BRNO UNIVERSITY OF TECHNOLOGY

FACULTY OF MECHANICAL ENGINEERING
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THESIS TITLE

GENERALIZED LOGISTIC MAPS

DIPLOMA THESIS

AUTHOR RAEES KHAN

SUPERVISOR Prof. Mgr. PAVEL ŘEHÁK, Ph.D.

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Assignment Master's Thesis

Institut:	Institute of Mathematics
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Branch:	no specialisation
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As provided for by the Act No. 111/98 Coll. on higher education institutions and the BUT Study and Examination Regulations, the director of the Institute hereby assigns the following topic of Master's Thesis:

Generalized logistic maps

Brief Description:

The classical logistic map is related to the discrete logistic equation. In contrast to the continuous case, the logistic difference equation exhibits very complicated dynamics, incl. chaotic behavior. Its applications are, for example, in biology, chemistry, or cryptography.
There is a certain disadvantage of the usual logistic maps, namely, they allow just one variable parameter. On the other hand, the number of degrees of freedom can be increased, and therefore the generalized logistic maps can be applied in a wider spectrum of problems. Also, their analysis becomes more challenging.

Master's Thesis goals:

1. Generalized logistic differential and difference equations as models.
2. Explanation of selected concepts from the theory of (discrete) dynamics, such as equilibrium, stability, periodic cycle, bifurcation, Lyapunov exponent, (deterministic) chaos, etc.
3. Analysis of generalized logistic maps.
4. Numerical testing and graphical interpretations of the results related to generalized logistic maps.

Recommended bibliography:

DA COSTA, D. G., MEDRANO, R. O. and E. D. Leonel. Route to chaos and some properties in the boundary crisis of a generalized logistic mapping. Phys. A 486 (2017), 674–680.

ELAYDI, S. N. Discrete Chaos. 2nd ed., Chapman & Hall/CRC, Boca Raton, FL, 2008.

RADWAN, A. G. On some generalized discrete logistic maps. J. Adv. Res. 4 (2013), 163–171.

RAK, R. and E. Rak. Route to chaos in generalized logistic map. Acta Physica Polonica A Vol. 127 (2015), 113-117.

ROBINSON, R. C. An Introduction to Dynamical Systems — Continuous and Discrete. 2nd ed., American Mathematical Society, Providence, 2012.

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doc. Mgr. Petr Vašík, Ph.D. Director of the Institute	doc. Ing. Jiří Hlinka, Ph.D. FME dean

Abstract
The logistic map is a classical model used to describe population dynamics in discrete time. A natural extension of this model is the generalized logistic map, which introduces additional flexibility through an extra exponent parameter. In this study, we focus on a particular case of the generalized logistic map known as the Richard's logistic map, which is closely related to Richard's difference equation. To support this analysis, we begin by reviewing the foundational concepts of differential and difference equations, providing a basis for understanding the behavior of systems in both continuous and discrete time settings.

We analyze the properties of these generalized logistic maps with emphasis on the special case, which simplifies to the Richard's logistic model. Through detailed algebraic manipulation, equilibrium points and their stability are investigated using derivative tests and the Schwarzian derivative. The maximum value of the growth parameter μ is also computed to identify when chaotic behavior begins.

The work further examines period-2 cycles and their bifurcation behavior as the growth parameter increases. Numerical simulations, cobweb plots, and bifurcation diagrams are implemented in Python to visualize transitions from stability to periodicity and chaos. The analysis reveals that for the Richard's case studied, the map becomes chaotic when μ exceeds approximately 2.59, which is notably lower than the simple logistic case.

Lastly, comparisons with the classical logistic map demonstrate that increasing the nonlinearity accelerates convergence to equilibrium and narrows the range of μ that results in bounded behavior. These findings underline how small changes in model structure can significantly affect long-term dynamics, contributing to a deeper understanding of discrete chaos in nonlinear systems.

Keywords
Logistic differential equation, logistic difference equation, Richard's logistic map, iteration of functions, equilibrium point, stability, k-periodic cycles, Schwarzian derivative, bifurcation, cobweb diagram, period doubling, Lyapunov exponent, chaos, Feigenbaum constant.

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I declare that I wrote the diploma thesis *Generalized Logistic Maps* independently under the guidance of *Prof. Mgr. Pavel Řehák, Ph.D.* using the literature included in the list of references.

Raees Khan

I am sincerely grateful to all the scholars, teachers, friends, and family members who supported me throughout my academic journey. I owe special thanks to my supervisor, whose invaluable guidance and insightful mentorship have truly shaped this thesis. I feel privileged to have worked under the direction of such a knowledgeable and inspiring academic.

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As the saying goes, there is light at the end of every tunnel. May the Almighty continue to guide our paths and protect us all. To everyone who contributed in any way, your impact will forever remain in my heart. Thank you.

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Contents

1 Introduction

1.1 General Information about Differential Equation and Difference Equation

Dynamical systems are fundamental to comprehending the evolution of quantities governed by deterministic or stochastic laws, providing insights into events in both natural and social sciences. These systems are classified based on their temporal treatment: continuous-time dynamics are represented by differential equations (e.g., planetary motion or chemical reactions), whereas discrete-time dynamics are dictated by difference equations, which iteratively transition a system's state from one step to the next. Dynamical systems theory offers a cohesive framework for examining complexity, stability, and unpredictability across various domains, including population dynamics in ecology, chaotic behaviour in physics, and volatility in financial markets.

A mathematical equation describing the evolution of a system over time can be categorized into either a continuous or discrete equation, depending on whether the variable changes continuously or discretely.

1.1.1 Differential Equations

A differential equation is an equation involving an unknown function and its derivatives. It describes the rate of change of a system and is given by

$$\frac{d^n x}{dt^n} = F\left(t, x, \frac{dx}{dt}, \frac{d^2x}{dt^2}, \ldots, \frac{d^{n-1}x}{dt^{n-1}}\right). \tag{1.1}$$

A first-order differential equation takes the form:

$$\frac{dx}{dt} = f(t, x). \tag{1.2}$$

Examine a singular population quantified by a variable x. The time derivative $\frac{dx}{dt}$ quantifies the population growth rate, while the ratio $\frac{1}{x} \cdot \frac{dx}{dt}$ indicates the growth rate per unit of population. The logistic model asserts that the relative growth rate $\frac{1}{x} \cdot \frac{dx}{dt}$ diminishes when x nears the environmental carrying capacity K. The relevant equation is known as the *logistic differential equation*:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right). \tag{1.3}$$

It is also known as the *logistic differential equation*.

On the other hand, the generalized logistic differential equation is an extension of the logistic differential equation that introduces more flexibility in modeling growth processes. It takes the form:

$$\frac{dx}{dt} = rx^p\left(1 - \left(\frac{x}{K}\right)^q\right). \tag{1.4}$$

where p, q are the parameters that generalize the growth term.

However, the generalized logistic differential equation (1.4) is significantly more difficult to solve.

Therefore, we will restrict ourselves to the special case where the growth parameters are set to $p = 1$ and $q = 2$. This equation is also known as **Richards' Differential Equation**, which is the main focus of our study. It is given by:

$$\frac{dx}{dt} = rx\left(1 - \left(\frac{x}{K}\right)^2\right), \tag{1.5}$$

We start with the equation

$$\frac{dx}{x\left(1 - \left(\frac{x}{K}\right)^2\right)} = r\,dt$$

or equivalently

$$\frac{K^2}{x(K-x)(K+x)}\,dx = r\,dt$$

where

$$\frac{1}{x(K-x)(K+x)} = \frac{A}{x} + \frac{B}{K-x} + \frac{C}{K+x}. \tag{1.6}$$

By solving the partial fractions in the equation (1.6), we get the values of the constants

$$A = \frac{1}{K^2}, \quad B = \frac{1}{2K^2}, \quad C = -\frac{1}{2K^2}.$$

We now substitute A, B, and C back into the partial fraction decomposition, the equation (1.6) becomes

$$\frac{1}{x(K-x)(K+x)} = \frac{\frac{1}{K^2}}{x} + \frac{\frac{1}{2K^2}}{K-x} - \frac{\frac{1}{2K^2}}{K+x},$$

$$\left(\frac{1}{x} + \frac{1}{2(K-x)} - \frac{1}{2(K+x)}\right)dx = r\,dt. \tag{1.7}$$

Now, we integrate both sides of the equation (1.7)

After integrating, the left-hand side of the equation (1.7) becomes

$$\int \left(\frac{1}{x} + \frac{1}{2(K-x)} - \frac{1}{2(K+x)}\right)dx = \ln|x| - \frac{1}{2}\ln|K-x| - \frac{1}{2}\ln|K+x|.$$

and the right-hand side of the equation (1.7) becomes

$$\int r\,dt = rt + C.$$

So, the equation (1.7) becomes

$$\ln|x| - \frac{1}{2}\ln|K-x| - \frac{1}{2}\ln|K+x| = rt + C.$$

Apply the initial condition, let $x(0) = x_0$. Then

$$\ln|x_0| - \frac{1}{2}\ln|K - x_0| - \frac{1}{2}\ln|K + x_0| = C.$$

Thus, the constant C becomes

$$C = \ln|x_0| - \frac{1}{2}\ln|K - x_0| - \frac{1}{2}\ln|K + x_0|.$$

Back to the general solution

$$\ln|x| - \frac{1}{2}\ln|K - x| - \frac{1}{2}\ln|K + x| = rt + C.$$

Combine the logarithms

$$\ln\left(\frac{|x|}{\sqrt{|K - x||K + x|}}\right) = rt + C,$$

Exponentiate both sides

$$\frac{|x|}{\sqrt{K^2 - x^2}} = C_1 e^{rt}, \quad \text{where } C_1 = e^C > 0.$$

Assuming $x > 0$, we drop the absolute value

$$\frac{x}{\sqrt{K^2 - x^2}} = C_1 e^{rt},$$

Square both sides

$$\frac{x^2}{K^2 - x^2} = C_1^2 e^{2rt}.$$

Multiply both sides

$$x^2 = C_1^2 e^{2rt}(K^2 - x^2),$$

$$x^2 + C_1^2 e^{2rt} x^2 = C_1^2 K^2 e^{2rt},$$

$$x^2(1 + C_1^2 e^{2rt}) = C_1^2 K^2 e^{2rt},$$

$$x(t) = \frac{C_1 K e^{rt}}{\sqrt{1 + C_1^2 e^{2rt}}}.$$

From initial condition

$$\frac{x_0}{\sqrt{K^2 - x_0^2}} = C_1.$$

Thus the explicit solution

$$x(t) = \frac{x_0 K e^{rt}}{\sqrt{K^2 - x_0^2 + x_0^2 e^{2rt}}}.$$

1.1.2 Difference Equations

A difference equation, on the other hand, describes the evolution of a sequence, where changes occur at discrete intervals. It has the general form:

$$y_{n+k} = F(y_{n+k-1}, y_{n+k-2}, \ldots, y_n, n), \tag{1.8}$$

where $k \in \mathbb{N}$ indicates the order of the difference equation, and F is a function defining the relation between the current and previous terms. In the special case where the function does not explicitly depend on n, the equation is called autonomous.

A first-order autonomous difference equation can be written as:

$$y_{n+1} = f(y_n), \tag{1.9}$$

where y_n represents the state of the system at the discrete time step n.

These equations play a fundamental role in modeling real-world phenomena, where differential equations are used for continuous processes (e.g., population growth, heat transfer), while difference equations are used for discrete processes (e.g., iterative algorithms, population growth models, financial models).

Discretization of the Richard's Differential Equation

We start with the given differential equation

$$\frac{dx}{dt} = rx\left(1 - \left(\frac{x}{K}\right)^2\right). \tag{1.10}$$

Using the forward Euler method, we shall continue as outlined below: taking into account the distinct collection of points $t_0, t_1, t_2, \ldots, t_n, \ldots$ with $h = t_{n+1} - t_n$ as the step size.

For $t_n \le t \le t_{n+1}$, we approximate $x(t)$ by $x(t_n)$, and the derivative $\frac{dx}{dt}$ by the finite difference

$$\frac{dx}{dt} \approx \frac{x(t_{n+1}) - x(t_n)}{h}.$$

Substituting this into the given differential equation

$$\frac{x(t_{n+1}) - x(t_n)}{h} = rx(t_n)\left(1 - \left(\frac{x(t_n)}{K}\right)^2\right).$$

Rearranging, we obtain

$$x(t_{n+1}) = x(t_n) + hrx(t_n)\left(1 - \left(\frac{x(t_n)}{K}\right)^2\right).$$

which may be written in the compact form

$$x_{n+1} = x_n + hrx_n\left(1 - \frac{x_n^2}{K^2}\right).$$

Define the normalized variable

$$y_n = \frac{x_n}{K}.$$

which implies

$$x_n = Ky_n.$$

Substituting this into the discretized equation

$$Ky_{n+1} = Ky_n + hrKy_n\left(1 - y_n^2\right).$$

Dividing both sides by K

$$y_{n+1} = y_n + hry_n(1 - y_n^2).$$

Factor out $(1 + rh)$

$$y_{n+1} = (1 + rh)y_n[1 - y_n^2]. \tag{1.11}$$

Define $\mu = (1 + rh)$, so

$$y_{n+1} = \mu y_n[1 - y_n^2]. \tag{1.12}$$

Unlike the Richard's logistic differential equation (continuous-time case) solved in Section 1.1.1, solving the Richard difference equation (discrete-time case), it is impossible to find a solution in a closed form. As a result, we turn to iterative methods to analyze and explore the dynamics of the system.

