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**Nonlinear Differential Equations and Applications NoDEA**

# Regularly varying solutions of subhomogeneous differential equations with $p(t)$-Laplacian

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**Abstract.** This paper investigates the asymptotic behavior of increasing solutions to subhomogeneous differential equations involving the $p(t)$-Laplacian operator. Specifically, we consider the quasilinear equation $(a(t)|y'|^{p(t)} \operatorname{sgn} y')' = b(t)|y|^{q(t)} L_G(|y|) \operatorname{sgn} y$ where $p(t)$ and $q(t)$ are variable exponents and $L_G$ is a slowly varying perturbation. Our focus is on regularly varying solutions under the subhomogeneity condition $p(t) > q(t)$ for large $t$. We show that all increasing solutions are regularly varying, derive asymptotic formulas for these solutions, and demonstrate their examples. This work contributes to the understanding of nonoscillatory solutions and shows how regular variation can be useful in studying differential equations involving variable exponents.

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## 1. Introduction

We consider second order nonlinear differential equations of the form

$$(a(t)F(t,y'))' = b(t)G(t,y), \quad t \geq t_0 > 0, \tag{1.1}$$

where $F(t,z) = |z|^{p(t)} \operatorname{sgn} z$, $G(t,z) = |z|^{q(t)} L_G(|z|) \operatorname{sgn} z$ with a continuous slowly varying function $L_G$, and such that $G(t,0) = 0$, and $a(t)$, $b(t)$, $p(t)$, $q(t)$ are positive continuous functions satisfying

$$\lim_{t\to\infty} p(t) = \alpha, \quad \lim_{t\to\infty} q(t) = \beta \tag{1.2}$$

with $\alpha > 0$ and $\beta > 0$. Here, we recall that a measurable function $f\colon [0,\infty) \to (0,\infty)$ is said to be *regularly varying of index* $\rho \in \mathbb{R}$ (we write $f \in \mathcal{RV}(\rho)$) if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\rho}$$

for any $\lambda > 0$, and it is said to be *slowly varying* (we write $f \in \mathcal{SV}$) if it satisfies

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = 1$$

for any $\lambda > 0$.

A function $y(t)$ is said to be a *solution* of equation (1.1) defined on $[t_0, \infty)$, if $y(t)$ and its quasiderivative

$$y^{[1]}(t) = a(t)F(t, y'(t))$$

are continuously differentiable, and $y(t)$ satisfies equation (1.1) on $[t_0, \infty)$. We define the class of eventually positive increasing solutions and its subclass of *strongly increasing solutions* by

$$\mathcal{IS} = \{y \mid y \text{ is a solution of equation (1.1)}, y(t) > 0 \text{ and } y'(t) > 0 \text{ for large } t\},$$
$$\mathcal{SIS} = \{y \in \mathcal{IS} \mid y(t) \to \infty \text{ and } y^{[1]}(t) \to \infty \text{ as } t \to \infty\}.$$

When $L_G(z) \equiv 1$, equation (1.1) becomes the equation

$$(a(t)|y'|^{p(t)} \operatorname{sgn} y')' = b(t)|y|^{q(t)} \operatorname{sgn} y. \tag{1.3}$$

It is worth noting that even in this case, our results are new. The differential operator in equation (1.3) is referred to as the $p(t)$-Laplacian, representing the one-dimensional analogue of the $p(x)$-Laplacian partial differential operator. The $p(x)$-Laplacian is known to arise in various mathematical models, including those related to image processing and electrorheological fluids. As a result, a lot of research papers have been dedicated to exploring elliptic partial differential equations involving the $p(x)$-Laplacian (see, for instance, [3,21,22,24] and references therein). Many results in elliptic partial differential equations are grounded in radialization techniques, allowing various methods and results from ordinary differential equations to be utilized. In light of this, there has been growing interest in studying the asymptotic behavior of solutions to equations featuring the $p(t)$-Laplacian (see, for example, [1,8,9,19,20,23]).

