathbb{R}^n$.

This chapter considers a system of nonlinear functional differential equations

$$\dot{y}(t) = f(t, y_{tI}) \tag{2.1}$$

where the functional $f\colon \mathbb{R} \times \mathcal{C}(I, \mathbb{R}^n) \to \mathbb{R}^n$ is continuous and quasi-bounded. Throughout the chapter, we assume that, if t varies within a compact interval $J_t \subset \mathbb{R}$, then $y_{tI}(\theta) = y(t+\theta)$ in $f(t, y_{tI})$ uses the values of argument $t+\theta$ from a compact interval $J_t^* \subset \mathbb{R}$. Equations (2.1) with argument y_{tI} are called mixed-type (delayed-advanced-type). Below, we study the existence of so-called global solutions to (2.1) within the meaning of the following definition.

Definition 2.1.3. A continuously differentiable function $y\colon \mathbb{R} \to \mathbb{R}^n$ satisfying (2.1) on $\mathbb{R}$ is called a global solution to (2.1).

2.2 PROBLEM DESCRIPTION

Assume that there exist vector-functions $\beta, \gamma : \mathbb{R} \to \mathbb{R}^n$ continuous everywhere except at a countable number of points of discontinuity of some of their co-ordinates and such that

$$\beta(t) \leqslant \gamma(t), \quad t \in \mathbb{R}. \tag{2.2}$$

Moreover, assume that the possible discontinuities of the co-ordinates of β and γ are only of the first order. If $t = t^* \in \mathbb{R}$ is such a point, then assume

$$\beta_i(t^* - 0) \leqslant \gamma_i(t^* + 0), \quad \beta_i(t^* + 0) \leqslant \gamma_i(t^* - 0), \quad i = 1, \dots, n. \tag{2.3}$$

Inequality (2.3) says that the intersection of intervals

$$[\beta_i(t^* - 0), \gamma_i(t^* - 0)] \cap [\beta_i(t^* + 0), \gamma_i(t^* + 0)]$$

is nonempty containing at least one point. Throughout this chapter, for a given function $\omega\colon \mathbb{R} \to \mathbb{R}$, define its jump $J(\omega(t^*))$ at a point $t^* \in \mathbb{R}$ as

$$J(\omega(t^*)) := \omega(t^* + 0) - \omega(t^* - 0).$$

By $\mathcal{C}(\mathbb{R}, \mathbb{R}^n)$, denote the set of all continuous real vector-functions defined on $\mathbb{R}$ and define its subset

$$\mathcal{S} := \{y \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n)\colon \beta(t) \leqslant y(t) \leqslant \gamma(t), t \in \mathbb{R}\}. \tag{2.4}$$

Obviously, $\mathcal{S} \neq \emptyset$ by (2.2), (2.3).

This chapter establishes the existence of a global solution $y\colon \mathbb{R} \to \mathbb{R}^n$ to (2.1) such that $y \in \mathcal{S}$. Sufficient conditions are formulated implying the existence of such a solution. The proof is based on a version of the Schauder-Tychonoff fixed point theorem [12, Corollary 1] covering interval $[0, \infty)$ which, after a trivial generalization, is applied on the entire $\mathbb{R}$. Below we refer to this source having in mind the above generalization.

2.3 EXISTENCE OF GLOBAL SOLUTIONS TO MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

This section brings the main theorem of the chapter and its proof.

Theorem 2.3.1. *Let $\beta, \gamma\colon \mathbb{R} \to \mathbb{R}^n$ be vector-functions satisfying* (2.2)*, continuous and continuously differentiable everywhere on $\mathbb{R}$ except at a countable number of points. Let us assume that the possible points of discontinuities of the co-ordinates of $\beta(t)$, $\gamma(t)$, $\beta'(t)$ and $\gamma'(t)$ are at most of the first order and that condition* (2.3) *is satisfied. Let there be a constant-vector $k \in \mathbb{R}^n$ and a number $l \in \{0, \dots, n\}$ such that*

$$\beta_i(+\infty) = \gamma_i(+\infty) = k_i, \quad i = 1, 2, \dots, l, \tag{2.5}$$
$$\beta_i(-\infty) = \gamma_i(-\infty) = k_i, \quad i = l+1, \dots, n \tag{2.6}$$

where condition (2.5) *is omitted if $l = 0$, and condition* (2.6) *is omitted if $l = n$ (the meaning of l in further conditions is similar). Moreover, for every $t \in \mathbb{R}$, $y \in \mathcal{S}$, assume that*

$$f_i\left(t, \beta_{tI}\right) \geqslant f_i\left(t, y_{tI}\right) \geqslant f_i\left(t, \gamma_{tI}\right), \quad i = 1, 2, \dots, l, \tag{2.7}$$
$$f_i\left(t, \beta_{tI}\right) \leqslant f_i\left(t, y_{tI}\right) \leqslant f_i\left(t, \gamma_{tI}\right), \quad i = l+1, \dots, n, \tag{2.8}$$

that, almost everywhere on $\mathbb{R}$,

$$\beta_i'(t) \geqslant f_i\left(t, \beta_{tI}\right), \quad i = 1, 2, \dots, l, \tag{2.9}$$
$$\beta_i'(t) \leqslant f_i\left(t, \beta_{tI}\right), \quad i = l+1, \dots, n, \tag{2.10}$$
$$\gamma_i'(t) \leqslant f_i\left(t, \gamma_{tI}\right), \quad i = 1, 2, \dots, l, \tag{2.11}$$
$$\gamma_i'(t) \geqslant f_i\left(t, \gamma_{tI}\right), \quad i = l+1, \dots, n, \tag{2.12}$$

and that, for arbitrary $t \in \mathbb{R}$,

$$J(\beta_i(t)) \geqslant 0, \quad \beta_i(t+0) = \beta_i(t), \quad i = 1, 2, \dots, l, \tag{2.13}$$
$$J(\beta_i(t)) \leqslant 0, \quad \beta_i(t-0) = \beta_i(t), \quad i = l+1, \dots, n, \tag{2.14}$$
$$J(\gamma_i(t)) \leqslant 0, \quad \gamma_i(t+0) = \gamma_i(t), \quad i = 1, 2, \dots, l, \tag{2.15}$$
$$J(\gamma_i(t)) \geqslant 0, \quad \gamma_i(t-0) = \gamma_i(t), \quad i = l+1, \dots, n. \tag{2.16}$$

Then, there exists a global solution $y\colon \mathbb{R} \to \mathbb{R}^n$ to (2.1) *satisfying*

$$y_i(+\infty) = k_i, \quad i = 1, 2, \dots, l, \tag{2.17}$$
$$y_i(-\infty) = k_i, \quad i = l+1, \dots, n, \tag{2.18}$$

and

$$\beta(t) \leqslant y(t) \leqslant \gamma(t), \quad t \in \mathbb{R}. \tag{2.19}$$

2.4 A LINEAR VARIANT OF THE MAIN RESULT

Let $A(t) = \{a_{ij}(t)\}_{i,j=1}^n$, $B(t) = \{b_{ij}(t)\}_{i,j=1}^n$ be matrices continuous on $\mathbb{R}$, $r_1 > 0$, $r_2 > 0$ and let $\omega(t) = (\omega_1(t), \ldots, \omega_n(t))^T$ be a vector continuous on $\mathbb{R}$. Consider a linear system

$$\dot{y}(t) = A(t)y(t - r_1) + B(t)y(t + r_2) + \omega(t). \tag{2.20}$$

The following theorem is a linear variant of Theorem 2.3.1 where $I = [-r_1, r_2]$.

Theorem 2.4.1. *Let $\beta, \gamma\colon \mathbb{R} \to \mathbb{R}^n$ be vector-functions satisfying* (2.2)*, continuous and continuously differentiable everywhere on $\mathbb{R}$ except at a countable number of points. Let us assume that the possible points of discontinuities of the coordinates of $\beta(t)$, $\gamma(t)$, $\beta'(t)$, and $\gamma'(t)$ are at most of the first order, and that condition* (2.3) *is satisfied. Let there be a constant-vector $k \in \mathbb{R}^n$ and a number $l \in \{0, \ldots, n\}$ such that*

$$\begin{aligned} &\beta_i(+\infty) = \gamma_i(+\infty) = k_i, \quad i = 1, 2, \ldots, l, \\ &\beta_i(-\infty) = \gamma_i(-\infty) = k_i, \quad i = l + 1, \ldots, n. \end{aligned} \tag{2.21}$$

Moreover, for every $t \in \mathbb{R}$, $y \in \mathcal{S}$, assume that

$$\begin{aligned} \sum_{j=1}^n a_{ij}(t)\beta_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\beta_j(t + r_2) &\geqslant \sum_{j=1}^n a_{ij}(t)y_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)y_j(t + r_2) \\ &\geqslant \sum_{j=1}^n a_{ij}(t)\gamma_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\gamma_j(t + r_2), \;\; i = 1, 2, \ldots, l, \\ \sum_{j=1}^n a_{ij}(t)\beta_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\beta_j(t + r_2) &\leqslant \sum_{j=1}^n a_{ij}(t)y_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)y_j(t + r_2) \\ &\leqslant \sum_{j=1}^n a_{ij}(t)\gamma_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\gamma_j(t + r_2), \;\; i = l + 1, \ldots, n, \end{aligned} \tag{2.22}$$

that, almost everywhere on $\mathbb{R}$,

$$\begin{aligned} \beta_i'(t) &\geqslant \sum_{j=1}^n a_{ij}(t)\beta_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\beta_j(t + r_2) + \omega_i(t), \;\; i = 1, 2, \ldots, l, \\ \beta_i'(t) &\leqslant \sum_{j=1}^n a_{ij}(t)\beta_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\beta_j(t + r_2) + \omega_i(t), \;\; i = l + 1, \ldots, n, \end{aligned} \tag{2.23}$$

$$\begin{aligned} \gamma_i'(t) &\leqslant \sum_{j=1}^n a_{ij}(t)\gamma_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\gamma_j(t + r_2) + \omega_i(t), \;\; i = 1, 2, \ldots, l, \\ \gamma_i'(t) &\geqslant \sum_{j=1}^n a_{ij}(t)\gamma_j(t - r_1) + \sum_{j=1}^n b_{ij}(t)\gamma_j(t + r_2) + \omega_i(t), \;\; i = l + 1, \ldots, n, \end{aligned} \tag{2.24}$$

and that, for arbitrary $t \in \mathbb{R}$,

$$\begin{aligned} &J(\beta_i(t)) \geqslant 0, \quad \beta_i(t + 0) = \beta_i(t), \quad i = 1, 2, \ldots, l, \\ &J(\beta_i(t)) \leqslant 0, \quad \beta_i(t - 0) = \beta_i(t), \quad i = l + 1, \ldots, n, \end{aligned} \tag{2.25}$$

$$\begin{aligned} &J(\gamma_i(t)) \leqslant 0, \quad \gamma_i(t + 0) = \gamma_i(t), \quad i = 1, 2, \ldots, l, \\ &J(\gamma_i(t)) \geqslant 0, \quad \gamma_i(t - 0) = \gamma_i(t), \quad i = l + 1, \ldots, n \end{aligned} \tag{2.26}$$