1.2 Iteration of Functions as Dynamics

In mathematical analysis and dynamical systems, iteration of a function refers to the process of repeatedly applying the same function to its own output. This process defines the evolution of a system over discrete time steps and provides insight into its long-term behavior. A function $f : X \to X$ is iterated by successively applying it to an initial value x_0, generating a sequence of values that follow the rule:

$$x_n = f(x_{n-1}), \quad n \in \mathbb{N},$$

where x_0 is the initial condition.

Definition 1.1 Function composition is an operation $\circ$ that takes two functions f and g, and produces a function $h = g \circ f$ such that

$$h(x) = g(f(x)).$$

The iteration of a function is denoted using function composition as

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}. \tag{1.13}$$

In this study, we utilize the notation for function composition as mentioned in (1.13).

This iterative process is fundamental in studying dynamical systems, where the objective is to analyze how sequences evolve over time under repeated function application.

1.3 Example

Consider a function $f : \mathbb{R} \to \mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Assume

$$f(x) = \lambda x, \quad \forall x \in \mathbb{R} \tag{1.14}$$

where λ is a constant. If we introduce a sequence $\{x_n\}_{n=0}^{k} \subset \mathbb{R}$, such that

$$x_n = f(x_{n-1}), \quad n \in \mathbb{N}$$

where x_0 represents a specified initial value, then we obtain the following iterates,

$$x_1 = f(x_0) = \lambda x_0.$$

By iteration, we get

$$x_2 = f(x_1) = f \circ f(x_0) = f^2(x_0) = \lambda^2 x_0$$

$$\vdots$$

$$\begin{aligned} x_n &= f(x_{n-1}) = f \circ f \circ \cdots \circ f(x_0) \\ &= f^n(x_0) = \lambda^n x_0. \end{aligned} \tag{1.15}$$

The behavior of the aforementioned relation can be anticipated with different values or ranges of λ, given that $x_0 > 0$. We derived a notable formula for x_n, which is, however, quite exceptional. Let $\lambda = 1$; in this case, the iterates remain constant. If $\lambda < 0$, the iterates approach zero as n tends to infinity. Conversely, if $\lambda > 0$, the iterates diverge to infinity as n approaches infinity. The logistic map $f(y) = \mu y(1 - y)$ defines a discrete dynamical system through iteration of the function. Another example is the generalized logistic map given by

$$f(y) = \mu y^p(1 - y^q).$$

2 Generalized Logistic Maps

2.1 General Overview

The logistic map gained significant attention in 1976 following its introduction by biologist Robert May, who applied it to population models. While the concept itself was already present in earlier works, particularly in the logistic equation developed by Pierre François Verhulst, May's approach brought it into the spotlight. His model provided a mathematical framework for understanding how populations evolve over time. Although the logistic map has been widely recognized in the context of population dynamics, much of the literature focuses on its role in modeling the population of a species within a specific generation, using the population of a previous generation as a reference point.

2.2 Model

Let x_n denote the population size of a specific species at time n, and let $\mu > 0$ represent the rate of population growth between generations. The equation presented serves as a straightforward mathematical model for depicting the temporal evolution of a population,

$$x_{n+1} = \mu x_n. \tag{2.1}$$

Here, the population size at time $n + 1$ is determined by multiplying the current population x_n by the growth rate μ. Assuming an initial population x_0, we can iteratively calculate the population at any time step. The general solution to this equation, as demonstrated in the preceding chapter, is

$$x_n = \mu^n x_0.$$

This equation describes the population size at any generation n.

When $\mu > 1$, the population x_n grows exponentially, increasing without bound as n approaches infinity. If $\mu = 1$, the population remains constant, so $x_n = x_0$ for all n. On the other hand, if $\mu < 1$, the population will decrease over time, and as n tends to infinity, the population approaches zero $\lim_{n\to\infty} x_n = 0$. In such a scenario, the population would eventually become extinct.

However, in real-life scenarios, these simple models are often not sufficient to describe species populations, as populations do not increase indefinitely or decline to zero. Instead, populations typically grow until they reach a maximum value, after which limited resources lead to competition among individuals. This competition is proportional to the square of the population size, which can be represented as x_n^2.

To account for this, a more realistic model incorporates a term that represents this competition, resulting in the following equation

$$x_{n+1} = \mu x_n - b x_n^2. \tag{2.2}$$

Here, $b > 0$ is a constant that represents the intensity of interaction or competition between individuals in the population.

To simplify the Equation (2.2), we introduce a new variable y_n, defined as

$$y_n = \left(\frac{b}{\mu}\right) x_n.$$

Substituting this into the model, we get

$$y_{n+1} = \left(\frac{b}{\mu}\right) x_{n+1}.$$

This gives us the following relationship

$$\left(\frac{\mu}{b}\right) y_{n+1} = \left(\frac{\mu}{b}\right) (\mu y_n) [1 - y_n],$$

$$y_{n+1} = \left(\frac{b}{\mu}\right) \left(\mu^2 \left(\frac{y_n}{b}\right)\right) [1 - y_n].$$

Simplifying further, we obtain

$$y_{n+1} = \mu y_n [1 - y_n]. \tag{2.3}$$

Th equation (2.3) shows the *logistic difference equation*, and the function

$$f(y) = \mu y(1 - y)$$

is called the *logistic map*.

To make the logistic map more applicable to real-world scenarios, we extend the logistic model, where the values of p and q are 1, to a generalized logistic map. This extension allows for greater flexibility and can better capture the complexities of various systems, including population dynamics, where factors like resource limitations, interaction rates, and environmental influences may vary. The following equation defines the generalized logistic map

$$x_{n+1} = \mu x_n^p (1 - x_n^q),$$

where $x_n \in [0, 1]$, $p, q > 0$, and $n = 0, 1, 2, 3, \ldots$, with p and q as parameters that modify the dynamics of the system, adding further complexity and versatility to the classical logistic model. This equation serves as the focal point of our research. This study examines the complex dynamics exhibited by the map through variations in the values of μ, p, and q. Due to the complexity in the general case, we will primarily focus on $p = 1$ and $q = 2$, which correspond to Richard's logistic equation. On the other hand, if we set $p = 1$ and $q = 1$, the generalized logistic map reduces to the classical logistic map

$$x_{n+1} = \mu x_n (1 - x_n),$$

which has been extensively studied for its rich dynamical behavior including stability, bifurcations, and chaos.

2.3 Equilibrium Points

This section examines the qualitative behavior of the orbit(s) of the logistic map.

Definition 2.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a map, and let $y_0 \in \mathbb{R}$ be an initial condition. The *positive orbit* of the point y_0 under f, denoted by $O(y_0)$, is the set of all forward iterates of y_0, i.e.,

$$O(y_0) = \{y_0, f(y_0), f^2(y_0), \dots\}.$$

In the examined map, only positive iterates were taken into account, therefore, the positive orbit will be referred to as *orbit*.

Definition 2.2. Given a map $f : \mathbb{R} \to \mathbb{R}$, a point y^* is an *equilibrium point* for f provided that $f(y^*) = y^*$. The set of all equilibrium points of f is denoted by

$$\text{Fix}(f) = \{y^* : f(y^*) = y^*\}.$$

Remark 2.3. The term fixed point is synonymous with equilibrium point and is used interchangeably in this study.

Definition 2.4. A point $\bar{y}$ is a *period-n point* or *n-periodic point* for f provided that

$$f^n(\bar{y}) = \bar{y}.$$

The positive integer n is called the *minimal period* for $\bar{y}$ if $\bar{y}$ is a period-n point for f but $f^j(\bar{y}) \neq \bar{y}$ for $0 < j < n$. The set of all period-n points of f with a minimal period is denoted by

$$\text{Per}(n, f) = \{\bar{y} : f^n(\bar{y}) = \bar{y} \text{ and } f^j(\bar{y}) \neq \bar{y} \text{ for } 0 \leq j < n\}.$$

If $\bar{y}$ is a period-n point, then

$$O(\bar{y}) = \{f^j(\bar{y}) : 0 \leq j < n\}$$

is called a *period-n orbit* and it is also called *n-period cycle*.

Definition 2.5. A point y is said to be an *eventually fixed point* of a map $f : \mathbb{R} \to \mathbb{R}$ if there exists a positive integer r and a fixed point y^* of f such that $f^r(y) = y^*$, but

$$f^{r-1}(y) \neq y^*.$$

From Equation (1.12), we will find the fixed points

$$y = f(y) = \mu y[1 - y^2],$$

$$\mu y^3 - \mu y + y = 0,$$

$$\mu y^3 + y(1-\mu) = 0,$$

$$y(\mu y^2 + (1-\mu)) = 0,$$

$$y = 0, \qquad \mu y^2 + (1-\mu) = 0$$

Solving for y^2,

$$\mu y^2 = (\mu - 1),$$

$$y^2 = \frac{\mu - 1}{\mu}.$$

Taking the square root on both sides,

$$\sqrt{y^2} = \sqrt{\frac{\mu - 1}{\mu}},$$

$$y = \pm\sqrt{\frac{\mu - 1}{\mu}}.$$

Thus, the equilibrium points are,

$$y^* = 0 \quad , \quad y^* = \pm\sqrt{\frac{\mu - 1}{\mu}}.$$

Thus, the set of fixed points is

$$\mathrm{Fix}(f) = \left\{0, +\sqrt{\frac{\mu - 1}{\mu}}, -\sqrt{\frac{\mu - 1}{\mu}}\right\}. \tag{2.4}$$

Now, we do calculations to determine the maximum value of the growth parameter μ because beyond that value, the system reaches chaos which we will see later. Thus for that, we first find the local maximum values for y.

The given function,

$$f_\mu(y) = \mu \cdot y^p \cdot (1 - y^q).$$

Expanding,

$$f_\mu(y) = \mu \cdot y^p - \mu \cdot y^{p+q}.$$

Taking the derivative,

$$f'_\mu(y) = \mu \left[p y^{p-1} - (p+q) y^{p+q-1} \right] = 0.$$

Rearranging,

$$p y^{p-1} - (p+q) y^{p+q-1} = 0.$$

Factoring,

$$y^{p-1} \left(p - y^q (p+q) \right) = 0.$$

Solving for y,

$$p = y^q (p+q),$$

$$y^q = \frac{p}{p+q},$$

$$y = \left(\frac{p}{p+q} \right)^{\frac{1}{q}}.$$

Substituting back into the function,

$$f_\mu \left(\left(\frac{p}{p+q} \right)^{\frac{1}{q}} \right) = \frac{\mu q p^{p/q}}{(p+q)^{(p+q)/q}}.$$

The parameter μ must satisfy the inequality (2.5) because if μ exceeds the value of μ_{max}, the system exhibits chaotic behavior as we will see later

$$\mu \leq \mu_{\max} \tag{2.5}$$

$$\mu \in (0, \mu_{\max}]$$

We are interested in finding the maximum possible value that the function $f_\mu(y) = \mu y^p (1 - y^q)$ can attain.

In order for the function to remain within the interval $[0, 1]$, we require,

$$f_\mu(y) \leq 1 \quad \text{for all } y \in [0, 1]$$

So, to find the largest possible value of μ such that $f_\mu(y)$ still maps values from $[0, 1]$ to $[0, 1]$, we maximize $f_\mu(y)$ and set that maximum equal to 1. For the function $f_\mu(y) = \mu y^p (1 - y^q)$ to define a valid discrete dynamical system on the interval $[0, 1]$, it must map any value $y \in [0, 1]$ back into the same interval. That is, we require $f_\mu(y) \in [0, 1]$ for all $y \in [0, 1]$, so that the iterates remain bounded. This ensures that population values do not exceed the normalized carrying capacity 1 or drop below 0, both of which would be non-physical.

Thus,

$$f_\mu\left(\left(\frac{p}{p+q}\right)^{\frac{1}{q}}\right) = \frac{\mu q p^{p/q}}{(p+q)^{(p+q)/q}} = 1.$$

Solving for $\mu_{\max}$

$$\mu_{\max} = \frac{\left(\frac{p+q}{p}\right)^{p/q} \cdot (p+q)}{q} \tag{2.6}$$

For the general case where p and q are any natural numbers, we derived equation (2.6). However, for our Richard's logistic map where $p = 1$ and $q = 2$, we use the equation (2.6)

Substitute $p = 1$ and $q = 2$, to obtain the maximum value of μ,

$$\mu_{\max} = \frac{\left(\frac{1+2}{1}\right)^{1/2} \cdot (1+2)}{2},$$
$$\mu_{\max} = \frac{(3)^{1/2} \cdot 3}{2},$$
$$\mu_{\max} = \frac{3\sqrt{3}}{2}.$$

Thus, the maximum value of μ is

$$\mu_{\max} = \frac{3\sqrt{3}}{2}.$$

Therefore, the range of the growth parameter μ is

$$\mu \in \left(0, \frac{3\sqrt{3}}{2}\right] \approx (0, 2.59]\,.$$

Now we will generate the values of iterations corresponding to the different values of μ in order to study the basin of attraction and draw the cobweb plot to analyse the dynamics. See Figure 1, 2, and 3 presented at the end of this section.

