CONSTRUCTION OF FRAMES BY ORTHOGONAL PROJECTION

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This short contribution is aimed at proving the following:
"*Any (Parseval) frame in a separable Hilbert space H can be obtained by orthogonal projection of a suitable Riesz (orthonormal) basis after embedding H into a suitable hyperspace*".

1. Notation and preliminary remarks

- X, X_1, X_2, Y, Y_1, Y_2 ... real or complex inner-product spaces (IP-spaces) with inner product $\langle\cdot,\circ\rangle$, induced norm $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$ and convergence $x_n \to x$ meaning $\|x - x_n\| \to 0$.
- I_X ... identity operator on X.
- H, H_1, H_2 ... Hilbert spaces (H-spaces)=complete real or complex IP-spaces (every Cauchy sequence is convergent); in what follows we deal with separable H-spaces: in the space there exists at most countable dense subset, or equivalently there exists at most countable complete orthonormal system — orthonormal basis (ONB).
- $\ell^2(J)$ ... the H-space of absolutely square-summable sequences $\{x_n\}_{n\in J}$ (J is at most countable index set): $\sum\limits_{n\in J}|x_n|^2 < \infty$.
- $\mathcal{E} := \{\varepsilon_n\}_{n\in J}$ ... standard (canonical) orthonormal basis in $\ell^2(J)$: $\varepsilon_n := \{\delta_{n,k}\}_{k\in J}\ \forall\, n \in J$.
- $\mathcal{L}(M)$, $M \subseteq X$ ... linear span of M in X.
- $\overline{M}$, $M \subseteq X$ ... (topological) closure of M in X (limits of sequences in M convergent in X).
- $\overline{\mathcal{L}(M)}$, $M \subseteq X$ ... closure of linear span of M in X = minimal closed subspace in X containing M.
- $X_1\times X_2$ ... direct product of IP-spaces X_1 and X_2 which is clearly IP-space with inner product $\langle(x_1,x_2),(y_1,y_2)\rangle{:=}\langle x_1,y_1\rangle+\langle x_2,y_2\rangle$ and induced norm $\|(x_1,x_2)\| = \sqrt{\|x_1\|^2+\|x_2\|^2}$.
- $X_1 \oplus X_2$ ... orthogonal sum of X_1 and X_2.
- $\mathcal{L}(X,Y)$ ... the linear space of linear operators $X \to Y$.
- $X \overset{L}{\simeq} Y$ ... X and Y are linearly isomorphic (exists linear bijection X onto Y).
- $\mathcal{B}(X,Y) \subseteq \mathcal{L}(X,Y)$ ... the normed-linear space (NL-space) of bounded linear operators $X \to Y$ where boundedness (operator maps bounded subsets in X to bounded subsets in Y) is equivalent with continuity on X (operator preserves convergence by mapping convergent (or Cauchy) sequence in X to convergent (or Cauchy) sequence in Y, or equivalently preimages of open sets in Y are open sets in X).
- $X \overset{TLI}{\simeq} Y$ ... X and Y are topologically and linearly isomorphic (TLI): exists linear isomorphism X onto Y the inverse of which is bounded as well.

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- $X \overset{UI}{\simeq} Y$ … X and Y are unitary isomorphic (UI): exists linear bijection X onto Y preserving norm which also preserves scalar products (angles) by the *Polarization identity.* Clearly UI is a special case of TLI.
- $\mathcal{R}(T)$, $T \in \mathcal{L}(X,Y)$ … range space of T (subspace in Y).
- $\mathcal{N}(T)$, $T \in \mathcal{L}(X,Y)$ … null space of T (subspace in X).
- $T^* \in \mathcal{B}(H_2,H_1)$ … adjoint operator to $T \in \mathcal{B}(H_1,H_2)$: by the *Riesz Representation Theorem* it holds $\langle Tx,y\rangle = \langle x,T^*y\rangle\ \forall\, x \in H_1, y \in H_2$. T is self-adjoint if $H_1 = H_2$ and $T = T^*$.
- $T_1 \times T_2 \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$ … direct product of linear operators $T_1 \in \mathcal{L}(X_1,Y_1)$ and $T_2 \in \mathcal{L}(X_2,Y_2)$ acting component-wise: $(T_1 \times T_2)(x_1,x_2) := (T_1x_1, T_2x_2)$.
- P_H … operator of orthogonal projection onto H from its H-hyperspace; we denote $\widehat{x} := P_Hx$ and $x^\perp := x - \widehat{x} \perp H$.