A particular case of equation (1.3) occurs when $p(t) \equiv p > 1$ and $q(t) \equiv q > 1$, leading to the quasilinear equation

$$(a(t)|y'|^{p} \operatorname{sgn} y')' = b(t)|y|^{q} \operatorname{sgn} y, \tag{1.4}$$

which serves as a key example for our study. In this case, the $p(t)$-Laplacian reduces to the classical $p$-Laplacian. It is worth noting that the results where we show that all $\mathcal{SIS}$ solutions are regularly varying and satisfy an asymptotic formula are new even in the case $(a(t)|y'|^p \operatorname{sgn} y')' = b(t)|y|^q L_G(|y|) \operatorname{sgn} y$. The investigation of equation (1.4) is motivated, among others, by a partial differential equation that models the concentration of a substance disappearing via an isothermal reaction within a finite catalytic slab. It is known that the asymptotic behavior of solutions to equation (1.4) is related to the ratio between the characteristic reaction rate and the characteristic diffusion rate approaches infinity (see [17]). Hence, considerable research has focused on examining the asymptotic behavior of solutions to equation (1.4). In particular, the concept of regular variation, which was introduced by Karamata in 1930, has been widely applied to the study of differential equations including equation (1.4). Relevant results can be found, for instance, in [4–7, 11–16, 18] and the references therein.

We also remark that, if $p = q$ then equation (1.4) becomes

$$(a(t)|y'|^p \operatorname{sgn} y')' = b(t)|y|^p \operatorname{sgn} y, \tag{1.5}$$

which is called half-linear equation. This is because the solution space of equation (1.5) has just one of the properties which characterize linearity, namely homogeneity, i.e. if $y(t)$ is its solution, then $\lambda y(t)$ ($\lambda \in \mathbb{R}$) is its solution too (see [5]).

In this paper, we assume that

$$(\varphi_i(\lambda t) - \varphi_i(t)) \log t \to 0 \tag{1.6}$$

as $t \to \infty$ for any $\lambda > 0$ and $i = 1, 2$, where

$$\varphi_1(t) = p(t) - \alpha, \quad \varphi_2(t) = q(t) - \beta.$$

Examples of functions satisfying (1.6) are given Section 4. Furthermore, we assume

$$\inf_{t\in[t_0,\infty)} p(t) > \sup_{t\in[t_0,\infty)} q(t), \tag{1.7}$$

which implies $\alpha > \beta$, namely subhomogeneity. We note that equation (1.3) has no singular solution of the second kind under (1.7) (see [1, Theorem 2.2]); later (in Lemma 2.8) we will see that the same holds for (1.1). The main purpose of this paper is to show that all increasing solutions of equation (1.1) under the subhomogeneity condition are strongly increasing and regularly varying, and to establish an asymptotic formula for these solutions. The following theorem also shows that the set of $\mathcal{SIS}$ regularly varying solutions of equation (1.1) is nonempty.

**Theorem 1.1.** *Assume* (1.2), (1.6), (1.7), *and*

$$a \in \mathcal{RV}(\gamma), \quad b \in \mathcal{RV}(\delta), \tag{1.8}$$

*where* $\delta \in \mathbb{R}$. *Suppose that*

$$L_G(tL(t)) \sim L_G(t) \text{ as } t \to \infty \text{ for all } L \in \mathcal{SV}, \tag{1.9}$$

$$z \mapsto G(\cdot, z) \text{ is eventually monotone,} \tag{1.10}$$

*and*

$$1 + \beta + \delta - \frac{\beta\gamma}{\alpha} > 0, \quad 1 + \alpha - \gamma + \delta > 0 \tag{1.11}$$

*hold. Then,* $\emptyset \neq \mathcal{IS} = \mathcal{SIS} \subset \mathcal{RV}(\vartheta)$, *where*

$$\vartheta = \frac{1 + \alpha - \gamma + \delta}{\alpha - \beta}.$$

*Moreover, for any $y \in \mathcal{IS}$ it holds*

$$y(t) \sim \left( \frac{1}{\alpha\vartheta - \alpha + \gamma} \cdot \frac{1}{\vartheta^\alpha} \cdot \frac{b(t)}{a(t)} t^{p(t)+1} L_G(t^\vartheta) \right)^{\frac{1}{p(t)-q(t)}}$$

*as $t \to \infty$.*

**Remark 1.1.** The notation $f(t) \sim g(t)$ as $t \to \infty$ means that

$$\lim_{t\to\infty} \frac{f(t)}{g(t)} = 1.$$

**Remark 1.2.** (i) A wide class of slowly varying functions satisfy condition (1.9), e.g., any $L_G$ of the form $L_G(t) \sim (\log t)^\xi (\log\log t)^\eta$ as $t \to \infty$, with $\xi, \eta \in \mathbb{R}$.