Then, there exists a global solution $y\colon \mathbb{R} \to \mathbb{R}^n$ *to* (2.20) *satisfying*

$$y_i(+\infty) = k_i, \quad i = 1, 2, \ldots, l,$$
$$y_i(-\infty) = k_i, \quad i = l+1, \ldots, n,$$

and

$$\beta(t) \leqslant y(t) \leqslant \gamma(t), \quad t \in \mathbb{R}. \tag{2.27}$$

3 LOWER AND UPPER ESTIMATES OF SEMI-GLOBAL AND GLOBAL SOLUTIONS TO MIXED-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

3.1 PRELIMINARIES

Let $\mathcal{I} \subseteq \mathbb{R}$ be an interval. By $\mathcal{C}(\mathcal{I}, \mathbb{R}^n)$ denote the Banach space of bounded continuous functions from $\mathcal{I}$ to $\mathbb{R}^n$ equipped with the norm $\|\psi\|_r = \sup_{\alpha \in \mathcal{I}} |\psi(\alpha)|$ where $\psi \in \mathcal{C}(\mathcal{I}, \mathbb{R}^n)$ with the last norm (used throughout the chapter) being defined by $|\psi(\alpha)| := \max\{|\psi_1(\alpha)|, \ldots, |\psi_n(\alpha)|\}$. If $\mathcal{I} := [0, r]$, where $r > 0$ is fixed, the relevant Banach space is denoted by $\mathcal{C}_r$.

For a function y, continuous on an interval $[t - r_1, t]$, $t \in \mathbb{R}$, $r_1 > 0$, define a delayed-type function $y_t \in \mathcal{C}_{r_1}$ by $y_t(\tau) = y(t - \tau)$ where $\tau \in [0, r_1]$. Similarly, for a function y, continuous on an interval $[t, t + r_2]$, $t \in \mathbb{R}$, $r_2 > 0$, define an advanced-type function $y^t \in \mathcal{C}_{r_2}$ by $y^t(\sigma) = y(t + \sigma)$ where $\sigma \in [0, r_2]$.

Let $t_0 \in \mathbb{R}$ be fixed and let $\mathcal{J} \subseteq \mathbb{R}$ be a set having one of the forms $\mathcal{J} = \mathcal{J}^+ := [t_0, \infty)$, $\mathcal{J} = \mathcal{J}^- := (-\infty, t_0]$ or $\mathcal{J} = \mathbb{R}$. In the chapter, we will consider a system of mixed-type functional differential equations

$$\dot{y}(t) = f(t, y_t, y^t) \tag{3.1}$$

where $y = (y_1, \ldots, y_n)$, $f = (f_1, \ldots, f_n)\colon \mathcal{J} \times \mathcal{C}_{r_1} \times \mathcal{C}_{r_2} \to \mathbb{R}^n$ is a continuous quasi-bounded functional which satisfies a local Lipschitz condition with respect to the second and the third arguments in the domains considered. The well-known definitions of quasi-boundedness and local Lipschitz condition can be found, e.g., in [23] or in Section 2.1 of the thesis.

We say that a continuous function $y\colon \mathcal{J}^+_{-1} := [t_0 - r_1, \infty) \to \mathbb{R}^n$ is a right semi-global solution to (3.1) on $\mathcal{J}^+_{-1}$ if it is continuously differentiable on $\mathcal{J}^+$ and satisfies (3.1) on $\mathcal{J}^+$. Similarly, a continuous function $y\colon \mathcal{J}^-_{+1} := (-\infty, t_0 + r_2] \to \mathbb{R}^n$ is said to be a left semi-global solution of (3.1) on $\mathcal{J}^-_{+1}$ if it is continuously differentiable on $\mathcal{J}^-$ and satisfies (3.1) on $\mathcal{J}^-$. Finally, for $\mathcal{J} = \mathbb{R}$, a continuous function $y\colon \mathcal{J} \to \mathbb{R}^n$ is said to be a global solution of (3.1) on $\mathbb{R}$ if it is continuously differentiable on $\mathbb{R}$ and satisfies (3.1) on $\mathbb{R}$.

3.2 PROBLEM DESCRIPTION

Concerned with the problems of existence of semi-global and global solutions to (3.1), the chapter has the following structure. The next Section 3.3 is about the existence of right semi-global solutions while Section 3.4 deals with the existence of left semi-global solutions. In these sections we use a monotone iterative method (for rudiments of this method, we refer, e.g., to [52]). Section 3.5 discusses the existence of semi-global solutions without assuming monotonicity of the relevant operators using Schauder-Tychonoff fixed point theorem instead (for applications of Schauder-Tychonoff fixed point theorem in functional differential equations, we refer, e.g., to [29, 44]). The outcomes concerning the existence of global solutions are described in Section 3.6.

In the chapter, some particular linear variants of general nonlinear statements are considered as well. By the methods and technique used, upper and lower estimates by exponential-type functions can be found of the co-ordinates of semi-global and global solutions.

3.3 RIGHT SEMI-GLOBAL SOLUTIONS

In this section we prove a general theorem on the existence of right semi-global solutions to mixed-type system (3.1). We use the iterative process to derive sequences of functions converging to these solutions.

Define a mapping

$$I\colon \mathbb{R}^n_{>0} \times \mathcal{C}(\mathcal{J}^+_{-1}, \mathbb{R}^n) \to \mathcal{C}(\mathcal{J}^+_{-1}, \mathbb{R}^n)$$

where $I(k,\lambda) = (I_1(k,\lambda), \dots, I_n(k,\lambda))$, $k \in \mathbb{R}^n_{>0}$ is a constant vector, $\lambda \in \mathcal{C}\left(\mathcal{J}^+_{-1}, \mathbb{R}^n\right)$ is a vector-function, $\lambda = (\lambda_1, \dots, \lambda_n)$, and

$$I_i(k,\lambda)(t) := k_i \exp\left(\int_{t_0-r_1}^{t} \lambda_i(s)\,\mathrm{d}s\right), \quad i = 1, \dots, n, \quad t \in \mathcal{J}^+_{-1}. \tag{3.2}$$

Below we assume that a solution of system (3.1) on $\mathcal{J}^+_{-1}$ is representable in the form

$$y(t) = I(k,\lambda)(t), \quad t \in \mathcal{J}^+_{-1} \tag{3.3}$$

with suitable k and λ. Substituting (3.3) into (3.1), for $t \in \mathcal{J}^+$ we get

$$\lambda(t)\,(\mathrm{diag}(I(k,\lambda)(t))) = f(t, I(k,\lambda)_t, I(k,\lambda)^t)$$

or, since the matrix $\mathrm{diag}(I(k,\lambda)(t))$ with entries defined by (3.2), (3.3) is regular,

$$\lambda(t) = f(t, I(k,\lambda)_t, I(k,\lambda)^t)(\mathrm{diag}(I(k,\lambda)(t)))^{-1}. \tag{3.4}$$

Similar transformations are used, without detailed explanation, in the sequel. Equation (3.4) is an operator equation with respect to λ. A function $\lambda \in \mathcal{C}\left(\mathcal{J}^+_{-1}, \mathbb{R}^n\right)$ is called a solution of equation (3.4) on $\mathcal{J}^+_{-1}$ if (3.4) holds for all $t \in \mathcal{J}^+$.

Define an operator $T\colon \mathcal{C}(\mathcal{J}^+_{-1}, \mathbb{R}^n) \to \mathcal{C}(\mathcal{J}^+, \mathbb{R}^n)$ where

$$(T\lambda)(t) = f(t, I(k,\lambda)_t, I(k,\lambda)^t)(\mathrm{diag}(I(k,\lambda)(t)))^{-1}, \quad t \in \mathcal{J}^+.$$

The following theorem gives conditions sufficient for the existence of a right semi-global solution to equation (3.4).