2. STANDARD STATEMENTS (WITHOUT DETAILED PROOF)

Proposition 2.1. *It holds:*

1. $X_1 \times X_2$ *is complete (H-space) iff both* X_1 *and* X_2 *are complete (H-spaces).*
2. $T_1 \times T_2 \in \mathcal{L}(X_1 \times X_2, Y_1 \times Y_2)$ *is bounded, TLI or UI iff both* T_1 *and* T_2 *are bounded, TLI or UI, respectively.*
3. *The coordinate mappings* $P_1 : H_1 \times H_2 \to H_1$ *and* $P_2 : H_1 \times H_2 \to H_2$ $(P_1(x_1,x_2) := x_1$ *and* $P_2(x_1,x_2) := x_2)$ *are orthogonal projections.*
4. $X_1 \oplus X_2 \overset{UI}{\simeq} X_1 \times X_2$ *by the one-to-one relationship* $x_1 + x_2 \longleftrightarrow (x_1,x_2)$.

Proposition 2.2. *Let* Y *be a subspace of* X. *Then it holds:*
(1) Y *complete* $\Rightarrow Y$ *closed in* X.
(2) X *complete,* Y *closed* $\Rightarrow Y$ *complete.*
(3) $Y := \mathcal{N}(T)$ *is closed in* X *for any bounded operator* T *on* X.

Proposition 2.3.
$\emptyset \neq M \subseteq X$ *subset* $\Rightarrow M^\perp := \{x \in X \mid x \perp y\ \forall y \in M\}$ *(orthogonal complement) is a closed linear subspace in* X *and it holds* $\overline{\mathcal{L}(M)}^\perp = M^\perp$ *and* $M^{\perp\perp} = \overline{\mathcal{L}(M)}$.
In particular X *H-space* $\overset{2.2(2)}{\Rightarrow} M^\perp$ *is H-subspace in* X.

Theorem 2.4. $T \in \mathcal{L}(X,Y)$ *is TLI* X *onto* $\mathcal{R}(T)$ *iff there exist* $0 < A \le B < \infty$ *such that* $A\|x\|^2 \le \|Tx\|^2 \le B\|x\|^2$ *holds* $\forall\, x \in X$. *In such a case it holds:*
(1) X *is H-space iff* $\mathcal{R}(T)$ *is H-subspace in* Y.
(2) T *is UI iff* $A = B = 1$.

Sketch of the proof. The upper inequality $\|Tx\| \le \sqrt{B}\|x\|$ is equivalent with boundedness of T. The lower inequality is equivalent with injectivity and boundedness of T^{-1} on $\mathcal{R}(T)$ in view of $\sqrt{A}\|T^{-1}y\| \le \|TT^{-1}y\| = \|y\|\ \forall\, y \in \mathcal{R}(T)$ which is equivalent with $\|T^{-1}y\| \le \frac{1}{\sqrt{A}}\|y\|$ (boundedness of T^{-1}). In such a case T is TLI preserving completeness which yields (1), while (2) is straightforward by the definition of UI. □

Theorem 2.5.
(1) $T : H_1 \to H_2$ *is TLI* $\Leftrightarrow T^* : H_2 \to H_1$ *is TLI, and it holds* $(T^*)^{-1} = (T^{-1})^*$.
(2) $T : H_1 \to H_2$ *is UI* $\Leftrightarrow T^* : H_2 \to H_1$ *is UI, and it holds* $T^* = T^{-1}$.

Theorem 2.6 (Open Mapping Theorem).
If $T \in \mathcal{B}(H_1,H_2)$ *is surjective then* T *is open (maps open sets to open sets).*

3. Key statements

Theorem 3.1. *$T \in \mathcal{B}(H_1, H_2)$ invokes mutually dual orthogonal decompositions*

(1) $H_1 = \overline{\mathcal{R}(T^*)} \oplus \mathcal{N}(T)$,

(2) $H_2 = \overline{\mathcal{R}(T)} \oplus \mathcal{N}(T^*)$.