Table 1a: Iteration Data for Richard's Logistic Map

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
$\mu = 1.5$	0.1	0.1485	0.2178	0.3112	0.4216	0.5200	0.5690	0.5771	0.5773	0.5773
$\mu = 2.0$	0.1	0.1980	0.3805	0.6508	0.7503	0.6558	0.7475	0.6596	0.7452	0.6627
$\mu = 2.5$	0.1	0.2475	0.5808	0.9622	0.1784	0.4319	0.8783	0.5019	0.9387	0.2790
$\mu = 1.5$	0.2	0.2880	0.3961	0.5009	0.5628	0.5768	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.2	0.3840	0.6548	0.7481	0.6588	0.7457	0.6620	0.7437	0.6647	0.7420
$\mu = 2.5$	0.2	0.4800	0.9235	0.3396	0.7512	0.8183	0.6759	0.9178	0.3617	0.7859
$\mu = 1.5$	0.3	0.4095	0.5112	0.5664	0.5770	0.5773	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.3	0.5460	0.7665	0.6324	0.7590	0.6436	0.7540	0.6506	0.7504	0.6557
$\mu = 2.5$	0.3	0.6825	0.9115	0.3856	0.8207	0.6699	0.9232	0.3408	0.7531	0.8150
$\mu = 1.5$	0.4	0.5040	0.5639	0.5768	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.4	0.6720	0.7371	0.6733	0.7362	0.6744	0.7353	0.6755	0.7346	0.6764
$\mu = 2.5$	0.4	0.8400	0.6182	0.9548	0.2107	0.5034	0.9396	0.2752	0.6358	0.9469
$\mu = 1.5$	0.5	0.5625	0.5767	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.5	0.7500	0.6563	0.7473	0.6600	0.7450	0.6630	0.7431	0.6655	0.7415
$\mu = 2.5$	0.5	0.9375	0.2838	0.6524	0.9368	0.2866	0.6577	0.9330	0.3019	0.6860

Remark 2.6. Table 1b reveals that the row corresponding to the initial value 0.6 for $\mu = 1.5$ consistently contains the value 0.5773 across nearly all entries. For $\mu = 1.5$, 0.5773 serves as a fixed point of the logistic map.

Definition 2.7. Let f be a function and y^* a fixed point. Then, the *basin of attraction* of y^* is the set of all initial conditions y_0 such that $f^n(y_0)$ converges to y^* as n goes to infinity. We denote the basin of attraction by $\mathcal{B}(y^*, f)$, so,

$$\mathcal{B}(y^*, f_\mu) = \bigcup_{n=0}^{\infty} f_\mu^{-n}(y^*)$$

where

$$f_\mu^{-n}(y^*) = \{y : f_\mu^n(y) = y^*\}$$

is a set of *eventually fixed point*.

From Table 1, it is seen that for $\mu = 1.5$, the basin of attraction for $y^* = 0.5773$ satisfies

$$\{0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9\} \subset \mathcal{B}(0.5773, f)$$

Table 1b: Iteration Data for Richard's Logistic Map

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
$\mu = 1.5$	0.6	0.5760	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.6	0.7680	0.6300	0.7599	0.6422	0.7547	0.6497	0.7509	0.6550	0.7480
$\mu = 2.5$	0.6	0.9600	0.1882	0.4537	0.9008	0.4246	0.8701	0.5284	0.9522	0.2223
$\mu = 1.5$	0.7	0.5355	0.5729	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.7	0.7140	0.7000	0.7140	0.7000	0.7140	0.7000	0.7140	0.7000	0.7140
$\mu = 2.5$	0.7	0.8925	0.4539	0.9010	0.4239	0.8694	0.5308	0.9531	0.2182	0.5196
$\mu = 1.5$	0.8	0.4320	0.5271	0.5710	0.5772	0.5773	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.8	0.5760	0.7698	0.6273	0.7609	0.6407	0.7554	0.6487	0.7514	0.6543
$\mu = 2.5$	0.8	0.7200	0.8669	0.5386	0.9559	0.2062	0.4935	0.9333	0.3009	0.6842
$\mu = 1.5$	0.9	0.2565	0.3594	0.4695	0.5490	0.5753	0.5773	0.5773	0.5773	0.5773
$\mu = 2.0$	0.9	0.3420	0.6040	0.7673	0.6311	0.7595	0.6428	0.7544	0.6501	0.7507
$\mu = 2.5$	0.9	0.4275	0.8734	0.5178	0.9474	0.2426	0.5708	0.9621	0.1790	0.4332

2.4 Stability of an Equilibrium Point

The definitions and lemmas presented herein are utilized in the stability analysis conducted in this study.

Definition 2.8 Let $f : \mathbb{R} \to \mathbb{R}$ be a map and y^* be a fixed point of f. Then,

- y^* is *stable* if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $y_0 \in \mathbb{R}$ with $|y_0 - y^*| < \delta$ we have $|f^n(y_0) - y^*| < \epsilon$ for all $n \in \mathbb{Z}^+$. Otherwise, the fixed point y^* is *unstable*.
- y^* is *attracting* if there exists $\eta > 0$ such that $|y_0 - y^*| < \eta$ implies $\lim_{n\to\infty} f^n(y_0) = y^*$. Otherwise, the fixed point y^* is a *repelling* fixed point.
- y^* is *asymptotically stable* if it is both stable and attracting. Furthermore, if in the preceding definition $\eta = \infty$, then y^* is *globally asymptotically stable*.
- y^* is *semistable* provided it is attracting from one side and repelling on the other side.

Definition 2.9 A fixed point y^* of a map f is said to be *hyperbolic* if $|f'(y^*)| \neq 1$. Otherwise, it is *nonhyperbolic*.

Next, we present tools for detecting local stability of *hyperbolic points*.

Lemma 2.10 Let f be a C^1 function at the point y^*.

- Assume that y^* is a fixed point for f. The following three statements give the relationship between the absolute value of the derivative and the stability of y^*:
  - If $|f'(y^*)| < 1$, then y^* is an attracting fixed point.
  - If $|f'(y^*)| > 1$, then y^* is a repelling fixed point.

  - If $|f'(y^*)| = 1$, then y^* can be attracting, repelling, semi-stable, or none of these.
- Assume that y^* is a period-n point, and $y_j = f^j(y^*)$. By the chain rule,

$$|(f^n)'(y^*)| = |f'(y_{n-1})| \cdots |f'(y_1)| \cdot |f'(y^*)|.$$

The following three statements give the relationship between the size of this derivative and the stability of y^*:

  - If $|(f^n)'(y^*)| < 1$, then y^* is an attracting period-n point.
  - If $|(f^n)'(y^*)| > 1$, then y^* is a repelling period-n point.
  - If $|(f^n)'(y^*)| = 1$, then the periodic orbit can be attracting, repelling, semi-stable, or none of these.

The definition and lemmas below are employed in describing the qualitative behaviour of the *nonhyperbolic* fixed points, i.e., for the case when $|f'(y^*)| = 1$.

Definition 2.11 Let f be a C^3 function from $\mathbb{R}$ to $\mathbb{R}$. The *Schwarzian derivative* of f is defined by

$$S_f(y) = \frac{f'''(y)}{f'(y)} - \frac{3}{2}\frac{(f''(y))^2}{(f'(y))^2}.$$

Lemma 2.12 Let y^* be a fixed point of a map f such that $f'(y^*) = -1$. If $f'''(y)$ is continuous at y^*, the following statements hold:

- If $S_f(y^*) < 0$, then y^* is *asymptotically stable*.
- If $S_f(y^*) > 0$, then y^* is *unstable*.

Lemma 2.13 Let y^* be a fixed point of a map f such that $f'(y^*) = 1$. If $f'''(y)$ is continuous at y^*, then the following statements hold:

- If $f''(y^*) \neq 0$, then y^* is *unstable* (semistable).
- If $f''(y^*) = 0$ and $f'''(y^*) > 0$, then y^* is *unstable*. If $f''(x) = 0$ and $f'''(y^*) < 0$, then y^* is *asymptotically stable*.

Lemma 2.14 If f and g are C^3 functions from $\mathbb{R}$ to $\mathbb{R}$, then

$$S_{g \circ f}(y) = S_g(f(y)) \cdot |f'(y)|^2 + S_f(y).$$

Taking the first derivative of the map under study, we have,

$$f(y) = \mu y(1 - y^2)$$

$$f'(y) = \mu - \mu(3y^2) = \mu(1 - 3y^2). \tag{2.7}$$

The second and third derivatives of the map are also computed for future use,

$$f''(y) = -6\mu y,$$

$$f'''(y) = -6\mu.$$

Recall that

$$\text{Fix}(f) = \left\{0, +\sqrt{\frac{\mu - 1}{\mu}}, -\sqrt{\frac{\mu - 1}{\mu}}\right\}.$$

Considering the stability of the point 0, put $y = 0$ in the equation (2.7).

$$f'(0) = \mu(1 - 3(0)^2) = \mu$$

implies

$$f'(0) = \mu$$

Therefore, from Lemma 2.10, we conclude.

- $y^* = 0$ is asymptotically stable if $0 < \mu < 1$.
- $y^* = 0$ is unstable if $\mu > 1$.

The scenario in which $\mu = 1$ requires particular consideration, as it results in $f'(0) = 1$ and $f''(0) = 0$. Applying Lemma 2.13 allows us to conclude that $y^* = 0$ exhibits asymptotic stability. Continuing with the stability analysis of the alternative fixed point $y^* = \pm\sqrt{\frac{\mu-1}{\mu}}$.

By using Equation (2.7),

$$f'(y) = \mu(1 - 3y^2).$$

$$f'\left(\pm\sqrt{\frac{\mu - 1}{\mu}}\right) = \mu\left(1 - 3\left(\pm\sqrt{\frac{\mu - 1}{\mu}}\right)^2\right),$$

$$f'\left(\pm\sqrt{\frac{\mu - 1}{\mu}}\right) = \mu\left(1 - 3\left(\frac{\mu - 1}{\mu}\right)\right),$$

$$f'\left(\pm\sqrt{\frac{\mu - 1}{\mu}}\right) = \mu\left(1 - \left(\frac{3\mu - 3}{\mu}\right)\right),$$

$$f'\left(\pm\sqrt{\frac{\mu - 1}{\mu}}\right) = \mu\left(\frac{\mu - 3\mu + 3}{\mu}\right),$$

$$f'\left(\pm\sqrt{\frac{\mu - 1}{\mu}}\right) = \mu - 3\mu + 3,$$

$$f'\left(\pm\sqrt{\frac{\mu-1}{\mu}}\right) = 3 - 2\mu. \tag{2.8}$$

We assume $\mu \geq 1$ to ensure that the expression $\sqrt{\frac{\mu-1}{\mu}}$ is real-valued. This condition guarantees that the fixed point lies in the real domain.