Theorem 3.3.1. *Let us assume that $k \in \mathbb{R}^n_{>0}$ and that the following holds:*

(i) For any fixed $M \geqslant 0$, $\theta > t_0$ there exists a constant K such that, for all $t, t' \in [t_0, \theta]$ and for any continuous function $\lambda : \mathcal{J}^+_{-1} \to \mathbb{R}^n$ with $|\lambda| \leqslant M$,

$$|(T\lambda)(t) - (T\lambda)(t')| \leqslant K\,|t - t'|\,. \tag{3.5}$$

(ii) There exist bounded continuous functions $\mathcal{L}, \mathcal{R}\colon \mathcal{J}^+_{-1} \to \mathbb{R}^n$ satisfying $\mathcal{L}(t) \leqslant \mathcal{R}(t)$, $t \in \mathcal{J}^+_{-1}$ and

$$\mathcal{L}(t) \leqslant (T\mathcal{L})(t), \quad \mathcal{R}(t) \geqslant (T\mathcal{R})(t), \quad t \in \mathcal{J}^+. \tag{3.6}$$

(iii) There exists a Lipschitz continuous function $\varphi\colon [t_0 - r_1, t_0] \to \mathbb{R}^n$ *satisfying* $\varphi(t_0) = 0$ *and*

$$\mathcal{L}(t) \leqslant (T\mathcal{L})(t_0) + \varphi(t), \quad \mathcal{R}(t) \geqslant (T\mathcal{R})(t_0) + \varphi(t), \quad t \in [t_0 - r_1, t_0]. \tag{3.7}$$

(iv) For any locally integrable functions $\lambda^*, \mu^*\colon \mathcal{J}^+_{-1} \to \mathbb{R}^n$*, the inequality*

$$\mathcal{L}(t) \leqslant \lambda^*(t) \leqslant \mu^*(t) \leqslant \mathcal{R}(t), \ \ t \in \mathcal{J}^+_{-1}$$

implies

$$(T\lambda^*)(t) \leqslant (T\mu^*)(t), \quad t \in \mathcal{J}^+. \tag{3.8}$$

Then, there exists a right semi-global solution $y\colon \mathcal{J}^+_{-1} \to \mathbb{R}^n$ *of* (3.1) *satisfying* $y(t_0 - r_1) = k$ *such that, for arbitrary indexes* $i \geqslant 0$, $j \geqslant 0$,

$$I(k, \nu_i)(t) \leqslant y(t) \leqslant I(k, \mu_j)(t), \quad t \in \mathcal{J}^+_{-1} \tag{3.9}$$

where $\nu_i(t) \leqslant \nu_{i+1}(t)$, $\mu_{j+1}(t) \leqslant \mu_j(t)$, $\nu_i(t) \leqslant \mu_j(t)$, $\nu_0(t) := \mathcal{L}(t)$, $\mu_0(t) := \mathcal{R}(t)$, $t \in \mathcal{J}^+_{-1}$, *and, for* $i > 0$, $j > 0$,

$$\nu_i(t) := \begin{cases} (T\nu_{i-1})(t), & t \in [t_0, \infty), \\ (T\nu_{i-1})(t_0) + \varphi(t), & t \in [t_0 - r_1, t_0], \end{cases} \tag{3.10}$$

$$\mu_j(t) := \begin{cases} (T\mu_{j-1})(t), & t \in [t_0, \infty), \\ (T\mu_{j-1})(t_0) + \varphi(t), & t \in [t_0 - r_1, t_0]. \end{cases} \tag{3.11}$$

Moreover, there exist continuous limits

$$\nu(t) = \lim_{i \to \infty} \nu_i(t), \ \ \mu(t) = \lim_{j \to \infty} \mu_j(t), \ \ \nu(t) \leqslant \mu(t), \ \ t \in \mathcal{J}^+_{-1},$$

defining right semi-global solutions $y_\nu(t) = I(k, \nu)(t)$, $y_\mu(t) = I(k, \mu)(t)$, $y_\nu, y_\mu\colon \mathcal{J}^+_{-1} \to \mathbb{R}^n$ *of* (3.1), *satisfying* $y_\nu(t_0 - r_1) = y_\mu(t_0 - r_1) = k$ *and inequalities*

$$I(k, \nu_i)(t) \leqslant y_\nu(t) \leqslant y_\mu(t) \leqslant I(k, \mu_j)(t), \ \ t \in \mathcal{J}^+_{-1}$$

where $i \geqslant 0$ *and* $j \geqslant 0$ *are arbitrary.*

3.3.1 Right Semi-Global Solutions in a Linear Case

In this section we will consider a scalar linear equation as a particular case of equation (3.1),

$$\dot{y}(t) = f(t, y_t, y^t) := -c(t)y(t - \tau(t)) + d(t)y(t + \sigma(t)) \tag{3.12}$$

where functions $c, d\colon \mathcal{J}^+ \to [0, \infty)$, $\tau\colon \mathcal{J}^+ \to [0, r_1]$ and $\sigma\colon \mathcal{J}^+ \to [0, r_2]$ are Lipschitz continuous.

Theorem 3.3.2. *Consider bounded continuous functions* $\mathcal{L}, \mathcal{R}\colon \mathcal{J}^+_{-1} \to \mathbb{R}$, $\mathcal{L}(t) \leqslant \mathcal{R}(t)$, $t \in \mathcal{J}^+_{-1}$ *and a Lipschitz continuous function* $\varphi\colon [t_0 - r_1, t_0] \to \mathbb{R}$ *satisfying* $\varphi(t_0) = 0$. *Moreover, let*

$$\mathcal{L}(t) \leqslant -c(t) \exp\left(\int_t^{t-\tau(t)} \mathcal{L}(s)\,\mathrm{d}s\right) + d(t) \exp\left(\int_t^{t+\sigma(t)} \mathcal{L}(s)\,\mathrm{d}s\right), \tag{3.13}$$

$$\mathcal{R}(t) \geqslant -c(t) \exp\left(\int_t^{t-\tau(t)} \mathcal{R}(s)\,\mathrm{d}s\right) + d(t) \exp\left(\int_t^{t+\sigma(t)} \mathcal{R}(s)\,\mathrm{d}s\right) \tag{3.14}$$

on $\mathcal{J}^+$ *and*

$$\mathcal{L}(t) \leqslant -c(t_0)\exp\left(\int_{t_0}^{t_0-\tau(t_0)} \mathcal{L}(s)\,\mathrm{d}s\right) + d(t_0)\exp\left(\int_{t_0}^{t_0+\sigma(t_0)} \mathcal{L}(s)\,\mathrm{d}s\right) + \varphi(t), \tag{3.15}$$

$$\mathcal{R}(t) \geqslant -c(t_0)\exp\left(\int_{t_0}^{t_0-\tau(t_0)} \mathcal{R}(s)\,\mathrm{d}s\right) + d(t_0)\exp\left(\int_{t_0}^{t_0+\sigma(t_0)} \mathcal{R}(s)\,\mathrm{d}s\right) + \varphi(t) \tag{3.16}$$

on $[t_0 - r_1, t_0]$. *Then, there exists a right semi-global solution* $y(t)$ *of* (3.12) *on* $\mathcal{J}^+_{-1}$ *such that* $y(t_0 - r_1) = 1$ *and, for arbitrary indices* $i \geqslant 0$, $j \geqslant 0$,

$$\exp\left(\int_{t_0-r_1}^{t} \nu_i(s)\,\mathrm{d}s\right) \leqslant\ y(t) \leqslant \exp\left(\int_{t_0-r_1}^{t} \mu_j(s)\,\mathrm{d}s\right),\quad t \in \mathcal{J}^+_{-1} \tag{3.17}$$

where $\nu_i(t) \leqslant \nu_{i+1}(t)$, $\mu_{j+1}(t) \leqslant \mu_j(t)$, $\nu_i(t) \leqslant \mu_j(t)$, $\nu_0(t) := \mathcal{L}(t)$, $\mu_0(t) := \mathcal{R}(t)$, $t \in \mathcal{J}^+_{-1}$, *and, for* $i > 0$, $j > 0$,

$$\nu_i(t) := \begin{cases} -c(t)\exp\left(\int_{t}^{t-\tau(t)} \nu_{i-1}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_{t}^{t+\sigma(t)} \nu_{i-1}(s)\,\mathrm{d}s\right), \ t \in [t_0, \infty), \\ \nu_i(t_0),\ \ t \in [t_0 - r_1, t_0), \end{cases}$$

$$\mu_j(t) := \begin{cases} -c(t)\exp\left(\int_{t}^{t-\tau(t)} \mu_{j-1}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_{t}^{t+\sigma(t)} \mu_{j-1}(s)\,\mathrm{d}s\right), \ t \in [t_0, \infty), \\ \mu_j(t_0),\ \ t \in [t_0 - r_1, t_0). \end{cases}$$

Moreover, there exist continuous limits $\nu(t) = \lim_{i\to\infty} \nu_i(t)$, $\mu(t) = \lim_{j\to\infty} \mu_j(t)$, $\nu(t) \leqslant \mu(t)$, $t \in \mathcal{J}^+_{-1}$ *defining right semi-global solutions* $y_\nu, y_\mu \colon \mathcal{J}^+_{-1} \to \mathbb{R}$ *of* (3.12) *and satisfying* $y_\nu(t_0 - r_1) = y_\mu(t_0 - r_1) = 1$ *by formulas*

$$y_\nu(t) = \exp\left(\int_{t_0-r_1}^{t} \nu(s)\,\mathrm{d}s\right),\quad y_\mu(t) = \exp\left(\int_{t_0-r_1}^{t} \mu(s)\,\mathrm{d}s\right),\quad t \in \mathcal{J}^+_{-1}$$

and, for arbitrary indexes $i \geqslant 0$, $j \geqslant 0$,

$$\exp\left(\int_{t_0-r_1}^{t} \nu_i(s)\,\mathrm{d}s\right) \leqslant y_\nu(t) \leqslant y_\mu(t) \leqslant \exp\left(\int_{t_0-r_1}^{t} \mu_j(s)\,\mathrm{d}s\right),\quad t \in \mathcal{J}^+_{-1},$$

3.4 LEFT SEMI-GLOBAL SOLUTIONS

The goal of this section is to prove the existence of left semi-global solutions to mixed-type system (3.1). Theorem 3.4.1 below is a modification of Theorem 3.3.1 for the existence of a left semi-global solution of (3.1). A linear particular case is considered in section 3.4.1 as well.