Proof.
(1) $x \in \mathcal{N}(T) \;\Leftrightarrow\; Tx = 0 \;\Leftrightarrow\; \langle Tx, y\rangle = 0 \,\forall y \in H_2$ (where implication $\Leftarrow$ follows when putting $y = Tx$) $\;\Leftrightarrow\; \langle x, T^*y\rangle = 0 \,\forall y \in H_2 \;\Leftrightarrow\; x \in \mathcal{R}(T^*)^\perp$. $\mathcal{N}(T)$ is closed in the complete space H_1 by 2.2(3) $\overset{2.2(2)}{\Rightarrow}$ $\mathcal{N}(T)$ is H-subspace in H_1 and orthogonal projection $P_{\mathcal{N}(T)}$ can be applied yielding $\forall\, x \in H_1$ unique decomposition $x = \widehat{x} + x^\perp$ where $x^\perp \in \mathcal{N}(T)^\perp = \mathcal{R}(T^*)^{\perp\perp} \overset{2.3}{=} \overline{\mathcal{R}(T^*)}$.
(2) follows from (1) when interchanging the roles of T and T^* along with $T^{**} = T$. ☐

Corollary 3.2. *$T \in \mathcal{B}(H_1, H_2)$ is surjective iff T^* is TLI H_2 onto $\mathcal{R}(T^*)$. In such a case $\mathcal{R}(T^*)$ is H-subspace in H_1 (also closed in view of 2.2(1))*[1].

Proof.
$\Rightarrow$: T surjective $\Rightarrow$ T is open on H_1 by the *Open Mapping Theorem* 2.6 $\Rightarrow$ $\dot{T} := T|_{\overline{\mathcal{R}(T^*)}} \in \mathcal{B}(\overline{\mathcal{R}(T^*)}, H_2)$ is an open bijection considering 3.1(1) $\Rightarrow$ $\dot{T}$ is TLI $\overset{2.5(1)}{\Rightarrow}$ $\dot{T}^* = T^*$ is TLI H_2 onto $\mathcal{R}(T^*)$ as well in view of 2.5(1) where $\mathcal{R}(T^*)$ is H-subspace in H_1 because H_2 is H-space — see 2.4(1).
$\Leftarrow$: If T^* is TLI H_2 onto $\mathcal{R}(T^*)$ then $\mathcal{R}(T^*) = \overline{\mathcal{R}(T^*)}$ is H-subspace (see 2.4(1) and 2.2(1)). It holds also $\mathcal{N}(T^*) = \{0\}$ by injectivity of T^*. Then $\dot{T} = T^{**}|_{\mathcal{R}(T^*)}$ is TLI from the H-subspace $\mathcal{R}(T^*)$ onto H_2 in view of 2.5(1) and thus $\mathcal{R}(T)$ is (closed) H-subspace in H_2 — again by 2.4(1) and 2.2(1). Finally using decomposition 3.1(2) we get $H_2 = \mathcal{R}(T) + \{0\} = \mathcal{R}(T)$ confirming surjectivity of T. ☐

Theorem 3.3. *Let $\Phi = \{\Phi_n\}_{n\in J}$ be a sequence in H-space H and $\mathcal{E} \subset \ell^2(J)$ standard canonical ONB. Then the following statements are equivalent:*

(1) *Φ is a frame in H.*
(2) *Synthesis operator $T := T_\Phi : \ell^2(J) \to H$ ($T\xi := \sum_{n\in J} \xi_n \Phi_n$) is bounded and surjective ($\Phi_n = T\varepsilon_n \;\forall n \in J$).*
(3) *Analysis (Bessel) operator $L := L_\Phi := T^*_\Phi$ ($Lx = \{\langle x, \Phi_n\rangle\}_{n\in J}$) is TLI H onto closed H-subspace $H^* := \mathcal{R}(L) \subseteq \ell^2(J)$.*
(4) *There exist $0 < A \le B < \infty$ such that $A\|x\|^2 \le \|Lx\|^2 \le B\|x\|^2$ holds $\forall\, x \in H$.*

In such a case Φ is Parseval frame (1-tight frame: $A = B = 1$) iff L is UI H onto H^*.