Three values of μ were considered, consistent with the data collected for this investigation, as shown in Table 1. For $\mu = 1.5$,

$$|f'(1/\sqrt{3})| = 3 - 2(3/2) = 0,$$

Since this value is less than 1, the application of Lemma 2.8 indicates that the point is an attracting fixed point. Let $\mu = 2$,

$$|f'(1/\sqrt{2})| = |2 - 3| = |-1| = 1,$$

This point is non-hyperbolic and therefore does not satisfy the conditions of Lemma 2.8. We utilize the Schwarzian derivative in the following manner,

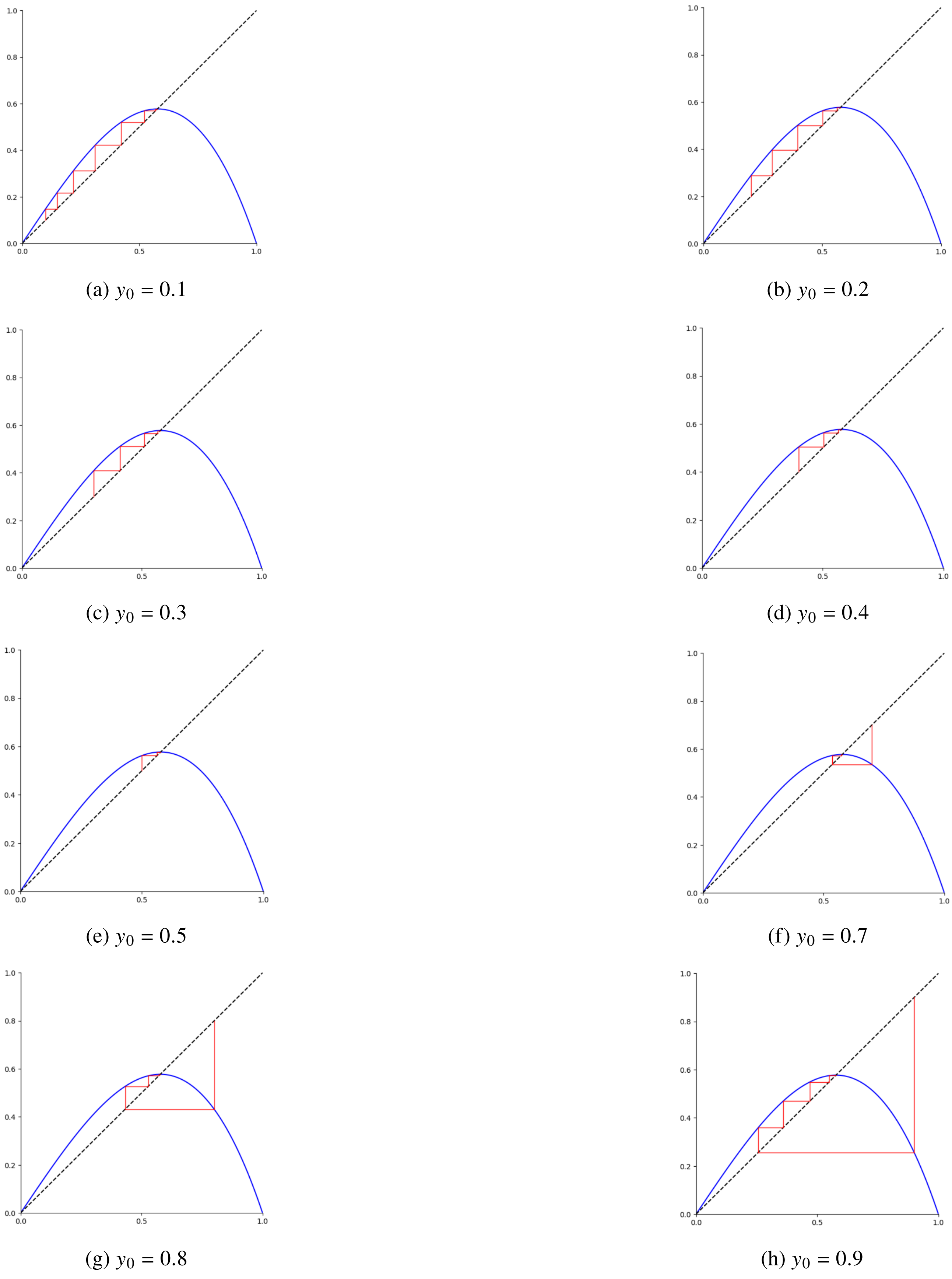
$$\begin{aligned} S_f(1/\sqrt{2}) &= -f'''(2/3) - \frac{3}{2}[f''(1/\sqrt{2})]^2, \\ &= 12 - \frac{3}{2}[36(2)], \\ &= -96. \end{aligned}$$

By the application of Lemma 2.12, the fixed point $1/\sqrt{2}$ is asymptotically stable. For $\mu = 2.5$,

$$\left|f'\left(\sqrt{\frac{3}{5}}\right)\right| = |3 - 2(5/2)| = |3 - 5| = |-2| = 2 > 1.$$

According to Lemma 2.10, the fixed point exhibits repelling behavior. We focus on attracting fixed points because they represent stable long-term behavior of the system, where nearby values converge under iteration. Repelling points, in contrast, do not influence the eventual outcome for most initial conditions. Therefore, we exclude the outcome at $\mu = 2.5$.

Figures 1 through 3 illustrate the iterations conducted for different values of μ, taking into account various initial conditions. Under these varied conditions, the figures illustrate distinct behaviors: for $\mu = 1.5$, the orbit of the initial point converges. In contrast, for $\mu = 2$, the orbit of the initial condition oscillates around the equilibrium point before stabilizing in a manner that suggests the emergence of a 2-periodic cycle. Also, for $\mu = 2.5$, the initial conditions keep bouncing in multiple periodic cycles and does not settle, this is because $\mu = 2.5$ is very near to the point of chaos which is $\mu > 2.59$.



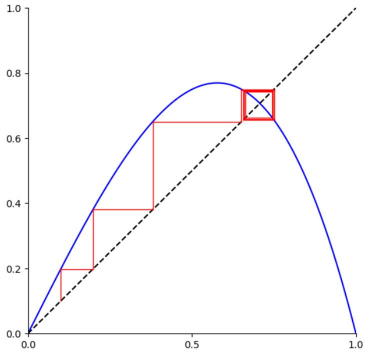
(a) $y_0 = 0.1$ (b) $y_0 = 0.2$

(c) $y_0 = 0.3$ (d) $y_0 = 0.4$

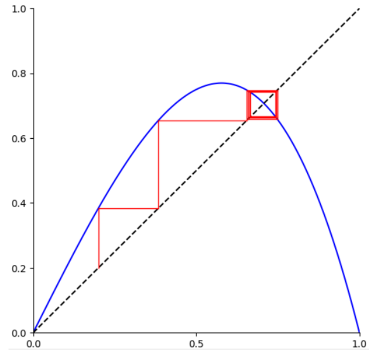
(e) $y_0 = 0.5$ (f) $y_0 = 0.7$

(g) $y_0 = 0.8$ (h) $y_0 = 0.9$

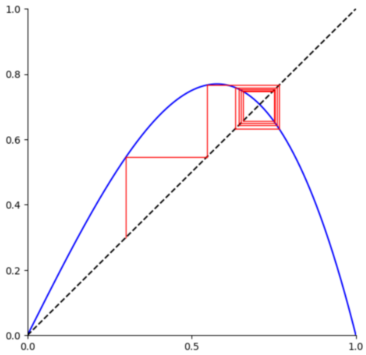
Figure 1: The Cobweb diagram for the Richard's logistic map, when $\mu = 1.5$ and varying values of y_0.



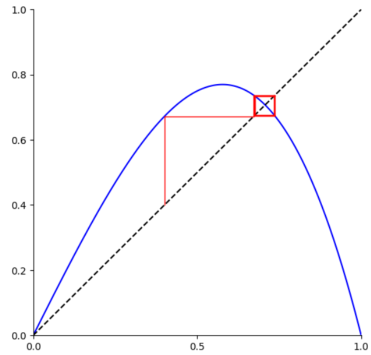
(a) $y_0 = 0.1$



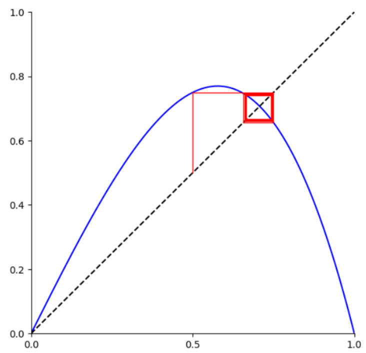
(b) $y_0 = 0.2$



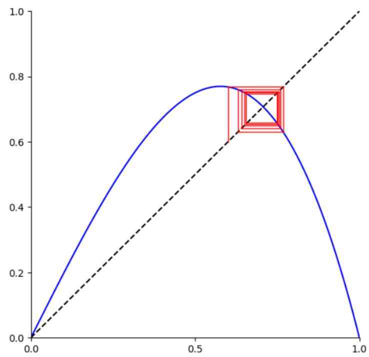
(c) $y_0 = 0.3$



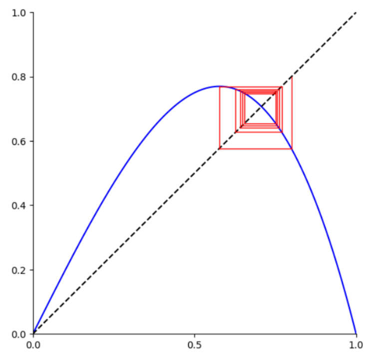
(d) $y_0 = 0.4$



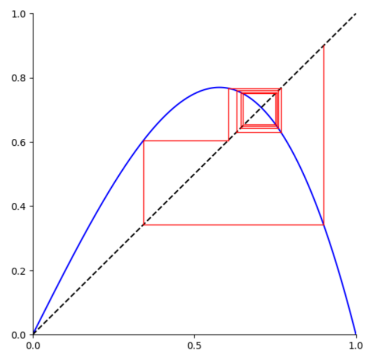
(e) $y_0 = 0.5$



(f) $y_0 = 0.6$

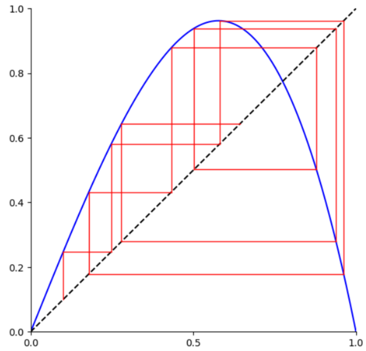


(g) $y_0 = 0.8$

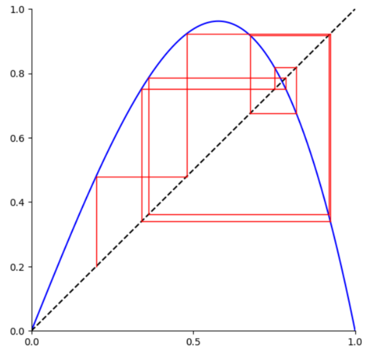


(h) $y_0 = 0.9$

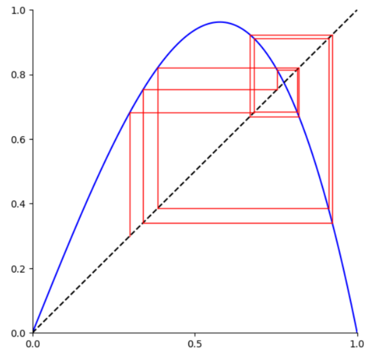
Figure 2: The Cobweb diagram for the Richard's logistic map, when $\mu = 2.0$ and varying values of y_0.



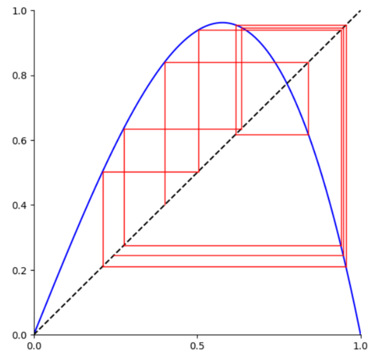
(a) $y_0 = 0.1$



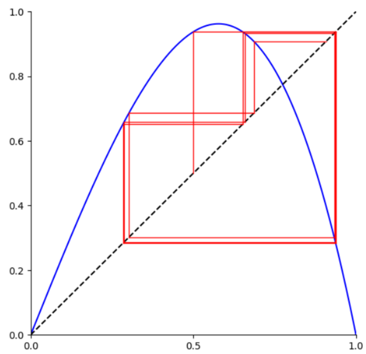
(b) $y_0 = 0.2$



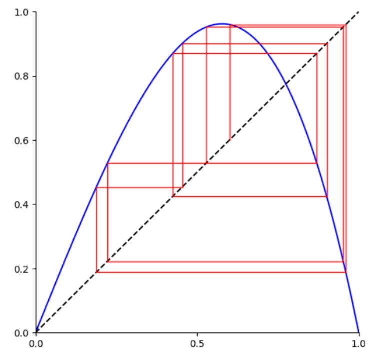
(c) $y_0 = 0.3$



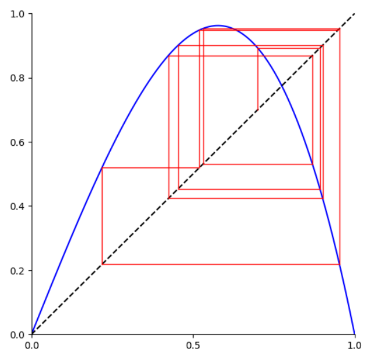
(d) $y_0 = 0.4$



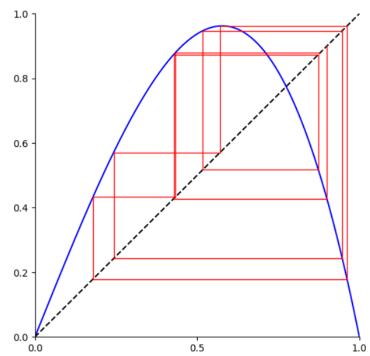
(e) $y_0 = 0.5$



(f) $y_0 = 0.6$



(g) $y_0 = 0.7$



(h) $y_0 = 0.9$

Figure 3: The Cobweb diagram for the Richard's logistic map, when $\mu = 2.5$ and varying values of y_0.

2.5 2-Periodic Cycles

This subsection addresses the period-2 points and their stability. Considering

$$f(y) = \mu y(1 - y^2).$$

We can calculate the period-2 points in the following manner

$$f(f(y)) = y$$

but are not fixed points (which satisfy $f(y) = y$). Thus, period-2 points must satisfy $f(f(y)) = y$ but $f(y) \neq y$.