Define a mapping

$$I^* \colon \mathbb{R}^n_{>0} \times \mathcal{C}(\mathcal{J}^-_{+1}, \mathbb{R}^n) \to \mathcal{C}(\mathcal{J}^-_{+1}, \mathbb{R}^n)$$

where $I^*(k,\lambda) = (I^*_1(k,\lambda), I^*_2(k,\lambda), \dots, I^*_n(k,\lambda))$, $k \in \mathbb{R}^n_{>0}$ is a constant vector, $\lambda \in \mathcal{C}\left(\mathcal{J}^-_{+1}, \mathbb{R}^n\right)$ is a vector-function and

$$I^*_i(k,\lambda)(t) := k_i \exp\left(\int_{t}^{t_0+r_2} \lambda_i(s)\,\mathrm{d}s\right),\ \ i = 1, \dots, n,\ \ t \in \mathcal{J}^-_{+1}.$$

We are looking for a solution of system (3.1) in the form $y(t) = I^*(k,\lambda)(t)$, $t \in \mathcal{J}^-_{+1}$ with suitable k and λ. This leads to the operator equation

$$\lambda(t) = (T^*\lambda)(t) := -f(t, I^*(k,\lambda)_t, I^*(k,\lambda)^t)(\mathrm{diag}(I^*(k,\lambda)(t)))^{-1},\quad t \in \mathcal{J}^- \tag{3.18}$$

where $T^* \colon \mathcal{C}(\mathcal{J}^-_{+1}, \mathbb{R}^n) \to \mathcal{C}(\mathcal{J}^-, \mathbb{R}^n)$.

Theorem 3.4.1. *Let us assume that $k \in \mathbb{R}^n_{>0}$ and that the following holds:*

(i) For any $M \geqslant 0$, $\theta < t_0$, there exists a constant K, such that, for all $t, t' \in [\theta, t_0]$ and for any continuous function $\lambda : \mathcal{J}^-_{+1} \to \mathbb{R}^n$ with $|\lambda| \leqslant M$,

$$|(T^*\lambda)(t) - (T^*\lambda)(t')| \leqslant K\,|t - t'|\,.$$

(ii) There exist bounded continuous functions $\mathcal{L}, \mathcal{R}\colon \mathcal{J}^-_{+1} \to \mathbb{R}^n$ satisfying $\mathcal{L}(t) \leqslant \mathcal{R}(t)$, $t \in \mathcal{J}^-_{+1}$ and

$$\mathcal{L}(t) \leqslant (T^*\mathcal{L})(t), \quad \mathcal{R}(t) \geqslant (T^*\mathcal{R})(t), \quad t \in \mathcal{J}^-. \tag{3.19}$$

(iii) There exists a Lipschitz continuous function $\varphi\colon [t_0, t_0 + r_2] \to \mathbb{R}^n$ satisfying $\varphi(t_0) = 0$ and

$$\mathcal{L}(t) \leqslant (T^*\mathcal{L})(t_0) + \varphi(t), \quad \mathcal{R}(t) \geqslant (T^*\mathcal{R})(t_0) + \varphi(t), \quad t \in [t_0, t_0 + r_2].$$

(iv) For any locally integrable functions $\lambda^*, \mu^*\colon \mathcal{J}^-_{+1} \to \mathbb{R}^n$, the inequality $\lambda^*(t) \leqslant \mu^*(t)$, $t \in \mathcal{J}^-_{+1}$ implies

$$(T^*\lambda^*)(t) \leqslant (T^*\mu^*)(t), \quad t \in \mathcal{J}^-. \tag{3.20}$$

Then, there exists a left semi-global solution $y\colon \mathcal{J}^-_{+1} \to \mathbb{R}^n$ of (3.1) *satisfying $y(t_0 + r_2) = k$ and such that, for arbitrary indices $i \geqslant 0$, $j \geqslant 0$,*

$$I^*(k, \nu_i)(t) \leqslant y(t) \leqslant I^*(k, \mu_j)(t), \quad t \in \mathcal{J}^-_{+1} \tag{3.21}$$

where $\nu_i(t) \leqslant \nu_{i+1}(t)$, $\mu_{j+1}(t) \leqslant \mu_j(t)$, $\nu_i(t) \leqslant \mu_j(t)$, $\nu_0(t) := \mathcal{L}(t)$, $\mu_0(t) := \mathcal{R}(t)$, $t \in \mathcal{J}^-_{+1}$, and, for $i > 0$, $j > 0$,

$$\nu_i(t) := \begin{cases} (T^*\nu_{i-1})(t), & t \in (-\infty, t_0], \\ (T^*\nu_{i-1})(t_0) + \varphi(t), & t \in (t_0, t_0 + r_2]. \end{cases}$$

$$\mu_j(t) := \begin{cases} (T^*\mu_{j-1})(t), & t \in (-\infty, t_0], \\ (T^*\mu_{j-1})(t_0) + \varphi(t), & t \in (t_0, t_0 + r_2]. \end{cases}$$

Moreover, there exist continuous limits

$$\nu(t) = \lim_{i\to\infty} \nu_i(t), \quad \mu(t) = \lim_{j\to\infty} \mu_j(t), \quad \nu(t) \leqslant \mu(t), \quad t \in \mathcal{J}^-_{+1},$$

defining left semi-global solutions $y_\nu(t) = I^*(k, \nu)(t)$, $y_\mu(t) = I^*(k, \mu)(t)$, $y_\nu, y_\mu\colon \mathcal{J}^-_{+1} \to \mathbb{R}^n$ of (3.1)*, satisfying $y_\nu(t_0 - r_1) = y_\mu(t_0 - r_1) = k$ and inequalities*

$$I^*(k, \nu_i)(t) \leqslant y_\nu(t) \leqslant y_\mu(t) \leqslant I^*(k, \mu_j)(t), \quad t \in \mathcal{J}^-_{+1}$$

where $i > 0$, $j > 0$ are arbitrary.

3.4.1 Left Semi-Global Solutions in a Linear Case

In this section, linear equation (3.12) is considered assuming that functions $c, d\colon \mathcal{J}^- \to [0, \infty)$, $\tau\colon \mathcal{J}^- \to [0, r_1]$ and $\sigma\colon \mathcal{J}^- \to [0, r_2]$ are Lipschitz continuous.

Theorem 3.4.2. *Let there be bounded continuous functions $\mathcal{L}, \mathcal{R}\colon \mathcal{J}^-_{+1} \to \mathbb{R}$, $\mathcal{L}(t) \leqslant \mathcal{R}(t)$, $t \in \mathcal{J}^-_{+1}$ and a Lipschitz continuous function $\varphi\colon [t_0, t_0{+}r_2] \to \mathbb{R}$ satisfying $\varphi(t_0) = 0$. Moreover, let*

$$\mathcal{L}(t) \leqslant c(t) \exp\left(\int_{t-\tau(t)}^{t} \mathcal{L}(s)\,\mathrm{d}s\right) - d(t) \exp\left(\int_{t+\sigma(t)}^{t} \mathcal{L}(s)\,\mathrm{d}s\right), \tag{3.22}$$

$$\mathcal{R}(t) \geqslant c(t) \exp\left(\int_{t-\tau(t)}^{t} \mathcal{R}(s)\,\mathrm{d}s\right) - d(t) \exp\left(\int_{t+\sigma(t)}^{t} \mathcal{R}(s)\,\mathrm{d}s\right) \tag{3.23}$$

on $\mathcal{J}^-$ and

$$\mathcal{L}(t) \leqslant c(t_0)\exp\left(\int_{t_0-\tau(t_0)}^{t_0} \mathcal{L}(s)\,\mathrm{d}s\right) - d(t_0)\exp\left(\int_{t_0+\sigma(t_0)}^{t_0} \mathcal{L}(s)\,\mathrm{d}s\right) + \varphi(t), \tag{3.24}$$

$$\mathcal{R}(t) \geqslant c(t_0)\exp\left(\int_{t_0-\tau(t_0)}^{t_0} \mathcal{R}(s)\,\mathrm{d}s\right) - d(t_0)\exp\left(\int_{t_0+\sigma(t_0)}^{t_0} \mathcal{R}(s)\,\mathrm{d}s\right) + \varphi(t) \tag{3.25}$$

on $[t_0, t_0 + r_2]$. Then, there exists a left semi-global solution $y(t)$ of (3.12) on $\mathcal{J}^-_{+1}$ such that $y(t_0 + r_2) = 1$ and, for arbitrary indexes $i \geqslant 0$, $j \geqslant 0$,

$$\exp\left(\int_t^{t_0+r_2} \nu_i(s)\,\mathrm{d}s\right) \leqslant y(t) \leqslant \exp\left(\int_t^{t_0+r_2} \mu_j(s)\,\mathrm{d}s\right), \quad t \in \mathcal{J}^-_{+1} \tag{3.26}$$

where $\nu_i(t) \leqslant \nu_{i+1}(t)$, $\mu_{j+1}(t) \leqslant \mu_j(t)$, $\nu_i(t) \leqslant \mu_j(t)$, $\nu_0(t) := \mathcal{L}(t)$, $\mu_0(t) := \mathcal{R}(t)$, $t \in \mathcal{J}^-_{+1}$, and, for $i > 0$, $j > 0$,

$$\nu_i(t) := \begin{cases} c(t)\exp\left(\int_{t-\tau(t)}^{t} \nu_{i-1}(s)\,\mathrm{d}s\right) - d(t)\exp\left(\int_{t+\sigma(t)}^{t} \nu_{i-1}(s)\,\mathrm{d}s\right), & t \in (-\infty, t_0], \\ \nu_i(t_0), \quad t \in (t_0, t_0 + r_2], \end{cases}$$

$$\mu_j(t) := \begin{cases} c(t)\exp\left(\int_{t-\tau(t)}^{t} \mu_{j-1}(s)\,\mathrm{d}s\right) - d(t)\exp\left(\int_{t+\sigma(t)}^{t} \mu_{j-1}(s)\,\mathrm{d}s\right), & t \in (-\infty, t_0], \\ \mu_j(t_0), \quad t \in (t_0, t_0 + r_2]. \end{cases}$$

Moreover, there exist continuous limits $\nu(t) = \lim_{i\to\infty} \nu_i(t)$, $\mu(t) = \lim_{j\to\infty} \mu_j(t)$, $\nu(t) \leqslant \mu(t)$, $t \in \mathcal{J}^-_{+1}$, defining left semi-global solutions $y_\nu, y_\mu \colon \mathcal{J}^-_{+1} \to \mathbb{R}$ of (3.12) satisfying $y_\nu(t_0 + r_2) = y_\mu(t_0 + r_2) = 1$, by the formulas

$$y_\nu(t) = \exp\left(\int_t^{t_0+r_2} \nu(s)\,\mathrm{d}s\right), \quad y_\mu(t) = \exp\left(\int_t^{t_0+r_2} \mu(s)\,\mathrm{d}s\right), \quad t \in \mathcal{J}^-_{+1}$$

and, for arbitrary indexes $i \geqslant 0$, $j \geqslant 0$,

$$\exp\left(\int_t^{t_0+r_2} \nu_i(s)\,\mathrm{d}s\right) \leqslant y_\nu(t) \leqslant y_\mu(t) \leqslant \exp\left(\int_t^{t_0+r_2} \mu_j(s)\,\mathrm{d}s\right), \quad t \in \mathcal{J}^-_{+1}.$$

3.5 SEMI-GLOBAL SOLUTIONS IN NON-ITERATIVE CASE

We can also derive a theorem on the existence of semi-global solutions of classes of equations without applying the monotone iterative method. In this case, we will get upper and below estimates of semi-global solutions without the possibility of improving them in an iterative process, that is, without using functions of the type $\nu_i(t)$, $\mu_i(t)$, $i = 0, 1, \dots$.