Proof.
(1) $\Leftrightarrow$ (2) is by definition.
(2) $\Leftrightarrow$ (3) by corollary 3.2 because $T^* = L$.
(3) $\Leftrightarrow$ (4) holds by theorem 2.4. ☐

Corollary 3.4 (Special cases of a frame)**.**
Keeping notation as of theorem 3.3 the following statements are equivalent:

(1) *Φ is a Riesz basis (ONB) in H.*
(2) *Synthesis operator $T := T_\Phi$ is TLI (UI) $\ell^2(J)$ onto H.*
(3) *Analysis (Bessel) operator $L := L_\Phi$ is TLI (UI) H onto $\ell^2(J)$.*

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[1] This statement can be also established as a direct consequence of a more general theorem valid in normed linear spaces (need not be complete in general) where adjoint of T operates between dual spaces.

(4) *There exist* $0 < A \le B < \infty$ *such that* $A\|x\|^2 \le \|Lx\|^2 \le B\|x\|^2$ *holds* $\forall\, x \in H$ *and* $\mathcal{N}(T) = \{0\}$ *(*$\|\varPhi_n\| = 1\,\forall\, n \in J$ *and Parseval identity* $\|x\|^2 = \|Lx\|^2$ *holds* $\forall\, x \in H$*).*

Proof.
(1) $\Leftrightarrow$ (2) is by definition.
(2) $\Leftrightarrow$ (3) follows by theorem 2.5 because $T^* = L$.
For the case of TLI (3) $\Leftrightarrow$ (4) holds by the theorem 3.3(3)(4) where the additional condition $\mathcal{N}(T) = \{0\}$ is equivalent with surjectivity of L onto $\ell^2(J)$ in view of theorem 3.1(1).
For the case of UI (3) $\Rightarrow$ (4) is straightforward by the unitarity of T and $L = T^*$ which implies Parseval identity and $\|\varPhi_n\| = \|T\varepsilon_n\| = \|\varepsilon_n\| = 1$.
The converse (3) $\Leftarrow$ (4) holds in this case as follows. Putting $x = \varPhi_m\ \forall\, m \in J$ in the Parseval identity we get $\forall\, m \in J$: $1 = \|\varPhi_m\|^2 = \sum\limits_{n\in J}|\langle\varPhi_m, \varPhi_n\rangle|^2 = |\langle\varPhi_m, \varPhi_m\rangle|^2 + \sum\limits_{m\ne n\in J}|\langle\varPhi_m, \varPhi_n\rangle|^2 = 1 + \sum\limits_{m\ne n\in J}|\langle\varPhi_m, \varPhi_n\rangle|^2 \Rightarrow \sum\limits_{m\ne n\in J}|\langle\varPhi_m, \varPhi_n\rangle|^2 = 0 \Rightarrow \langle\varPhi_m, \varPhi_n\rangle = 0$ $\forall\, n \in J, n \ne m \Rightarrow \varPhi_m \perp \varPhi_n\ \forall\, n, m \in J, n \ne m$.
Thus $\varPhi$ is orthonormal system in H which implies $\mathcal{N}(T) = \{0\}$ because $T\xi = 0 \Rightarrow \xi = 0\ \forall\, \xi \in \ell^2(J)$ by orthonormality and the Riesz-Fischer theorem which guarantees convergence of $T\xi\ \forall\, \xi \in \ell^2(J)$. As the Parseval identity $\|Lx\|^2 = \|x\|^2 < \infty$ holds $\forall\, x \in H$, we have unitarity of L which is also surjective in view of $\mathcal{N}(T) = \{0\}$ so as it was in the more general case of L being TLI. ☐

4. Constructing a frame by orthogonal projection

Theorem 4.1 (Constructing (Parseval) frame by means of orthogonal projection)**.** *Let* $P_H \in \mathcal{B}(H_1, H)$ *be the operator of orthogonal projection of the* H*-space* H_1 *onto its subspace* $H \subseteq H_1$*. Then* P_H *maps each Riesz basis* $\Psi = \{\varPsi_n\}_{n\in J}$ *in* H_1 *onto a frame* $\widehat{\Psi} = \{\widehat{\varPsi}_n\}_{n\in J}$ *in* H *and each ONB* $E = \{e_n\}_{n\in J}$ *in* H_1 *onto a Parseval frame* $\widehat{E} = \{\widehat{e}_n\}_{n\in J}$ *in* H*.*