Apply f again

$$f(f(y)) = \mu f(y)\left(1 - (f(y))^2\right).$$

Substitute $f(y) = \mu y(1 - y^2)$

$$f(f(y)) = \mu\left[\mu y(1 - y^2)\right]\left(1 - \left[\mu y(1 - y^2)\right]^2\right).$$

Simplifying

$$f(f(y)) = \mu^2 y(1 - y^2)\left[1 - \mu^2 y^2 (1 - y^2)^2\right].$$

Thus

$$\mu^2 y(1 - y^2)\left[1 - \mu^2 y^2 (1 - y^2)^2\right] = y,$$

$$(\mu^2 y - \mu^2 y^3)\left[1 - \mu^2 y^2 (1 + y^4 - 2y^2)\right] - y = 0,$$

$$(\mu^2 y - \mu^2 y^3)\left[1 - \mu^2 y^2 - \mu^2 y^6 + 2\mu^2 y^4\right] - y = 0,$$

Simplifying

$$y(\mu y^2 - \mu - 1)(\mu y^2 - \mu + 1)\left[\mu^2 y^4 - \mu^2 y^2 + 1\right] = 0. \tag{2.9}$$

The part $y(\mu y^2 - \mu - 1)(\mu y^2 - \mu + 1)$ of the equation (2.9) cannot be equal to zero because we have already found that these are the equilibrium or fixed points, so our interest lies in the expression below

$$\mu^2 y^4 - \mu^2 y^2 + 1 = 0.$$

Finally, we get the quadratic equation in $z = y^2$

$$\mu^2 z^2 - \mu^2 z + 1 = 0.$$

Solving the quadratic equation, we get

$$z = \frac{\mu \pm \sqrt{(\mu - 2)(\mu + 2)}}{2\mu}$$

Recall $z = y^2$, so

$$y^2 = \frac{\mu \pm \sqrt{(\mu - 2)(\mu + 2)}}{2\mu} \tag{2.10}$$

$$y = \pm\sqrt{\frac{\mu \pm \sqrt{(\mu - 2)(\mu + 2)}}{2\mu}}$$

Remark 2.13. *Given the assumption* $\mu > 0$*, it is evident that these roots are real and distinct just when* $\mu \geq 2$*.*

2.6 Stability of Period-2 points of the Richard's Logistic Map

The definitions and lemmas presented below were utilized in the stability analysis of the period-2 points examined in this study.

Definition 2.14. Given a map $f : \mathbb{R} \to \mathbb{R}$, let $\bar{y}$ be a periodic point of f with minimal period n. Then,

- $\bar{y}$ is *stable* if it is a stable fixed point of f^n,
- $\bar{y}$ is *asymptotically stable* if it is an asymptotically stable fixed point of f^n,
- $\bar{y}$ is *unstable* if it is an unstable fixed point of f^n.

In view of the above Remark 2.13, Lemma 2.15 holds.

Lemma 2.15. *The logistic map has a single period-2 orbit that occurs only for* $\mu \geq 2$*.*

Lemma 2.16. Let $O(\bar{y}) = \{\bar{y}, f(\bar{y}), \ldots, f^{n-1}(\bar{y})\}$ be the orbit of the period-n point $\bar{y}$, where f is a continuously differentiable function at $\bar{y}$. Then the following statements hold true:

- $\bar{y}$ is asymptotically stable if

$$\left|f'(\bar{y}_1) f'(f(\bar{y}_2)) \cdots f'(f^{n-1}(\bar{y}_n))\right| < 1$$

- $\bar{y}$ is unstable if

$$\left|f'(\bar{y}_1) f'(f(\bar{y}_2)) \cdots f'(f^{n-1}(\bar{y}_n))\right| > 1$$

Proceeding from Equation (2.10) the assertion of Lemma 2.15 can be seen as follows

$$\bar{y}^2 = \frac{\mu \pm \sqrt{(\mu - 2)(\mu + 2)}}{2\mu}$$

we denote the roots obtained above by:

$$\bar{y_1}^2 = \frac{\mu + \sqrt{(\mu-2)(\mu+2)}}{2\mu} \quad , \quad \bar{y_2}^2 = \frac{\mu - \sqrt{(\mu-2)(\mu+2)}}{2\mu} \quad .$$

For $\mu = 2$,

$$f(y) = \mu y(1-y^2).$$

$$f(y_1) = (2)(1/\sqrt{2})(1-1/2) = 1/\sqrt{2}.$$

$$f(y_2) = (2)(1/\sqrt{2})(1-1/2) = 1/\sqrt{2}.$$

Thus, the period-2 points y_1 and y_2 are the same for $\mu = 2$.

Hence, the period-2 orbit exists for $\mu > 2$. To check for the stability, we evaluate

$$(f^2)'(\bar{y}) = f'(\bar{y_1}) \cdot f'(\bar{y_2})$$

In view of Lemma (2.16) we need to examine the values of $f'(\bar{y_1})$ and $f'(\bar{y_2})$

Recall

$$f(y) = \mu y(1-y^2) \quad \Rightarrow \quad f'(y) = \mu(1-3y^2)$$

Thus

$$(f^2)'(\bar{y}) = \mu(1-3\bar{y_1}^2) \cdot \mu(1-3\bar{y_2}^2) = \mu^2(1-3\bar{y_1}^2)(1-3\bar{y_2}^2)$$

Using the earlier results

$$\bar{y_1}^2 = \frac{\mu - \sqrt{(\mu^2-4)}}{2\mu}, \quad \bar{y_2}^2 = \frac{\mu + \sqrt{(\mu^2-4)}}{2\mu}$$

So

$$(f^2)'(\bar{y}) = \mu^2 \left(1 - 3 \cdot \frac{\mu - \sqrt{\mu^2-4}}{2\mu}\right)\left(1 - 3 \cdot \frac{\mu + \sqrt{\mu^2-4}}{2\mu}\right). \tag{2.11}$$

Simplifying

$$(f^2)'(\bar{y}) = \mu^2 \left(\frac{2\mu - 3\mu + 3\sqrt{\mu^2-4}}{2\mu}\right)\left(\frac{2\mu - 3\mu - 3\sqrt{\mu^2-4}}{2\mu}\right)$$

$$(f^2)'(\bar{y}) = \mu^2 \left(\frac{-\mu + 3\sqrt{\mu^2-4}}{2\mu}\right)\left(\frac{-\mu - 3\sqrt{\mu^2-4}}{2\mu}\right)$$

$$(f^2)'(\bar{y}) = \mu^2 \cdot \frac{(-\mu)^2 - (3\sqrt{\mu^2-4})^2}{(2\mu)^2} = \mu^2 \cdot \frac{\mu^2 - 9(\mu^2-4)}{4\mu^2}$$

$$(f^2)'(\bar{y}) = \mu^2 \cdot \frac{\mu^2 - 9\mu^2 + 36}{4\mu^2} = \frac{-8\mu^2 + 36}{4\mu^2} = \frac{-8\mu^2 + 36}{4}$$

$$(f^2)'(\bar{y}) = -2\mu^2 + 9 \tag{2.12}$$

To find the range of μ for which the period-2 points are stable, set:

$$|(f^2)'(\bar{y})| < 1 \Rightarrow |-2\mu^2 + 9| < 1.$$

$$-1 < -2\mu^2 + 9 < 1 \Rightarrow 8 < 2\mu^2 < 10 \Rightarrow 4 < \mu^2 < 5 \Rightarrow \mu \in (\sqrt{4}, \sqrt{5}) \cup (-\sqrt{5}, -\sqrt{4})$$

Remark. The value $\mu < 0$ is disregarded, as this study focuses solely on positive orbits.

Therefore, the range being examined is:

$$2 < \mu < \sqrt{5}.$$

Select an arbitrary point μ within the interval, for instance, $\mu = 2.1$. According to Equation (2.12),

$$(f^2)'(\bar{y}) = -2(2.1)^2 + 9 = -2(4.41) + 9 = 9 - 8.82 = 0.18 < 1.$$

According to the implications of Lemma 2.16, $\bar{y}$ is identified as an asymptotically stable period-2 point. Therefore, we can say that the periodic orbit is attracting for the interval $2 < \mu < \sqrt{5}$.

When $\mu > \sqrt{5}$, the period-2 orbit loses stability and bifurcations may occur leading to higher-period or chaotic behavior.

To examine the basin of attraction of the period-2 orbit, two values of μ, specifically 2.1 and 2.22, were analysed. The map was simulated to produce the iterates corresponding to these values. Table 2 presents the generated data.

Table 2: Iteration Data for Richard's Logistic Map Period-2 Points

	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9
$\mu = 2.1$	0.1	0.4177	0.7233	0.7231	0.7227	0.7223	0.7217	0.7208	0.7194	0.7175
$\mu = 2.22$	0.1	0.4643	0.6214	0.5319	0.5321	0.5319	0.5320	0.5319	0.5320	0.5319
$\mu = 2.1$	0.2	0.7091	0.7025	0.6931	0.6797	0.6617	0.6394	0.6166	0.5998	0.5922
$\mu = 2.22$	0.2	0.7743	0.8041	0.8431	0.8496	0.8442	0.8488	0.8449	0.8483	0.8454
$\mu = 2.1$	0.3	0.8082	0.8078	0.8078	0.8078	0.8078	0.8077	0.8077	0.8077	0.8077
$\mu = 2.22$	0.3	0.8513	0.8424	0.8500	0.8437	0.8492	0.8446	0.8485	0.8452	0.8481
$\mu = 2.1$	0.4	0.7440	0.7521	0.7627	0.7754	0.7887	0.7994	0.8053	0.8072	0.8076
$\mu = 2.22$	0.4	0.7346	0.7272	0.7116	0.6786	0.6123	0.5245	0.5388	0.5265	0.5369
$\mu = 2.1$	0.5	0.6282	0.6075	0.5951	0.5908	0.5898	0.5896	0.5896	0.5895	0.5895
$\mu = 2.22$	0.5	0.5673	0.5129	0.5512	0.5187	0.5447	0.5224	0.5409	0.5250	0.5383
$\mu = 2.1$	0.6	0.5922	0.5901	0.5896	0.5895	0.5895	0.5895	0.5895	0.5895	0.5895
$\mu = 2.22$	0.6	0.5172	0.5464	0.5214	0.5419	0.5243	0.5390	0.5263	0.5371	0.5278
$\mu = 2.1$	0.7	0.6895	0.6748	0.6553	0.6323	0.6106	0.5966	0.5912	0.5898	0.5896
$\mu = 2.22$	0.7	0.6543	0.5716	0.5123	0.5520	0.5183	0.5452	0.5222	0.5411	0.5248
$\mu = 2.1$	0.8	0.8055	0.8073	0.8077	0.8077	0.8077	0.8077	0.8077	0.8077	0.8077
$\mu = 2.22$	0.8	0.8392	0.8519	0.8417	0.8504	0.8433	0.8495	0.8443	0.8487	0.8450
$\mu = 2.1$	0.9	0.6569	0.6340	0.6120	0.5973	0.5914	0.5899	0.5896	0.5895	0.5895
$\mu = 2.22$	0.9	0.7213	0.6990	0.6523	0.5687	0.5127	0.5515	0.5186	0.5449	0.5223

These iterations simulate the behavior of stability of Period-2 points of Richard's logistic map, helping to observe how different values of μ and initial conditions y_0 influence the system's evolution over time.

For $\mu = 2.1$, the equation (2.10) becomes

$$y^2 = \frac{[2.1 \pm \sqrt{(2.1-2)(2.1+2)]}}{2(2.1)},$$
$$y^2 = \frac{[2.1 \pm 0.64]}{4.2}$$

$$y_3^2 = 0.652, \quad y_4^2 = 0.347.$$

$$y_3 \approx 0.8074, \quad y_4 \approx 0.5890.$$

For $\mu = 2.22$,

$$y^2 = \frac{[2.22 \pm \sqrt{(2.22-2)(2.22+2)}]}{2(2.22)},$$
$$y^2 = \frac{[2.22 \pm 0.9635]}{4.44}$$

$$y_5^2 = 0.7170, \quad y_6^2 = 0.2829.$$

$$y_5 \approx 0.8467, \quad y_6 \approx 0.5318.$$

The basin of attraction for the orbit of the points under consideration therefore satisfies

for $\mu = 2.1$

$$\{0.3, 0.4, 0.8\} \subset \mathcal{B}(0.8074, f),$$

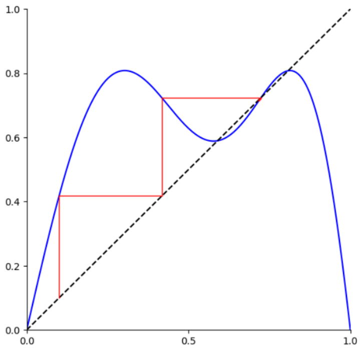
$$\{0.5, 0.6, 0.7, 0.9\} \subset \mathcal{B}(0.5890, f).$$

for $\mu = 2.22$

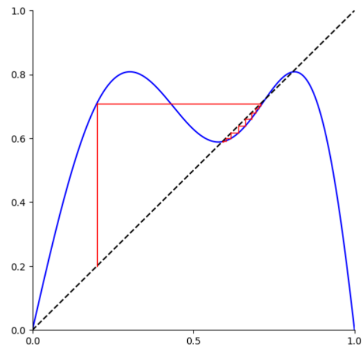
$$\{0.2, 0.3, 0.8\} \subset \mathcal{B}(0.8467, f)$$

$$\{0.1, 0.4, 0.5\} \subset \mathcal{B}(0.5318, f)$$

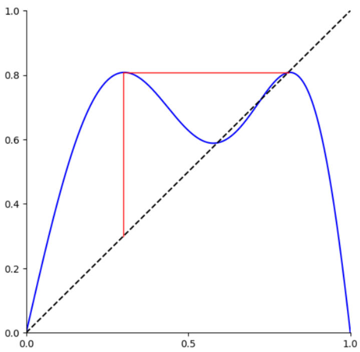
Based on the data, it is evident that only points attracted to the fixed point $\left\{0, +\sqrt{\frac{\mu-1}{\mu}}, -\sqrt{\frac{\mu-1}{\mu}}\right\}$ are those in the interval (0,1) that converge to the fixed point y^* after a finite number of iterations (i.e., they are eventually fixed points).