Theorem 3.5.1. *Let us assume that $k \in \mathbb{R}^n_{>0}$. Let the hypotheses (i), (ii) and (iii) of Theorem 3.3.1 hold. If, moreover, for any locally integrable function $\lambda \colon \mathcal{J}^+_{-1} \to \mathbb{R}^n$, the inequality*

$$\mathcal{L}(t) \leqslant \lambda(t) \leqslant \mathcal{R}(t), \ \ t \in \mathcal{J}^+_{-1}$$

implies

$$(T\mathcal{L})(t) \leqslant (T\lambda)(t) \leqslant (T\mathcal{R})(t), \quad t \in \mathcal{J}^+, \tag{3.27}$$

then there exists a right semi-global solution $y \colon \mathcal{J}^+_{-1} \to \mathbb{R}^n$ of (3.1) satisfying $y(t_0 - r_1) = k$ and

$$I(k, \mathcal{L})(t) \leqslant y(t) \leqslant I(k, \mathcal{R})(t), \quad t \in \mathcal{J}^+_{-1}.$$

Theorem 3.5.2. *Let us assume that* $k \in \mathbb{R}^n_{>0}$. *Let the hypotheses* (*i*), (*ii*) *and* (*iii*) *of Theorem 3.4.1 hold. If, moreover, for any locally integrable function* $\lambda^*\colon \mathcal{J}_+^{-1} \to \mathbb{R}^n$, *the inequality*

$$\mathcal{L}(t) \leqslant \lambda^*(t) \leqslant \mathcal{R}(t), \ \ t \in \mathcal{J}^-_{+1}$$

implies

$$(T^*\mathcal{L})(t) \leqslant (T^*\lambda^*)(t) \leqslant (T^*\mathcal{R})(t), \quad t \in \mathcal{J}^-,$$

then there exists a left semi-global solution $y\colon \mathcal{J}^-_{+1} \to \mathbb{R}^n$ *of* (3.1) *satisfying* $y(t_0 + r_2) = k$ *and*

$$I^*(k, \mathcal{L})(t) \leqslant y(t) \leqslant I^*(k, \mathcal{R})(t), \quad t \in \mathcal{J}^-_{+1}.$$

3.6 GLOBAL SOLUTIONS

This section is concerned with the existence of global solutions on the entire $\mathbb{R}$. The general case is treated in section 3.6.1 using the iterative method, and a particular linear case in section 3.6.2. This problem without the iterative method applied is discussed in section 3.6.3. Assume the existence of bounded continuous functions $\mathcal{L}, \mathcal{R}\colon \mathbb{R} \to \mathbb{R}^n$ which satisfy

$$\mathcal{L}(t) \leqslant \mathcal{R}(t), \ \ t \in \mathbb{R}. \tag{3.28}$$

By Ω we denote the set of the functions $\lambda \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n)$ with the property $\mathcal{L}(t) \leqslant \lambda(t) \leqslant \mathcal{R}(t)$, $t \in \mathbb{R}$, that is,

$$\Omega := \{\lambda \ \in \mathcal{C}(\mathbb{R}, \mathbb{R}^n)\colon \mathcal{L}(t) \leqslant \lambda(t) \leqslant \mathcal{R}(t)\}.$$

Define a mapping

$$I^{**}\colon \mathbb{R}^n_{>0} \times \mathbb{R} \times \mathcal{C}(\mathbb{R}, \mathbb{R}^n) \to \mathcal{C}(\mathbb{R}, \mathbb{R}^n)$$

where $I^{**}(k, t_*, \lambda) = (I^{**}_1(k, t_*, \lambda), I^{**}_2(k, t_*, \lambda), \dots, I^{**}_n(k, t_*, \lambda))$, $k \in \mathbb{R}^n_{>0}$ is a constant vector, $\lambda \in \mathcal{C}\left(\mathbb{R}, \mathbb{R}^n\right)$ is a vector-function, $t_* \in \mathbb{R}$ is fixed, and

$$I^{**}_i(k, t_*, \lambda)(t) := k_i \exp\left(\int_{t_*}^t \lambda_i(s)\, \mathrm{d}s\right), \ \ i = 1, \dots, n, \quad t \in \mathbb{R}. \tag{3.29}$$

Let us look for a solution of equation (3.1) in an exponential form

$$y(t) = I^{**}(k, t_*, \lambda)(t) \tag{3.30}$$

with suitable $k \in \mathbb{R}^n_{>0}$, $t_* \in \mathbb{R}$ and $\lambda \in \mathcal{C}\left(\mathbb{R}, \mathbb{R}^n\right)$. This leads to the operator equation

$$\lambda(t) = (T^{**}\lambda)(t) := f(t, I^{**}(k, t_*, \lambda)_t, I^{**}(k, t_*, \lambda)^t)(\mathrm{diag}(I^{**}(k, t_*, \lambda)(t)))^{-1} \tag{3.31}$$

where $t \in \mathbb{R}$ and $T^{**}\colon \mathcal{C}(\mathbb{R}, \mathbb{R}^n) \to \mathcal{C}(\mathbb{R}, \mathbb{R}^n)$.

3.6.1 Global Solutions Found by Monotone Iterative Method

With several modifications, the monotone iterative method used in the proof of Theorem 3.3.1, can be employed to prove the existence of global solutions as well. This is described by the below theorem.

Theorem 3.6.1. *Let* $k \in \mathbb{R}^n_{>0}$, $t_* \in \mathbb{R}$ *and let* $\mathcal{L}, \mathcal{R}\colon \mathbb{R} \to \mathbb{R}^n$ *be bounded continuous functions, satisfying* (3.28). *Let, moreover, the following hold.*

(*i*) *For any fixed* $a, b \in \mathbb{R}$, $a < b$, *there exists a constant* K *such that, for any function* $\lambda \in \Omega$,

$$\left|\left(T^{**}\lambda\right)(t) - \left(T^{**}\lambda\right)(t')\right| \leqslant K\left|t - t'\right|$$

for arbitrary $t, t' \in [a, b]$.

(ii) $\mathcal{L}(t) \leqslant (T^{**}\mathcal{L})(t)$, $\mathcal{R}(t) \geqslant (T^{**}\mathcal{R})(t)$, $t \in \mathbb{R}$.

(iii) *For any* $\lambda^*, \mu^* \in \Omega$, *the inequality*

$$\mathcal{L}(t) \leqslant \lambda^*(t) \leqslant \mu^*(t) \leqslant \mathcal{R}(t), \ \ t \in \mathbb{R}$$

implies

$$(T^{**}\mathcal{L})(t) \leqslant (T^{**}\lambda^*)(t) \leqslant (T^{**}\mu^*)(t) \leqslant (T^{**}\mathcal{R})(t), \quad t \in \mathbb{R}.$$

Then, there exists a global solution $y\colon \mathbb{R} \to \mathbb{R}^n$ *of* (3.1) *satisfying* $y(t_*) = k$ *and, for arbitrary indexes* $i \geqslant 0$, $j \geqslant 0$,

$$I^{**}(k, t_*, \nu_i)(t) \leqslant y(t) \leqslant I^{**}(k, t_*, \mu_j)(t) \quad \textit{for} \quad t_* < t, \tag{3.32}$$

$$I^{**}(k, t_*, \mu_j)(t) \leqslant y(t) \leqslant I^{**}(k, t_*, \nu_i)(t) \quad \textit{for} \quad t_* > t, \tag{3.33}$$

where $\nu_i(t) \leqslant \nu_{i+1}(t)$, $\mu_{j+1}(t) \leqslant \mu_j(t)$, $\nu_i(t) \leqslant \mu_j(t)$, $\nu_0(t) := \mathcal{L}(t)$, $\mu_0(t) := \mathcal{R}(t)$, $t \in \mathcal{J}_{-1}^{+}$, *and, for* $i > 0$, $j > 0$,

$$\nu_i(t) := (T^{**}\nu_{i-1})(t), \quad \mu_j(t) := (T^{**}\mu_{j-1})(t), \ \ t \in \mathbb{R}. \tag{3.34}$$

Moreover, there exist continuous limits

$$\nu(t) = \lim_{i\to\infty} \nu_i(t), \ \ \mu(t) = \lim_{j\to\infty} \mu_j(t), \ \ \nu(t) \leqslant \mu(t), \ \ t \in \mathbb{R}, \tag{3.35}$$

defining global solutions by the formulas $y_\nu(t) = I(k,\nu)(t)$, $y_\mu(t) = I(k,\mu)(t)$, $t \in \mathbb{R}$ *satisfying* $y_\nu(t_*) = y_\mu(t_*) = k$ *and inequalities*

$$I^{**}(k, t_*, \nu_i)(t) \leqslant y_\nu(t) \leqslant y_\mu(t) \leqslant I^{**}(k, t_*, \mu_j)(t) \quad \textit{for} \quad t_* < t,$$

$$I^{**}(k, t_*, \mu_j)(t) \leqslant y_\mu(t) \leqslant y_\nu(t) \leqslant I^{**}(k, t_*, \nu_i)(t) \quad \textit{for} \quad t_* > t,$$

where $i > 0$, $j > 0$ *are arbitrary.*

3.6.2 Global Solutions Found by Monotone Iterative Method - a Linear Case

Consider a linear equation (3.12) if $\mathcal{J} = \mathbb{R}$ and assume that the functions $c, d\colon \mathbb{R} \to [0,\infty)$, $\tau\colon \mathbb{R} \to [0, r_1]$ and $\sigma\colon \mathbb{R} \to [0, r_2]$ are Lipschitz continuous. Therefore, the result below is a linear variant of Theorem 3.6.1.