(ii) Condition (1.10) is not very restrictive. Indeed, recall that any differentiable normalized regularly varying function $y$ of index $\rho > 0$ satisfies $y'(t) \sim \rho y(t)/t$, thus it is eventually increasing.

This paper is organized as follows. In Section 2, we introduce some lemmas, such as the existence of solutions in certain classes, and we give the proof of Theorem 1.1. In Section 3, we give the necessary conditions for the existence of solutions in $\mathcal{SIS}$. In Section 4, we show some examples and propose some open problems.

## 2. Proof of the main theorem

In the first part, we propose some lemmas. To begin with, we prepare the following lemma, which is a tool for dealing with regularly varying functions.

**Lemma 2.1.** *Assume* (1.2), (1.6), (1.7), *and $f \in \mathcal{RV}(\rho)$ with $\rho \in \mathbb{R}$. Then, the following statements hold.*

*(i)* $f^p \in \mathcal{RV}(\alpha\rho)$, $f^q \in \mathcal{RV}(\beta\rho)$,

$$f^{\frac{1}{p-q}} \in \mathcal{RV}\left(\frac{\rho}{\alpha - \beta}\right).$$

*(ii) If $g(t) \asymp f(t)$ as $t \to \infty$, that is, there exist positive constants $M_1$ and $M_2$ such that $M_1 g(t) \le f(t) \le M_2 g(t)$ for large $t$, then for all $\lambda > 0$*

$$(g(t))^{p(\lambda t)-p(t)} \sim 1, \quad (g(t))^{q(\lambda t)-q(t)} \sim 1$$

*as $t \to \infty$.*

*Proof.* (i) Since $f \in \mathcal{RV}(\rho)$ and (1.2) holds, we can write $f(t) = t^\rho L_f(t)$, where $L_f \in \mathcal{SV}$. Let $\lambda > 0$. Then,

$$\frac{(f(\lambda t))^{p(\lambda t)}}{(f(t))^{p(t)}} = \left(\frac{f(\lambda t)}{f(t)}\right)^{p(\lambda t)} (f(t))^{p(\lambda t)-p(t)}.$$

Here, since $f \in \mathcal{RV}(\rho)$, we have

$$\left(\frac{f(\lambda t)}{f(t)}\right)^{p(\lambda t)} \to \lambda^{\alpha\rho}$$

as $t \to \infty$. Further, from (1.6),

$$t^{\rho(\varphi_1(\lambda t)-\varphi_1(t))} = \exp\{\rho(\varphi_1(\lambda t) - \varphi_1(t)) \log t\} \to 1$$

and

$$\begin{aligned}(L_f(t))^{\varphi_1(\lambda t)-\varphi_1(t)} &= \exp\{(\varphi_1(\lambda t) - \varphi_1(t)) \log L_f(t)\} \\ &= \exp\left\{(\varphi_1(\lambda t) - \varphi_1(t)) \log t \cdot \frac{\log L_f(t)}{\log t}\right\} \to 1\end{aligned}$$

as $t \to \infty$, which implies

$$(f(t))^{p(\lambda t)-p(t)} = t^{\rho(\varphi_1(\lambda t)-\varphi_1(t))}(L_f(t))^{\varphi_1(\lambda t)-\varphi_1(t)} \to 1$$

as $t \to \infty$. Noting that $\log L_f(t)/\log t = 0$ as $t \to \infty$, we thus get

$$\lim_{t\to\infty} \frac{(f(\lambda t))^{p(\lambda t)}}{(f(t))^{p(t)}} = \lambda^{\alpha\rho},$$