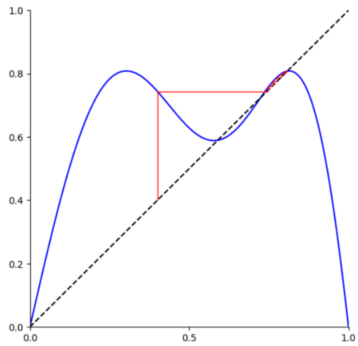
(a) $y_0 = 0.1$



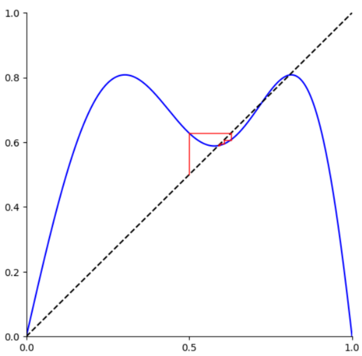
(b) $y_0 = 0.2$



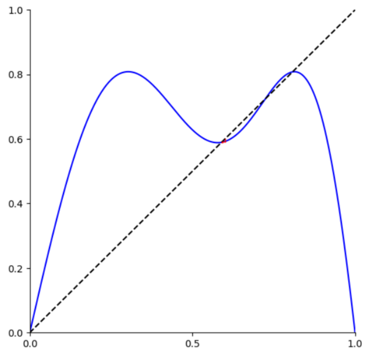
(c) $y_0 = 0.3$



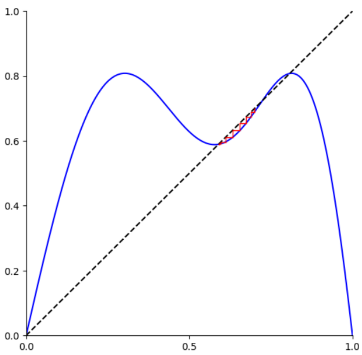
(d) $y_0 = 0.4$



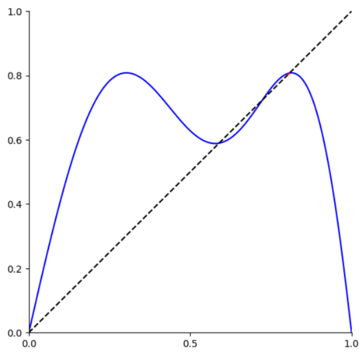
(e) $y_0 = 0.5$



(f) $y_0 = 0.6$

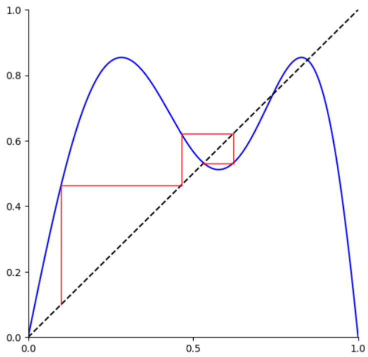


(g) $y_0 = 0.7$

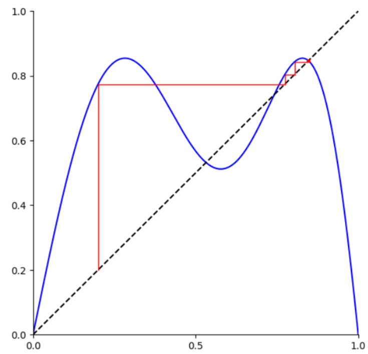


(h) $y_0 = 0.8$

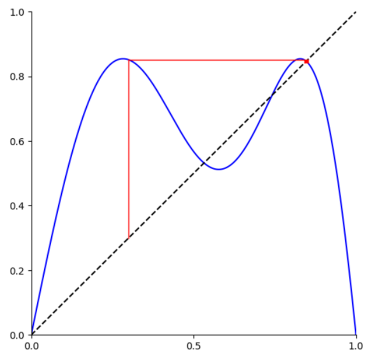
Figure 4: The cobweb diagram for Richard's map for period-2 points, when $\mu = 2.1$ and varying values of y_0.



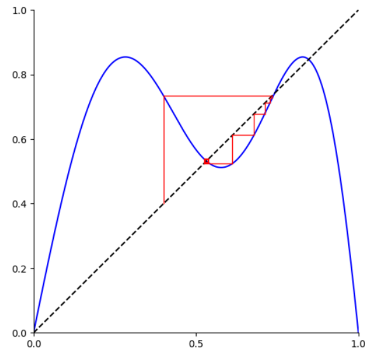
(a) $y_0 = 0.1$



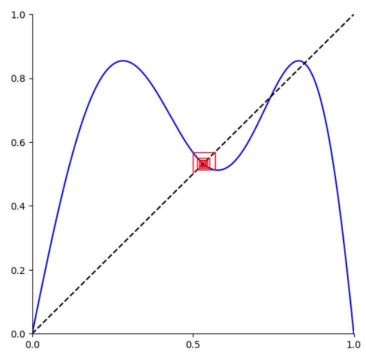
(b) $y_0 = 0.2$



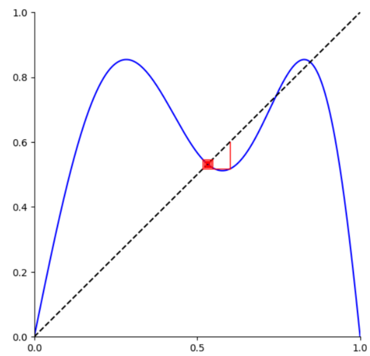
(c) $y_0 = 0.3$



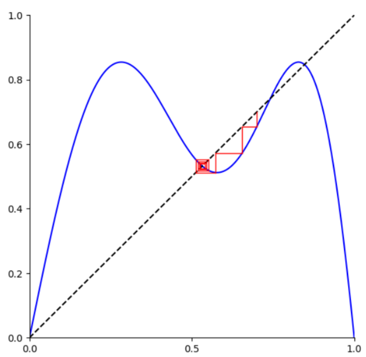
(d) $y_0 = 0.4$



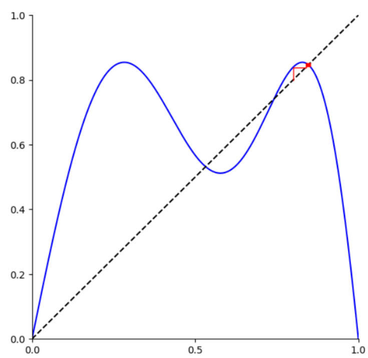
(e) $y_0 = 0.5$



(f) $y_0 = 0.6$



(g) $y_0 = 0.7$



(h) $y_0 = 0.8$

Figure 5: The cobweb diagram for Richard's map for period-2 points, when $\mu = 2.22$ and varying values of y_0.

The iterations for different values of μ, taking into account different initial conditions, are shown in Figures 4 and 5. The illustrations show the variable behavior that is presented for these varied conditions. For example, in the case where $\mu = 2.1$, the orbit of the beginning point under consideration converges to two distinct fixed/equilibrium positions. Likewise, we find that the orbit of the initial condition converges to two distinct fixed/equilibrium points for $\mu = 2.22$.

3 The Period-Doubling of Richard's Logistic Map

This section examines the *period-doubling* phenomenon in the logistic map. In *dynamical systems theory*, a period-doubling bifurcation occurs when a minor alteration in a system's parameters (in this instance, it is μ) leads to the emergence of a new periodic trajectory from an existing periodic trajectory.

Remark. The new periodic trajectory has a period that is double that of the original. Furthermore, with the doubled period, the duration required for the numerical values within the system to repeat is doubled, or in the context of a discrete dynamical system, it necessitates twice as many iterations.

Recall the Richard's logistic map

$$f(y) = \mu y[1 - y^2].$$

Also recall the equation that represents the second iterate used to identify period-2 points by solving the equation $f^2(y) = y$, under the condition $f(y) \neq y$.

$$f^2(y) = \mu^2 y(1 - y^2)\left[1 - \mu^2 y^2 (1 - y^2)^2\right].$$

3.1 2^2-Periodic Cycles

To find the 4-periodic cycles, we solve the equation $f^4(y) = y$, by proceeding as follows,

$$f^4 = f^2 \circ f^2.$$

We obtain,

$$f^4(y) = \mu^2 y(1 - y^2)\left[1 - \mu^2 y^2 (1 - y^2)^2\right] \cdot \mu^2 y(1 - y^2)\left[1 - \mu^2 y^2 (1 - y^2)^2\right]$$

$$f^4(y) = \left[\mu^2 y(1 - y^2)\left(1 - \mu^2 y^2 (1 - y^2)^2\right)\right]^2.$$

First compute

$$(1 - y^2)^2 = 1 - 2y^2 + y^4.$$

Then

$$1 - \mu^2 y^2 (1 - y^2)^2 = 1 - \mu^2 y^2 (1 - 2y^2 + y^4),$$

$$1 - \mu^2 y^2 (1 - y^2)^2 = 1 - \mu^2 y^2 + 2\mu^2 y^4 - \mu^2 y^6.$$

Thus

$$f^4(y) = \left[\mu^2 y(1 - y^2)\left(1 - \mu^2 y^2 + 2\mu^2 y^4 - \mu^2 y^6\right)\right]^2.$$

Now expand

$$(1 - y^2)(1 - \mu^2 y^2 + 2\mu^2 y^4 - \mu^2 y^6)$$

$$= (1)(1 - \mu^2 y^2 + 2\mu^2 y^4 - \mu^2 y^6) - y^2(1 - \mu^2 y^2 + 2\mu^2 y^4 - \mu^2 y^6)$$

$$= (1 - \mu^2 y^2 + 2\mu^2 y^4 - \mu^2 y^6) - (y^2 - \mu^2 y^4 + 2\mu^2 y^6 - \mu^2 y^8)$$

$$= 1 - y^2 - \mu^2 y^2 + 3\mu^2 y^4 - 3\mu^2 y^6 + \mu^2 y^8.$$

Now multiply this with $\mu^2 y$

$$\mu^2 y(1 - y^2 - \mu^2 y^2 + 3\mu^2 y^4 - 3\mu^2 y^6 + \mu^2 y^8),$$

$$= \mu^2 y - \mu^2 y^3 - \mu^4 y^3 + 3\mu^4 y^5 - 3\mu^4 y^7 + \mu^4 y^9,$$

$$f^4(y) = \left(\mu^2 y - \mu^2 y^3 - \mu^4 y^3 + 3\mu^4 y^5 - 3\mu^4 y^7 + \mu^4 y^9\right)^2.$$

Further expansion of $f^4(y)$ leads to increasingly complex expressions, making analytical treatment impractical. Hence, we rely on numerical methods and graphical tools such as the *bifurcation diagram* for further analysis.

3.2 Bifurcation

Definition 3.1. In dynamical systems, a *bifurcation* occurs when a small smooth change made to the *parameter values (the bifurcation values)* of a map causes a sudden qualitative change in its behaviour.

Remark. At a bifurcation, the local stability characteristics of fixed points and periodic orbits undergo changes.

The previous sections demonstrated that the fixed point of the logistic map becomes unstable at $\mu = 2$, leading to the emergence of an asymptotically 2-periodic orbit for the range $2 < \mu < \sqrt{5}$. The number and stability of periodic orbits change at $\mu = 2$, indicating that the logistic map undergoes a bifurcation at this point, which is referred to as a bifurcation value.

Remark. The bifurcation of the logistic map at $\mu = 2$ is identified as a period doubling bifurcation, as the orbit's period transitions from one to two, then from two to four, four to eight, and continues in this manner.