Theorem 3.6.2. *Let there exist bounded continuous functions* $\mathcal{L}, \mathcal{R}\colon \mathbb{R} \to \mathbb{R}$ *satisfying* (3.28) *and inequalities*

$$\mathcal{L}(t) \leqslant -c(t)\exp\left(\int_t^{t-\tau(t)} \mathcal{L}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_t^{t+\sigma(t)} \mathcal{L}(s)\,\mathrm{d}s\right), \tag{3.36}$$

$$\mathcal{R}(t) \geqslant -c(t)\exp\left(\int_t^{t-\tau(t)} \mathcal{R}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_t^{t+\sigma(t)} \mathcal{R}(s)\,\mathrm{d}s\right) \tag{3.37}$$

on $\mathbb{R}$. *Then, for any given* $t_* \in \mathbb{R}$, *there exists a global solution* $y = y^*(t)$ *of* (3.12) *on* $\mathbb{R}$ *such that* $y^*(t_*) = 1$ *and, for arbitrary indices* $i \geqslant 0$, $j \geqslant 0$,

$$\exp\left(\int_{t_*}^{t} \nu_i(s)\,\mathrm{d}s\right) \leqslant y^*(t) \leqslant \exp\left(\int_{t_*}^{t} \mu_j(s)\,\mathrm{d}s\right) \quad \textit{for} \quad t > t_*, \tag{3.38}$$

$$\exp\left(\int_{t_*}^{t}\mu_j(s)\,\mathrm{d}s\right) \leqslant y^*(t) \leqslant \exp\left(\int_{t_*}^{t}\nu_i(s)\,\mathrm{d}s\right) \quad \textit{for} \quad t < t_* \tag{3.39}$$

where $\nu_i(t) \leqslant \nu_{i+1}(t)$, $\mu_{j+1}(t) \leqslant \mu_j(t)$, $\nu_i(t) \leqslant \mu_j(t)$, $\nu_0(t) := \mathcal{L}(t)$, $\mu_0(t) := \mathcal{R}(t)$, $t \in \mathbb{R}$, *and, for* $i > 0$, $j > 0$,

$$\nu_i(t) := -c(t)\exp\left(\int_{t}^{t-\tau(t)}\nu_{i-1}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_{t}^{t+\sigma(t)}\nu_{i-1}(s)\,\mathrm{d}s\right), \quad t \in \mathbb{R}$$

$$\mu_j(t) := -c(t)\exp\left(\int_{t}^{t-\tau(t)}\mu_{j-1}(s)\,\mathrm{d}s\right) + d(t)\exp\left(\int_{t}^{t+\sigma(t)}\mu_{j-1}(s)\,\mathrm{d}s\right), \quad t \in \mathbb{R},$$

Moreover, there exist continuous limits $\nu(t) = \lim_{i\to\infty}\nu_i(t)$, $\mu(t) = \lim_{j\to\infty}\mu_j(t)$, $\nu(t) \leqslant \mu(t)$, $t \in \mathbb{R}$, *defining global solutions* $y_\nu, y_\mu \colon \mathbb{R} \to \mathbb{R}^n$ *of* (3.12) *such that* $y_\nu(t_*) = y_\mu(t_*) = k$, *by the formulas*

$$y_\nu(t) = \exp\left(\int_{t_*}^{t}\nu(s)\,\mathrm{d}s\right), \quad y_\mu(t) = \exp\left(\int_{t_*}^{t}\mu(s)\,\mathrm{d}s\right), \quad t \in \mathbb{R}$$

and, for arbitrary $i \geqslant 0$, $j \geqslant 0$,

$$\exp\left(\int_{t_*}^{t}\nu_i(s)\,\mathrm{d}s\right) \leqslant y_\nu(t) \leqslant y_\mu(t) \leqslant \exp\left(\int_{t_*}^{t}\mu_j(s)\,\mathrm{d}s\right) \quad \textit{for} \quad t > t_*,$$

$$\exp\left(\int_{t_*}^{t}\mu_j(s)\,\mathrm{d}s\right) \leqslant y_\mu(t) \leqslant y_\nu(t) \leqslant \exp\left(\int_{t_*}^{t}\nu_i(s)\,\mathrm{d}s\right) \quad \textit{for} \quad t < t_*.$$

3.6.3 Global Solutions in a Non-Iterative Case

Below we formulate a theorem on the existence of global solutions if hypothesis (*iii*) in Theorem 3.6.1 is replaced by a weaker one. While by this approach, the existence of a global solution can be proved for a rather wide class of equations, we lose the iterative process converging to such a global solution.

Theorem 3.6.3. *Let* $k \in \mathbb{R}^n_{>0}$, $t_* \in \mathbb{R}$ *and let* $\mathcal{L}, \mathcal{R} \colon \mathbb{R} \to \mathbb{R}^n$ *be bounded continuous functions, satisfying* (3.28)*. Let the hypotheses* (*i*) *and* (*ii*) *of Theorem 3.6.1 be fulfilled. If, moreover, for an arbitrary* $\lambda \in \Omega$,

$$\left(T^{**}\mathcal{L}\right)(t) \leqslant \left(T^{**}\lambda\right)(t) \leqslant \left(T^{**}\mathcal{R}\right)(t), \quad t \in \mathbb{R}, \tag{3.40}$$

then there exists a global solution $y \colon \mathbb{R} \to \mathbb{R}^n$ *of* (3.1) *satisfying* $y(t_*) = k$ *and*

$$I^{**}(k, t_*, \mathcal{L})(t) \leqslant y(t) \leqslant I^{**}(k, t_*, \mathcal{R})(t) \quad \textit{for} \quad t_* < t, \tag{3.41}$$

$$I^{**}(k, t_*, \mathcal{R})(t) \leqslant y(t) \leqslant I^{**}(k, t_*, \mathcal{L})(t) \quad \textit{for} \quad t_* > t. \tag{3.42}$$

4 CONCLUSION

As noted in the Introduction, the results of the thesis cover not only the cases of mixed type functional differential systems, but also their reductions to delayed, advanced or ordinary differential equations. The main result of Chapter 2 - Theorem 2.3.1 has been proved by the well-known Schauder-Tychonoff fixed-point theorem with a suitable technique, we refer to [12, Corollary 1]. A similar technique is used, for example, in [25, 29, 37] and in [44].

The closest to the topic of the thesis are the results derived in [4, 43]. The existence of solutions on intervals of the $[t_0, \infty)$ type and qualitative behavior of solutions to some classes of scalar linear advanced-delayed equations are considered in [43] by an iterative method and, in [4], qualitative properties of such types of equations are treated by the method of integral inequalities.

The conditions of Theorem 2.3.1 are sufficient for the existence of a global solution $y = y(t)$, $t \in \mathbb{R}$ to mixed-type functional differential equation (2.1) such that $y \in \mathcal{S}$. A simple analysis of the hypotheses of Theorem 2.3.1 shows that these conditions are also necessary for the existence of the above solution.

Chapter 3 is concerned with right and left semi-global solutions and global solutions to nonlinear mixed type functional differential equations giving existence criteria for each type. The main results are formulated by Theorems 3.3.1, 3.4.1, 3.6.1 and 3.6.3. Quite natural questions arise. One of them is, for example, if the statement of Theorem 3.6.1 can be derived by regarding its conclusion as the "union"of the conclusions of Theorems 3.3.1 and 3.4.1 or, vice versa, by splitting its conclusion into those of Theorems 3.3.1 and 3.4.1. Trying to find out whether the conclusions of Theorems 3.3.1 and 3.4.1, in a sense, imply the one of Theorem 3.6.1, we conclude that the hypotheses of Theorems 3.3.1, 3.4.1 differ from those of Theorem 3.6.1. Inequality (3.8) in hypothesis (*iv*) of Theorem 3.3.1 does not imply condition (*iii*) in Theorem 3.6.1 because the operator T used in Theorem 3.3.1 produces functions defined on $\mathcal{J}^+$ but not on $\mathcal{J}^+_{-1}$. Therefore, we conclude that semi-global solutions (existing by Theorems 3.3.1 and 3.4.1) cannot be automatically extended (connected) to global solutions. On the other hand, Theorem 3.6.1 cannot be split into two “semi-global” ones. Although, from a global solution $y = y(t)$, $t \in \mathbb{R}$ to equation (3.1) existing by Theorem 3.6.1, formally, a right semi-global solution y_R and a left semi-global solution y_L can be obtained to equation (3.1) by the restrictions

$$y_R = y_R(t) := y(t)|_{\mathcal{J}^+_{-1}}, \quad y_L = y_L(t) := y(t)|_{\mathcal{J}^-_{+1}},$$

the following objections against such a simple direct restriction can be raised. If there exists such a global solution, then the above restrictions are probably (in most practical situations) not needed. Moreover, such restrictions have been derived from the “global” assumptions formulated in Theorem 3.6.1 (that is its hypotheses must be fulfilled on the entire $\mathbb{R}$) in spite of the fact that “semi-global” Theorems 3.3.1 and 3.4.1 use only assumptions related to semi-global intervals $\mathcal{J}^+_{-1}$, $\mathcal{J}^-_{+1}$. Then, the above restrictions can be wrong in the sense that they do not give either right or left semi-global solution. Therefore, Theorems 3.3.1, 3.4.1 and 3.6.1 are mutually independent.