Proof.
Ψ RB $\overset{3.4(2)}{\Rightarrow}$ $T_\Psi \in \mathcal{B}(\ell^2(J), H_1)$ is TLI. As $P_H \in \mathcal{B}(H_1, H)$ is surjection, we have $\widehat{\varPsi}_n = P_H\varPsi_n = P_H T_\Psi \varepsilon_n$ yielding a composite surjective operator $P_H T_\Psi \in \mathcal{B}(\ell^2(J), H)$ being a synthesis operator for the frame $\widehat{\Psi} = \{\widehat{\varPsi}_n\}_{n\in J}$ in H in view of 3.3(2).
When $\Psi = E$ is an ONB in H_1, then Parseval identity $\sum\limits_{n\in J}|\langle x, e_n\rangle|^2 = \|x\|^2$ holds $\forall\, x \in H_1$. For $x \in H$ we have $\langle x, e_n\rangle = \langle x, \widehat{e}_n + e_n^\perp\rangle = \langle x, \widehat{e}_n\rangle + \langle x, e_n^\perp\rangle = \langle x, \widehat{e}_n\rangle$ because $e_n^\perp \perp x$. Then for $x \in H$ the Parseval identity attains the form $\|L_{\widehat{E}}x\|^2 = \sum\limits_{n\in J}|\langle x, \widehat{e}_n\rangle|^2 = \|x\|^2$ saying that $L_{\widehat{E}}$ is UI. Then $\widehat{E} = \{\widehat{e}_n\}_{n\in J}$ is a Parseval frame in H in view of 3.3. ☐

Finally it remains to show that also converse to the above theorem is true.

Theorem 4.2 (Every (Parseval) frame can be constructed as of theorem 4.1)**.** *Let* $\Phi = \{\varPhi_n\}_{n\in J}$ *be a (Parseval) frame in* H *with the synthesis operator* $T \in \mathcal{B}(\ell^2(J), H)$ *and the associated analysis operator* T^* *of* H *onto* H*-space* $H^* := \mathcal{R}(T^*)$*. Denote* $\varPsi_n := (\varPhi_n, \varepsilon_n^\perp) \in H \times \mathcal{N}(T)\ \forall\, n \in J$ *where* $\varepsilon_n^\perp = P_{\mathcal{N}(T)}\varepsilon_n$*. Then* $\Psi = \{\varPsi_n\}_{n\in J}$ *is a Riesz basis (ONB) in* $H \times \mathcal{N}(T)$ *and when embedding* H *into* $H \times \mathcal{N}(T)$ *it holds* $\varPhi_n = P_H\varPsi_n$ *for* $n \in J$*.*

Proof (see commutative diagram in figure 1).
As T^* is TLI $\ell^2(J)$ onto a closed H-subspace $H^* = \mathcal{R}(T^*)$, we have $\ell^2(J) = H^* \oplus \mathcal{N}(T)$ by 3.1(1). Hence we get for any $\xi \in \ell^2(J)$ an orthogonal decomposition $\xi = \widehat{\xi} + \xi^\perp$ where $\widehat{\xi} = P_{H^*}\xi$ and $\xi^\perp \in P_{\mathcal{N}(T)}\xi$. Then $T\xi = T(\widehat{\xi} + \xi^\perp) = T\widehat{\xi} + T\xi^\perp = T\widehat{\xi} = \dot{T}\widehat{\xi}$, $\dot{T} := T|_{H^*}$.