3.3 Bifurcation Scenarios

Figure 6 presents the bifurcation diagram of the Richard's logistic map. Upon examining sub-figure (a), it can be inferred that the band remains unitary until $\mu = 2$, at which point it bifurcates into two segments. At this juncture, a 2-periodic orbit is observed, as illustrated in sub-figure (b). Subsequently, it bifurcates into four distinct parts, as illustrated in sub-figure (c) at the point $\mu \approx 2$. Subsequently, it bifurcates into eight parts, as illustrated in sub-figure (d) at the point $\mu \approx 2.30$. The division persists without end. Additionally, *Feigenbaum's number* can be derived from the sequence of values at which continuous splitting occurs.

3.4 Feigenbaum's Number

The Feigenbaum's number, also known as the *Feigenbaum constant*, is a universal constant that characterizes the geometric rate at which period-doubling bifurcations occur in one-dimensional discrete dynamical systems as a parameter is varied.

Let μ_n be the parameter value at which a stable period-2^n orbit appears. Then the Feigenbaum number δ is defined as the limiting ratio of the difference between successive bifurcation parameter values:

$$\delta = \lim_{n\to\infty} \frac{\mu_{n-1} - \mu_{n-2}}{\mu_n - \mu_{n-1}}.$$

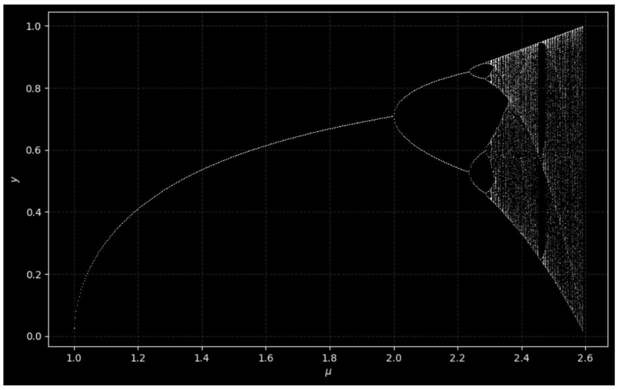
Figure 6 indicates specific values of μ at which successive band splitting occurs. The values of μ are presented in Table 3, along with the corresponding computed values. The expression $[\mu_{n-1} - \mu_{n-2}]$ represents the quotient of the differences between the two consecutive terms. The value of $\delta \approx 4.6692$ is approached by $[\mu_{n-1} - \mu_{n-2}]/[\mu_n - \mu_{n-1}]$ as n approaches infinity. It remains uncertain whether Feigenbaum's Number is irrational.

Table 3: Feigenbaum's Number

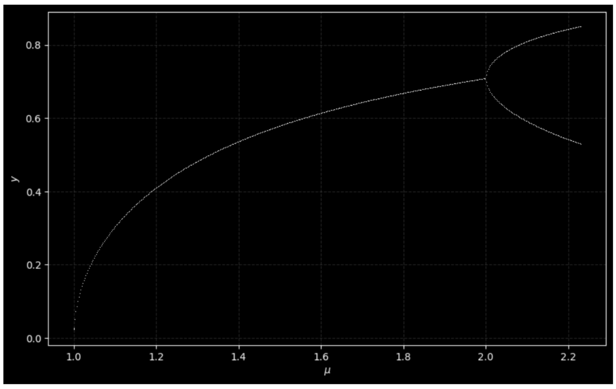
n	μ_n	$\mu_n - \mu_{n-1}$	$\mu_{n-1} - \mu_{n-2}$	$\frac{\mu_{n-1} - \mu_{n-2}}{\mu_n - \mu_{n-1}}$
1	2			
2	2.24	0.24		
3	2.29	0.05	0.24	4.8
4	2.30	0.01	0.05	5
5	2.302	0.002	0.01	5

Based on these computations, we can see that the ratio is stabilizing between 4.8 and 5, which is near to the Richard's logistic map's theoretical Feigenbaum constant value of 4.669.

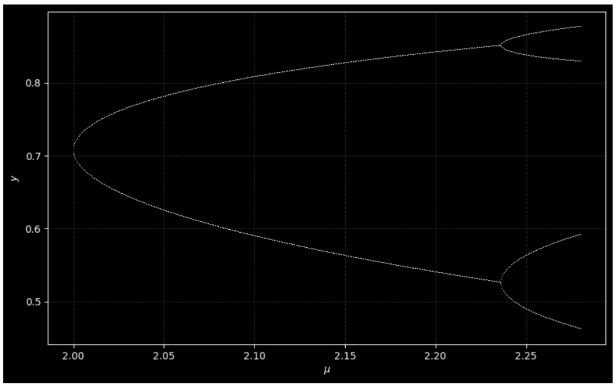
This indicates that the ratio of the differences between subsequent bifurcation points are convergent to the Feigenbaum constant. Thus, our system of Richard's logistic map is displaying period-doubling behavior.



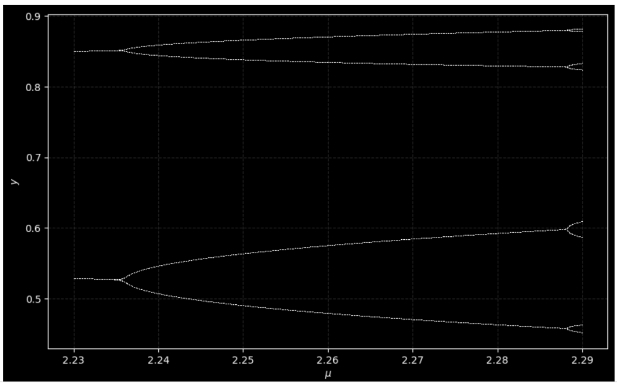
(a) The General Bifurcation Appearance



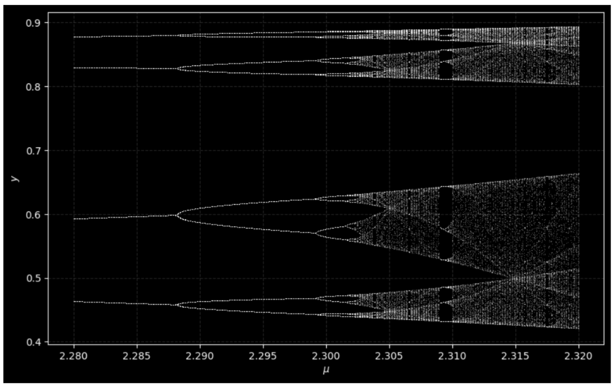
(b) The appearance of a 2-periodic cycle



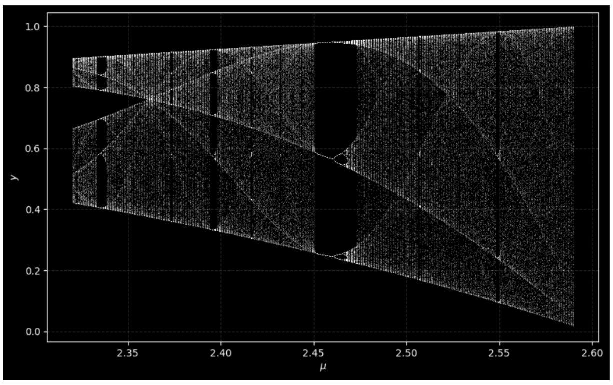
(c) The appearance of a 4-periodic cycle



(d) The appearance of a 8-periodic cycle



(e) The Bifurcation Diagram for $2.28 < \mu < 2.32$



(f) The Bifurcation Diagram for $2.32 < \mu < 2.59$

Figure 6: Bifurcation Diagram of the Richard's Logistic Map

4 Analysis of the Generalized Logistic Equation

4.1 Lyapunov Exponent

This subsection defines the Lyapunov exponent of an orbit, which can be computed numerically. A positive Lyapunov exponent indicates sensitive dependence in the system, whereas a negative Lyapunov exponent indicates convergence to an attracting periodic orbit. In light of this motivation, we present the definition.

Definition 4.1. Let f be a map from $\mathbb{R}$ to itself that has a derivative. The *Lyapunov multiplier* of an initial condition x_0 for the map f is defined to be

$$L(x_0; f) = \lim_{n\to\infty} |(f^n)'(x_0)|^{1/n},$$

when the limit exists. The *Lyapunov exponent* of an initial condition x_0 for the map f is defined to be the logarithm of this quantity.

$$\ell(x_0; f) = \lim_{n\to\infty} \frac{\ln(|(f^n)'(x_0)|)}{n},$$

when this limit exists.

By chain rule,

$$|(f^n)'(x_0)| = |f'(x_{n-1})| \cdot |f'(x_{n-2})| \cdots |f'(x_1)| \cdot |f'(x_0)|,$$
$$\ln(|(f^n)'(x_0)|) = \ln(|f'(x_{n-1})|) + \ln(|f'(x_{n-2})|) + \cdots + \ln(|f'(x_1)|) + \ln(|f'(x_0)|),$$

therefore,

$$\ell(x_0; f) = \lim_{n\to\infty} \frac{\ln(|f'(x_{n-1})|) + \ln(|f'(x_{n-2})|) + \cdots + \ln(|f'(x_1)|) + \ln(|f'(x_0)|)}{n},$$

and in a more compact manner, we have,

$$\ell(x_0; f) = \frac{1}{n} \lim_{n\to\infty} \sum_{j=0}^{n-1} \ln(|f'(x_j)|),$$

where $x_j = f^j(x_0)$.

Remark. *Notice that* $\ell(x_0; f) = \ln L(x_0; f)$

Consider the Richard's logistic map $f(y) = \mu y(1 - y^2)$, from which $f'(y) = \mu(1 - 3y^2)$. Recall that the fixed points of the logistic map are $\left\{0, +\sqrt{\frac{\mu-1}{\mu}}, -\sqrt{\frac{\mu-1}{\mu}}\right\}$, of which for $1 < \mu < 2$ all points in $[0, 1]$ are in the basin of attraction of the fixed point $y^* = \pm\sqrt{\frac{\mu-1}{\mu}}$. Also recall, that

$$f'(y^*) = 3 - 2\mu.$$

The Lyapunov exponent of y^* is

$$\ell(y^*; f) = \ln(f'(y^*)) = \ln|3 - 2\mu|.$$

Testing two random values of μ within the range $1 < \mu < 2$, specifically $\mu = 1.40$ and $\mu = 1.99$, the Lyapunov exponents are -1.6094 and -0.0202, respectively. This indicates that the orbit will be an attracting orbit. Therefore, within this range of μ, the logistic map exhibits no sensitive dependence on initial conditions.

Next is the period-2 orbit. Recall that the period-2 points are given by

$$\bar{y}^2 = \frac{\mu \pm \sqrt{(\mu-2)(\mu+2)}}{2\mu}.$$

From Equation (2.11),

$$(f^2)'(\bar{y}) = \mu^2 \left(1 - 3 \cdot \frac{\mu - \sqrt{\mu^2-4}}{2\mu}\right)\left(1 - 3 \cdot \frac{\mu + \sqrt{\mu^2-4}}{2\mu}\right)$$

Thus,

$$\begin{aligned}
\ell(\bar{y}; f) &= \frac{1}{2}\left[\ln \mu + \ln\left|\left(1 - 3 \cdot \frac{\mu - \sqrt{\mu^2-4}}{2\mu}\right)\right| + \ln \mu + \ln\left|\left(1 - 3 \cdot \frac{\mu + \sqrt{\mu^2-4}}{2\mu}\right)\right|\right] \\
&= \frac{1}{2}\left[\ln\left|\mu^2\left(\frac{-\mu + 3\sqrt{\mu^2-4}}{2\mu}\right)\left(\frac{-\mu - 3\sqrt{\mu^2-4}}{2\mu}\right)\right|\right] \\
&= \frac{1}{2}[\ln(|-2\mu^2+9)|].
\end{aligned}$$

It is important to note that, as demonstrated in equation (2.12), the stability of the period-2 orbit is confirmed for the range $2 < \mu < \sqrt{5}$. Testing $\ell(\bar{y}; f)$ for two random values within this range, specifically $\mu = 2.1$ and $\mu = 2.22$, yields Lyapunov exponents of -0.8573 and -0.0772, respectively. These results suggest that the logistic map exhibits no sensitive dependence on initial conditions within this range of μ. It is important to note that fixed points are included among the period-2 points. Consequently, we will examine the Lyapunov exponent $\ell(y^*; f)$ for $\mu = 2.1$ and $\mu = 2.22$. We obtained 0.1823 and 0.3646. This indicates the presence of sensitive dependence on initial conditions within this range.

Remark. For any point y_0 in $[0, 1]$ one of the following cases holds

- It lands exactly on y^* for some iterate, $f^n(y_0) = y^*$, and has $\ell(y^*, f) > 0$.
- It lands on the critical point 0.5 for some iterate, $f^n(y_0) = 0.5$.
- It is in the basin of attraction of the period-2 orbit and does not pass through 0.5.