Obviously, it is possible to consider also two particular cases of equation (3.1) if either the delayed or the advanced terms are missing. In the first case, the function $f(t, y_t, y^t)$ reduces to an advance-type function $f_a(t, y^t)$ so equation (3.1) reduces to

$$\dot{y}(t) = f_a(t, y^t). \tag{4.1}$$

In the second case, the function $f(t, y_t, y^t)$ reduces to a delayed-type function $f_d(t, y_t)$ and equation (3.1) reduces to

$$\dot{y}(t) = f_d(t, y_t). \tag{4.2}$$

Since, as mentioned above, systems (4.1), (4.2) are particular cases of (3.1), we can get results for these systems as restrictions of the results derived. To investigate the system (4.1), only the space $\mathcal{C}_{r_2}$ without $\mathcal{C}_{r_1}$ needs to be considered. For the system (4.2) this is true vice versa. Therefore, such results can be obtained from the general results for (3.1) by formally setting either $r_1 = 0$ in the case of system (4.1), or $r_2 = 0$ in the case of system (4.2).

In Chapter 2 we consider global solutions of nonlinear systems of mixed-type. To prove the existence of global solutions, a different approach and substantially different operators are constructed than those used in Chapter 3. Comparing the results, one sees that these are applicable to different classes of equations. Moreover, in Chapter 2 no iterative method is suggested.

REFERENCES

[1] K. A. Abell, C. E. Elmer, A. R. Humphries, E. S. V. Vleck, Computation of mixed type functional differential boundary value problems, SIAM J. Appl. Dyn. Syst. **4** (2005) 755–781, https://doi.org/10.1137/040603425.

[2] R. P. Agarwal, L. Berezansky, E. Braverman, A. Domoshnitsky, Nonoscillation Theory of Functional Differential Equations with Applications, Springer, New York, 2012, https://doi.org/10.1007/978-1-4614-3455-9.

[3] H. D' Albis, E. Augeraud-Véron, Competitive growth in a lifecycle model: Existence and dynamics, Internat. Econom. Rev. **50** (2009), No. 2, 459–484, https://doi.org/10.1111/j.1468-2354.2009.00537.x.

[4] L. Berezansky, E. Braverman, S. Pinelas, On nonoscillation of mixed advanced-delay differential equations with positive and negative coefficients, Comput. Math. Appl. **58** (2009), No. 4, 766–775, https://doi.org/10.1016/j.camwa.2009.04.010.

[5] Bocharov G. A. and Rihan F. A., Numerical modelling in biosciences using delay differential equations, J. Comput. Appl. Math. **125** (2000), No. 1-2, 183–199, https://doi.org/10.1016/S0377-0427(00)00468-4.

[6] E. Braverman, G. E. Chatzarakis, I. P. Stavroulakis, Iterative oscillation tests for differential equations with several non-monotone arguments, Adv. Differ. Equ. **2016**, Article Number 87, (2016),1–18, https://doi.org/10.1186/s13662-016-0817-3

[7] E. Braverman, B. Karpuz, On oscillation of differential and difference equations with non-monotone delays, Appl. Math. Comput. **218** (2011), No. 7, 3880–3887, https://doi.org/10.1016/j.amc.2011.09.035.

[8] G. E. Chatzarakis, T. Li, Oscillation criteria for delay and advanced differential equations with nonmonotone arguments, Complexity (2018), 1–18, https://doi.org/10.1155/2018/8237634.

[9] G. E. Chatzarakis, Ö. Öcalan, Oscillations of differential equations with several non-monotone advanced arguments, Dynamical Systems **30** (2015), No. 3, 310–323, https://doi.org/10.1080/14689367.2015.1036007.

[10] H. Chi, J. Bell, B. Hassard, Numerical solution of a nonlinear advance-delay-differential equation from nerve conduction theory, J. Math. Biol. **24** (1986), No. 5, 583–601, https://doi.org/10.1007/BF00275686.

[11] K-S./,Chiu, T./,Li, Oscillatory and periodic solutions of differential equations with piecewise constant generalized mixed arguments, Mathematische Nachrichten, **292** (2019), No. 10, 2153–2164, https://doi.org/10.1002/mana.201800053.

[12] L. Díaz, R. Naulin, A proof of the Schauder-Tychonoff theorem, Divulg. Mat. **14** (2006), No. 1, 47–57, https://www.emis.de/journals/DM/v14-1/art5.pdf.

[13] J. Diblík, Long-time behaviour of solutions of delayed-type linear differential equations, Electron. J. Qual. Theory Differ. Equ **2018**, No. 47, 1–23, https://doi.org/10.14232/ejqtde.2018.1.47.

[14] J. Diblík, N. Koksch, Existence of global solutions of delayed differential equations via retract approach, Nonlinear Anal. **64** (2006) 1153–1170, https://doi.org/10.1016/j.na.2005.06.030.

[15] J. Diblík, N. Koksch, Sufficient conditions for the existence of global solutions of delayed differential equations, J. Math. Anal. Appl. **318** (2006) 611–625, https://doi.org/10.1016/j.jmaa.2005.06.020.

[16] J. Diblík, M. Kúdelčíková, Two classes of asymptotically different positive solutions to advanced differential equations via two different fixed - point principles, Math. Methods Appl. Sci. **40** (2017), No. 5, 1422–1437.

[17] J. Diblík, G. Vážanová, Criteria for existence of solutions to linear advance-delay equations, In Matematika, informační technologie a aplikované vědy (2019), 1–5, ISBN: 978-80-7582-097-6.

[18] J. Diblík, G. Vážanová, Existence of global solutions to nonlinear mixed-type functional differential equations, Nonlinear Anal. **195** (2020), 111731, 22 pp.

[19] J. Diblík, G. Vážanová, Global solutions to mixed-type functional differential equations, In Matematika, informační technologie a aplikované vědy (2018), 1–5, ISBN: 978-80-7582-040-2.

[20] J. Diblík, G. Vážanová, Global solutions to mixed-type nonlinear functional differential equations, In Mathematics, Information Technologies and Applied Sciences 2018 post-conference proceedings of extended versions of selected papers (2018), 44–54, ISBN: 978-80-7582-065-5.

[21] J. Diblík, G. Vážanová, On the existence of global solutions of advanced differential equations, In Matematika, informační technologie a aplikované vědy (2016), 15–20, ISBN: 978-80-7231-464- 5.

[22] N. Dilna, M. Fečkan, M. Solovyov, J. Wang, Symmetric nonlinear functional differential equations at resonance, Electron. J. Qual. Theory Differ. Equ. **2019** (2019), 1-16, https://doi.org/10.14232/ejqtde.2019.1.76.

[23] R. D. Driver, Ordinary and Delay Differential Equations, Springer-Verlag, 1977.

[24] D. M. Dubois, Extension of the Kaldor-Kalecki model of business cycle with a computational anticipated capital stock, J. Organ. Trans. Soc. Change **1** (2004), No. 1, 63–80. https://doi.org/10.1386/jots.1.1.63/0

[25] M. Fečkan, J. Wang, W. Wei, Nonlocal Cauchy problems for fractional evolution equations involving Volterra-Fredholm type integral operators, Miskolc Mathematical Notes **13** (2012), No. 1, 127–147, https://doi.org/10.18514/MMN.2012.457.

[26] N. J. Ford, P. M. Lumb, Mixed-type functional differential equations: A numerical approach, J. Comput. Appl. Math. **229** (2009), No. 2, 471–479.

[27] U. Foryś, Marchuk's model of immune system dynamics with application to tumour growth, J. Theor. Med. **1** (2002), Vol. 4, pp. 85–93, https://doi.org/10.1080/10273660290052151.

[28] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Mathematics and Its Applications, Vol. 74, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1992, http://dx.doi.org/10.1007/978-94-015-7920-9.

[29] I. Györi, G. Ladas, Oscillation Theory of Delay Differential Equations, Clarendon Press, 1991, https://doi.org/10.1007/978-1-4684-9467-9.

[30] J. K. Hale, Theory of Functional Differential Equations, Springer-Verlag, New York, 1977.

[31] J. Hale, S. Lunel, Introduction to Functional Differential Equations, Applied Mathematical Sciences, Springer New York, 2013.

[32] S. I. Iakovlev, V. Iakovleva, Eigenvalue-eigenfunction problem for Steklov's smoothing operator and differential-difference equations of mixed type, Opuscula Math. **33** (2013), No. 1, 81–98, http://dx.doi.org/10.7494/OpMath.2013.33.1.81.

[33] S. I. Iakovlev, V. Iakovleva, Systems of advance-delay differential-difference equations and transformation groups, Electron. J. Differential Equations **2016**, No. 311, 1–16, https://ejde.math.txstate.edu/Volumes/2016/311/iakovlev.pdf.

[34] A. Kaddar, H. Talibi Alaoui, Fluctuations in a mixed $IS - LM$ business cycle model, Electron. J. Differential Equations **2008**, 134, 1–9. https://ejde.math.txstate.edu/Volumes/2008/134/kaddar.pdf.

[35] V. Kolmanovskii, A. Myshkis, Introduction to the Theory and Applications of Functional Differential Equations, Kluwer Academic Publishers, Dordrecht, 1999, https://doi.org/10.1007/978-94-017-1965-0.

[36] M. Kon, Y. G. Sficas, I. P. Stavroulakis, Oscillation criteria for delay equations, Proc. Am. Math. Soc. **128** (2000) No. 10, 2989-2997.