When denoting U the UI $\ell^2(J)$ onto $H^* \times \mathcal{N}(T)$ by proposition 2.1(4), we get also $\dot{T}\widehat{\xi} \overset{2.1(3)}{=} P_H(\dot{T}\widehat{\xi}, \xi^\perp) = P_H(\dot{T} \times I_{\mathcal{N}(T)})(\widehat{\xi}, \xi^\perp) = P_H(\dot{T} \times I_{\mathcal{N}(T)})U(\widehat{\xi} + \xi^\perp) = P_H(\dot{T} \times I_{\mathcal{N}(T)})U\xi$. Altogether $T = P_H(\dot{T} \times I_{\mathcal{N}(T)})U$. As $I_{\mathcal{N}(T)}$ is UI it follows by 2.1(2) that $(\dot{T} \times I_{\mathcal{N}(T)})U$ is TLI (UI) $\Leftrightarrow$ $\dot{T}$ is TLI (UI) $\overset{2.5}{\Leftrightarrow}$ T^* is TLI (UI) $\overset{3.3}{\Leftrightarrow}$ Φ is (Parseval) frame. Hence putting $\xi = \varepsilon_n$ $\forall n \in J$ we get $\Phi_n = P_H \Psi_n$ where $\Psi_n = (\dot{T} \times I_{\mathcal{N}(T)})U\varepsilon_n = (\dot{T}\widehat{\varepsilon_n}, \varepsilon^\perp) = (T\varepsilon_n, \varepsilon^\perp) = (\Phi_n, \varepsilon^\perp)$ is Riesz basis (ONB) by 3.4(1)(2). □

Remark 4.3 (Algorithm for matrix operators).

Let $\boldsymbol{T}$ be matrix of size $r \times m$ having rank $r = \mathrm{r}(\boldsymbol{T})$, $r \leq m$, representing surjective synthesis operator for the (Parseval) frame Φ with columns Φ_n, $n = 1, \dots, m$. Let $\boldsymbol{N}$ be of size $m \times (m - r)$, the columns of which form ONB for $\mathcal{N}(\boldsymbol{T})$ (all solutions of the homogeneous SLE $\boldsymbol{T\xi} = \boldsymbol{0} \Rightarrow \boldsymbol{TN} = \boldsymbol{0}$) — e.g. using MATLAB function `null.m` or bringing $\boldsymbol{T}$ to reduced row echelon form to find basis of $\mathcal{N}(\boldsymbol{T})$ and then orthogonalize it using QR-decomposition or Gram-Schmidt process.

Clearly $\mathrm{r}(\boldsymbol{N}) = m - r$. Then $m \times m$ matrix $\boldsymbol{U} := \left[\begin{smallmatrix} \boldsymbol{T} \\ \boldsymbol{N}^* \end{smallmatrix}\right]$ is nonsingular (unitary) because $\boldsymbol{UU}^* = \left[\begin{smallmatrix} \boldsymbol{TT}^* & \boldsymbol{TN} \\ \boldsymbol{N}^*\boldsymbol{T}^* & \boldsymbol{N}^*\boldsymbol{N} \end{smallmatrix}\right] = \left[\begin{smallmatrix} \boldsymbol{TT}^* & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{m-r} \end{smallmatrix}\right]$ and $\boldsymbol{TT}^*$ is nonsingular (representing TLI) for general frame (or $\boldsymbol{I}_r$ for Parseval frame). Of course, there is a freedom in choosing $\boldsymbol{N}$: $\boldsymbol{N} \rightsquigarrow \boldsymbol{NQ}$ where $\boldsymbol{Q}$ is an arbitrary unitary matrix of size $(m - r) \times (m - r)$.

Let us observe that in the case of a non-Parseval frame Φ it is sufficient for $\boldsymbol{N}$ to be a (Riesz) basis in $\mathcal{N}(\boldsymbol{T})$ and $\boldsymbol{Q}$ nonsingular. In such a case we get nonsingular matrix $\boldsymbol{N}^*\boldsymbol{N}$ instead of $\boldsymbol{I}_{m-r}$ guaranteeing nonsingularity of $\boldsymbol{U}$ as well.

$$\begin{tikzcd}
H^* \oplus \mathcal{N}(T) = \ell^2(J) \arrow[r, "U"] \arrow[d, "P_{H^*}"] \arrow[drr, "T"] & H^* \times \mathcal{N}(T) \arrow[r, "\dot{T} \times I_{\mathcal{N}(T)}"] & H \times \mathcal{N}(T) \arrow[d, "P_H"] \\
H^* \arrow[rr, "\dot{T}"] & & H
\end{tikzcd}$$

FIGURE 1. Construction of any frame by means of orthogonal projection