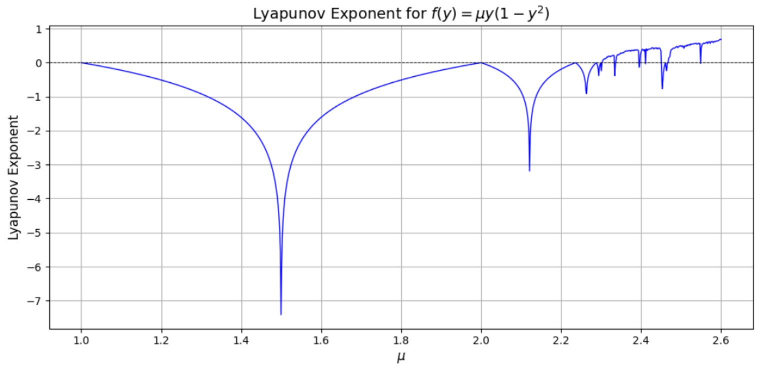


Figure 7: Plot of the Lyapunov exponent for the Richard's Logistic Map as a function of the paramter μ.

In this plot, we observe that for $\mu \in (1, 2)$, the Lyapunov exponent remains negative, indicating stable behavior. Around $\mu \approx 2.0$, the exponent becomes zero, indicating a bifurcation. At $\mu \approx 2.0$, the fixed point changes from stable to unstable and then a stable period-2 orbit is born, the Lyapunov exponent is 0. Similarly, at $\mu \approx 2.23$, the point changes from stable to unstable and then a stable period-4 orbit is born as it can also be observed from the Figure 6. For $\mu > 2$, the Lyapunov exponent becomes mostly positive, demonstrating the onset of chaos. Within this chaotic region, sharp dips (e.g., around $\mu \approx 2.2$) represent windows of periodic behavior embedded within chaos. These observations confirm the period-doubling route to chaos in the Richard's logistic map.

4.2 Path to chaos

In this subsection, we will go through our findings and conclude that until what value of μ, we can overcome the chaos and after that value the system behaves in chaotic behavior.

As the parameter μ increases, the system undergoes a sequence of bifurcations. Initially, the system converges to a stable fixed point. For $\mu = 1.5$, the Richard's logistic map converges to a point 0.5773 as it can be seen from Table $1b$ and Figure 1. For $\mu = 2$, the orbit of the initial values oscillates around the equilibrium point before stabilising in a pattern that alternates between two points, suggesting the emergence of a 2-periodic cycle. Also, for $\mu = 2.5$, the initial conditions keep bouncing in multiple periodic cycles and does not settle, this is because $\mu = 2.5$ is very near to the point of chaos which is $\mu > 2.59$.

The system converges as μ increases further, period-doubling bifurcations occur, leading to 2-cycle, 4-cycle, 8-cycle, and so on. This process continues rapidly until the onset of chaos, where the behavior becomes aperiodic and highly sensitive to initial conditions. This transition has been illustrated using the bifurcation diagram as it can be seen from the Figure 6.

Thus, we can conclude that for μ value greater than 2.59, the system starts behaving chaotic behavior, therefore, to study the dynamics of the system, the growth parameter value μ should be less than 2.59.

4.3 Graphical Interpretation of results.

In this subsection, we compare the findings of Richard's logistic map with the simple logistic map.

Recall the logistic map

$$f(y) = \mu y[1 - y].$$

Recall the Richard's logistic map

$$f(y) = \mu y[1 - y^2].$$

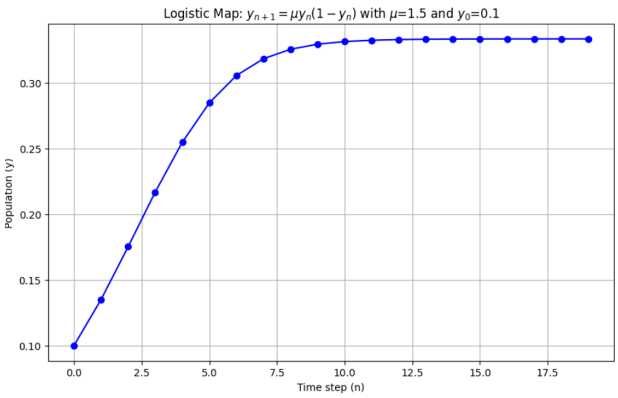


Figure 8a: Logistic Map

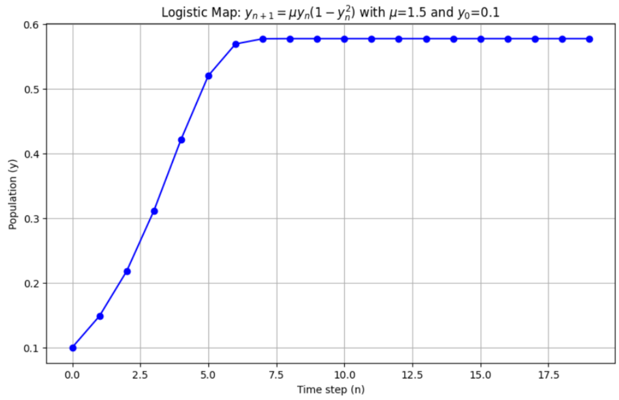


Figure 8b: Richard's Logistic Map

From Figure 8a and 8b, the difference between the convergence of both systems can be seen. We let the value of growth parameter $\mu = 1.5$ and initial iteration value $y_0 = 0.1$ for both systems. For this case, the *logistic map* converges to a point around 0.33, whereas, on the other hand, the *Richard's logistic model* converges to a point equal to 0.59

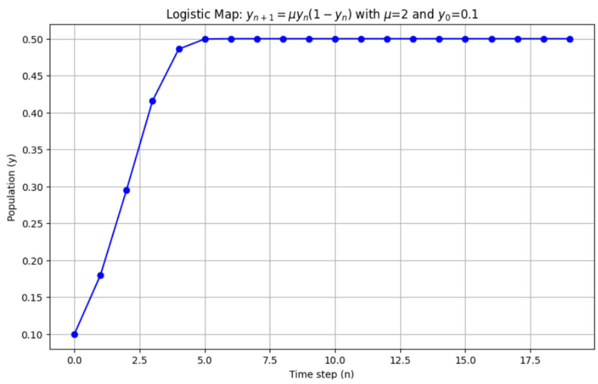


Figure 9a: Logistic Map

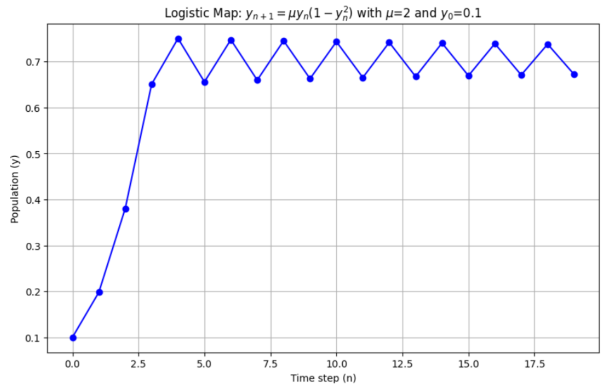


Figure 9b: Richard's Logistic Map

The case visualized in the Figure 9a and 9b is very interesting because if we set the growth parameter $\mu = 2$ and initial iteration value $y_0 = 0.1$, the logistic model converges to an equilibrium point at 0.5, while on the other hand, in the *Richard's logistic map*, the orbit of the initial condition oscillates around the equilibrium point before stabilizing in a pattern that alternates between two points, suggesting the emergence of a 2-periodic cycle. From this point, the system becomes sensitive to the initial values as we see in the following cases.

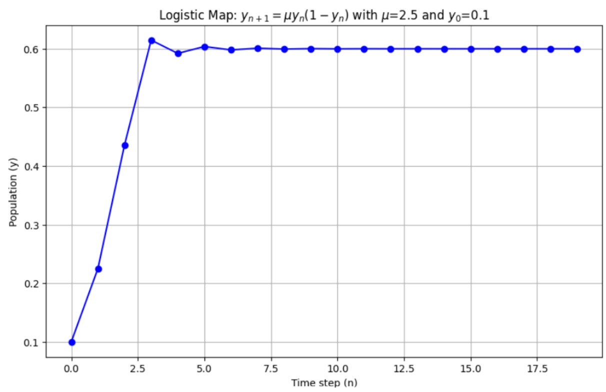


Figure 10a: Logistic Map

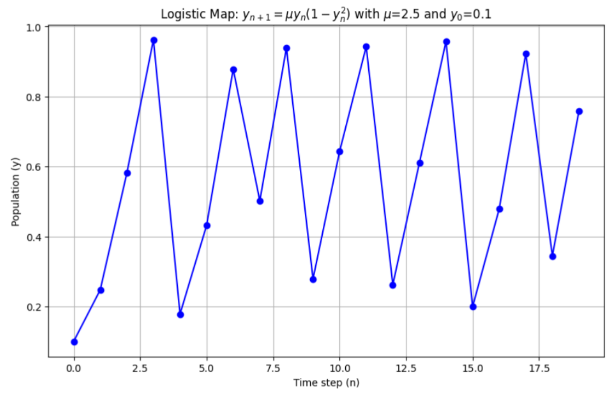


Figure 10b: Richard's Logistic Map

In Figure 10a and 10b, the value of growth parameter $\mu = 2.5$ and initial iteration value $y_0 = 0.1$, both systems give output which is completely different. As from Figure 9a, initially, it exhibits period-2 cycle but, ultimately, it converges to a point 0.6, whereas, in Figure 9b, the system start flipping around many points.

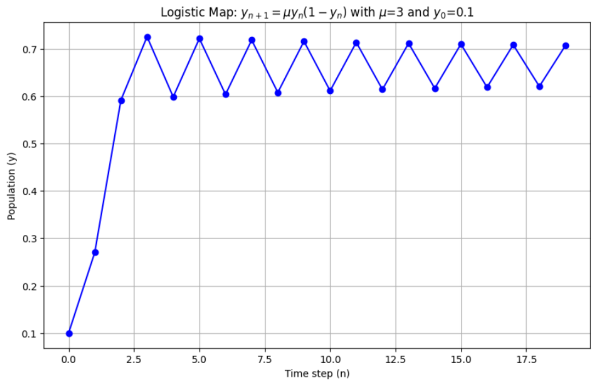


Figure 11a: Logistic Map

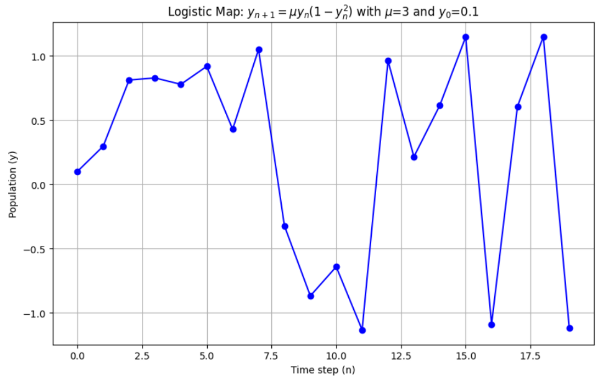


Figure 11b: Richard's Logistic Map

The sensitivity to initial values can be seen from the Figure $11a$ and $11b$. If we set the growth parameter $\mu = 3$ and initial iteration value $y_0 = 0.1$, in *logistic model*, the orbit of the initial condition oscillates around the equilibrium point before stabilising in a pattern that alternates between two points, suggesting the emergence of a 2-periodic cycle. While on the other hand, in the *Richard's logistic map*, the system exhibits *chaotic behavior* as for this system, the growth parameter μ must be less or equal to 2.59.

5 Conclusion

Conclusively, we analyzed a generalized form of the logistic map, commonly known as *Richard's Logistic Map*, and compared its dynamical behaviour with the classical *logistic map*. For the classical logistic map, the growth parameter μ lies in the interval $(1, 4)$, whereas in the Richard's logistic model with $p = 1$ and $q = 2$, our analysis shows that μ is restricted to the smaller interval $(1, 2.59)$. This implies that the onset of chaotic behavior occurs earlier, and the system reaches equilibrium more quickly in the Richard's case.

Moreover, preliminary simulations and phase plots for values $p > 1$ and $q > 2$ suggest that increasing the nonlinearity narrows the admissible range of μ even further. As a result, the system becomes more sensitive to parameter changes and settles into stability sooner. This highlights that the Richard's logistic map is more demanding in terms of parameter tuning, and its dynamics are more tightly regulated compared to the classical logistic map, making it a stronger candidate for modeling systems with sharper growth constraints or saturation effects.

References

[1] Elyadi, S. N., *Discrete Chaos: With Applications in Science and Engineering*, Chapman and Hall/CRC, 2007.

[2] Radwan, A. G., *On Some Generalized Logistic Maps*, Journal of Advanced Research, Engineering Mathematics Department, Faculty of Engineering, Cairo University, Egypt, 12613.

[3] Rak, R., Rak, E., *Route to Chaos in Generalized Logistic Map*, Faculty of Mathematics and Natural Sciences, University of Rzeszow, PL-35959 Rzeszow, Poland.

[4] Costa, D. R., Medrano-T, R. O., Leonel, E. D., *Route to Chaos and Some Properties in the Boundary Crisis of a Generalized Logistic Mapping*.

[5] Adeleke, J. Analysis of logistic maps [online]. Brno: Brno University of Technology. Faculty of Mechanical Engineering. 2022.