[37] R. G. Koplatadze, T. A. Chanturija, Oscillating and monotone solutions of first-order differential equations with deviating argument, Differential'nye Uravneniya **18** (1982) 1463–1465. (Russian)

[38] N. N. Krasovskii, Stability of Motion, Applications of Lyapunov Second Method to Differential Systems and Equations with Delay, Stanford University Press, Stanford, California, 1963. (The book is a translation, with alterations and additions, of N. N. Krasovskii, Nekotorye zadachi teorii ustojchivosti dvizheniya [Some Problems of the Stability of Motion], Moscow, Fizmatgiz Publ., 1959, (In Russian)

[39] T. Krisztin, Nonoscillation for functional differential equations of mixed type, J. Math. Anal. Appl. **245** (2000), No. 2, 326–345, https://doi.org/10.1006/jmaa.2000.6735.

[40] T. Kusano, On even-order functional differential equations with advanced and retarded arguments, J. Differential Equations **45** (1982), No. 1, 75–84, https://doi.org/10.1016/0022-0396(82)90055-9.

[41] H. A. El-Morshedy, E. R. Attia, New oscillation criterion for delay differential equations with non-monotone arguments, Appl. Math. Lett. **54** (2016), 54–59, http://dx.doi.org/10.1016/j.aml.2015.10.014.

[42] D. Otrocol, Systems of functional differential equations with maxima, of mixed type, Electron. J. Qual. Theory Differ. Equ. **2014** (2014), No. 5, 1–9, https://doi.org/10.14232/ejqtde.2014.1.5.

[43] S. Pinelas, Asymptotic behavior of solutions to mixed type differential equation, Electron. J. Differential Equations **2014**, 210, 1–9, https://www.emis.de/journals/EJDE/2014/210/pinelas.pdf.

[44] M. Pituk, G. Röst, Large time behavior of a linear delay differential equation with asymptotically small coefficient, Boundary Value Problems **2014**, 2014:114, 1–9, https://doi.org/10.1186/1687-2770-2014-114.

[45] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkreledze, E. F. Mischenko, The Mathematical Theory of Optimal Processes, Interscience, New York, 1962, https://doi.org/10.1002/zamm.19630431023.

[46] R. Qesmi, A Short Survey on Delay Differential Systems with Periodic Coefficients, Journal of Applied Analysis and Computation **8** (2018), 296–330, https://doi.org/10.11948/2018.296.

[47] F. A. Rihan, B. F. Rihan, Numerical modelling of biological systems with memory using delay differential equations, Appl. Math. Inf. Sci. **9** (2015), No. 3, 1-14.

[48] A. Rustichini, Functional differential equations of mixed type: the linear autonomous case, J. Dynam. Differential Equations **1** (1989), No. 2, 121–143, https://doi.org/10.1007/BF01047828.

[49] A. Rustichini, Hopf bifurcation for functional-differential equations of mixed type, J. Dynam. Differential Equations **1** (1989), No. 2, 145–177, https://doi.org/10.1007/BF01047829.

[50] G. Vážanová, Global solutions to linear delayed differential equations, In Proceedings of the 22nd Conference STUDENT EEICT 2016, Brno, Vysoké učení technické v Brně, Fakulta elektrotechniky a komunikačních (2016), 733–737, ISBN: 978-80-214-5350-0.

[51] G. Vážanová, Semi-global solutions to mixed-type functional differential equations, In Proceedings of the 24th Conference STUDENT EEICT 2018, Brno, Vysoké učení technické v Brně, Fakulta elektrotechniky a komunikačních (2018), 498–502, ISBN: 978-80-214-5614-3.

[52] E. Zeidler, Nonlinear functional analysis and its applications. I. Fixed-point theorems., Springer 1986. (Translated from the German by P. Wadsack.)

AUTHOR'S PUBLICATIONS

Papers published in journals

[1] J. Diblík, G. Vážanová, Existence of global solutions to nonlinear mixed-type functional differential equations, Nonlinear Anal. **195** (2020), 111731, 22 pp.

Papers published in conference proceedings

[2] J. Diblík, M. Růžičková, G. Vážanová, Exponential-type estimates of solutions linear discrete systems with constant coefficients and single delay, In AIP Conference Proceedings 2116, AIP conference proceedings, AIP (2019), 310002-1–310002-4, ISBN: 9780735418547, ISSN: 0094-243X.

[3] J. Diblík, M. Růžičková, G. Vážanová, Mathematical modeling of the IV-administered drug concentration level. In International conference of numerical analysis and applied mathematics (ICNAAM 2017), AIP conference proceedings, Amer Inst Physics, 2 Huntington Quadrangle, STE 1NO1, Melville, NY 11747-4501 USA, American Institute of Physics (2018), 430011-1–4300011-4, ISBN: 978-0-7354-1690-1, ISSN: 0094-243X.

[4] J. Diblík, G. Vážanová, Criteria for existence of solutions to linear advance-delay equations, In Matematika, informační technologie a aplikované vědy (2019), 1–5, ISBN: 978-80-7582-097-6.

[5] J. Diblík, G. Vážanová, Global solutions to mixed-type functional differential equations, In Matematika, informační technologie a aplikované vědy (2018), 1–5, ISBN: 978-80-7582-040-2.

[6] J. Diblík, G. Vážanová, Global solutions to mixed-type nonlinear functional differential equations, In Mathematics, Information Technologies and Applied Sciences 2018 post-conference proceedings of extended versions of selected papers (2018), 44–54, ISBN: 978-80-7582-065-5.

[7] J. Diblík, G. Vážanová, On the existence of global solutions of advanced differential equations, In Matematika, informační technologie a aplikované vědy (2016), 15–20, ISBN: 978-80-7231-464- 5.

[8] J. Diblík, G. Vážanová, Positive solutions to Dickman equation, In 7th Podlasie Conference on Mathematics, Bialystok, University of Bialystok, Poland (2016), 15-16, ISBN: 978-83-7431-478- 7.

[9] G. Vážanová, Global solutions to linear delayed differential equations, In Proceedings of the 22nd Conference STUDENT EEICT 2016, Brno, Vysoké učení technické v Brně, Fakulta elektrotechniky a komunikačních (2016), 733–737, ISBN: 978-80-214-5350-0.

[10] G. Vážanová, Semi-global solutions to mixed-type functional differential equations, In Proceedings of the 24th Conference STUDENT EEICT 2018, Brno, Vysoké učení technické v Brně, Fakulta elektrotechniky a komunikačních (2018), 498–502, ISBN: 978-80-214-5614-3.

Abstracts published in conference proceedings

[11] J. Baštinec, J. Diblík, G. Vážanová, Asymptotic behaviour of positive solutions of a one class of discrete delayed equation, In Dynamical systems, modelling and stability investigation, Kyjev, Ukraine, Taras Shevchenko National University of Kyiv, Ukraine (2017), 149-150, ISBN: 978-617-571-127-9.

[12] J. Baštinec, J. Diblík, G. Vážanová, Global solutions to advance-delay functional differential equations, In Dynamical systems, modelling and stability investigation, Kyjev, Ukraine, Taras Shevchenko National University of Kyiv, Ukraine (2019), 29–30, ISBN: 978-617-571-164-4.

[13] J. Diblík, G. Vážanová, Asymptotic behavor of positive solutions of differential equations with state delay. 6th Ariel Conference on Functional Differential Equations and Applications, Ariel, Israel, Ariel University (2017), 14–14.

[14] J. Diblík, G. Vážanová, Positive solutions of a differential equation $\dot{y}(t) = -c(t)y(t - \tau(t, y(t)))$, Abstracts CDDEA 2017, Poznan, Publishing House of Poznan University of Technology (2017), 17–17, ISBN: 978-83-7775-466-5.

[15] H. Demchenko, Z. Svoboda, G. Vincúrová, Represantation of solutions of linear differential systems of second-order with constant delays, by matrix exponential (2015), 32–32.

CURRICULUM VITÆ

Name:	Gabriela Vážanová
Date of birth:	November 5th, 1987
Country:	Slovak Republic
Nationality:	Slovak
Contact:	xvincu00@stud.feec.vutbr.cz

Education

2015 – 2018	**Brno University of Technology, Faculty of Electrical Engineering and Communication**, Mathematics in Electrical Engineering, Ph.D. study
2014 – 2015	**University of Žilina in Žilina, Faculty of Humanities**, Applied Mathematics, Ph.D. study
2007 – 2012	**Comenius University, Faculty of Science**, Mathematical Analysis, Master study

Languages

Slovak, English, a reading knowledge of German

ABSTRACT

This thesis focuses on functional differential equations of mixed type also referred to as advance-delay equations. It gives sufficient conditions for the existence of global and semi-global solutions.

The methods used in this thesis consist of building suitable operators for differential equations and proving the existence of their fixed points. These fixed points are then used to construct the solutions of advance-delay equations.

The monotone iterative method and Schauder-Tychonoff fixed point theorems are used in the proofs. In both cases, we also provide solution estimates. Moreover, with the monotone iterative method, these estimates may be improved by iterations.

In addition, criteria for linear equations and systems are derived and series of examples are provided. The results obtained are also applicable to ordinary, delayed or advanced differential equations.

ABSTRAKT

Dizertační práce se věnuje funkcionálním diferenciálním rovnicím smíšeného typu, které jsou také nazývány rovnicemi se zpožděným a zrychleným argumentem. Uvádí kritéria pro existenci globálních a semi-globálních řešení.

Metody použité v teto práci spočívají v sestavení vhodných operátorů pro diferenciální rovnice a prokázání existence jejich pevných bodů. Tyto pevné body jsou potom použity ke konstrukci řešení rovnic s předcházením a zpožděním.

V důkazech tvrzení jsou použity monotónní iterační metoda a Schauderovy-Tychonovovy věty o existenci pevného bodu. V obou případech jsou uvedeny také odhady řešení. Pokud je použita iterační metoda, lze tyto odhady zlepšit iterováním.

Kromě toho jsou odvozena kritéria pro lineární rovnice a systémy a je uvedena řada přikladů. Dosažené výsledky lze aplikovat také pro obyčejné diferenciální rovnice nebo diferenciální rovnice se zpožděním či s předcházením argumentu.