and so, $f^p \in \mathcal{RV}(\alpha\rho)$ by the definition. Proceeding in the same manner, we have $f^q \in \mathcal{RV}(\beta\rho)$. Similarly,

$$\begin{aligned}\frac{(f(\lambda t))^{1/(p(\lambda t)-q(\lambda t))}}{(f(t))^{1/(p(t)-q(t))}} &= \left(\frac{f(\lambda t)}{f(t)}\right)^{\frac{1}{p(\lambda t)-q(\lambda t)}} (f(t))^{\frac{1}{p(\lambda t)-q(\lambda t)}-\frac{1}{p(t)-q(t)}} \\ &= \left(\frac{f(\lambda t)}{f(t)}\right)^{\frac{1}{p(\lambda t)-q(\lambda t)}} \left(\frac{(f(t))^{q(\lambda t)-q(t)}}{(f(t))^{p(\lambda t)-p(t)}}\right)^{\frac{1}{(p(\lambda t)-q(\lambda t))(p(t)-q(t))}} \\ &\to \lambda^{\frac{\rho}{\alpha-\beta}}\end{aligned}$$

as $t \to \infty$. Here, we note that

$$\lim_{t\to\infty} \frac{(f(t))^{q(\lambda t)-q(t)}}{(f(t))^{p(\lambda t)-p(t)}} = 1, \qquad \lim_{t\to\infty} \frac{1}{(p(\lambda t) - q(\lambda t))(p(t) - q(t))} = \frac{1}{(\alpha-\beta)^2}$$

by the previous computations. Hence, we have $f^{1/(p-q)} \in \mathcal{RV}(\rho/(\alpha - \beta))$.

(ii) Let $\lambda > 0$. Then, we have

$$(g(t))^{p(\lambda t)-p(t)} = (g(t))^{\varphi_1(\lambda t)-\varphi_1(t)} = \exp\{(\varphi_1(\lambda t) - \varphi_1(t)) \log g(t)\}.$$

From $f(t) \asymp g(t)$ as $t \to \infty$, we get

$$\log M_1 f(t) \le \log g(t) \le \log M_2 f(t).$$

Since

$$\frac{\log M_i f(t)}{\log t} = \frac{\log M_i}{\log t} + \frac{\log f(t)}{\log t} = \frac{\log M_i}{\log t} + \frac{\rho \log t + \log L_F(t)}{\log t} \to \rho$$

as $t \to \infty$, we obtain that $\log g(t)/\log t$ is bounded. Consequently,

$$(\varphi_1(\lambda t) - \varphi_1(t)) \log g(t) = (\varphi_1(\lambda t) - \varphi_1(t)) \log t \cdot \frac{\log g(t)}{\log t} \to 0,$$

and hence, $(g(t))^{p(\lambda t)-p(t)} \to 1$ as $t \to \infty$. Similarly, we get $(g(t))^{q(\lambda t)-q(t)} \to 1$ as $t \to \infty$. □

Using Lemma 2.1, we give the following lemma, which guarantees the existence of solutions in the class $\mathcal{SIS} \cap \mathcal{RV}(\vartheta)$.

**Lemma 2.2.** *Assume* (1.2) *and* (1.6)–(1.11) *hold. Then,* $\mathcal{SIS} \cap \mathcal{RV}(\vartheta) \neq \emptyset$.

*Proof.* From (1.7) and the second inequality of (1.11), we have $\vartheta > 0$. The existence of the desired solution will be shown via the Schauder–Tychonoff fixed point theorem. Denote

$$H(t) = \left(\frac{b(t)}{a(t)} t^{p(t)+1} L_G(t^{\vartheta})\right)^{\frac{1}{p(t)-q(t)}}. \tag{2.1}$$

Then, from Lemma 2.1 and $L_G(t^{\vartheta}) \in \mathcal{SV}$ (see [2, Proposition 1.5.7 (ii)]), we get $H \in \mathcal{RV}(\vartheta)$. Hence, we can find $L_H \in \mathcal{SV}$ such that $H(t) = t^{\vartheta} L_H(t)$. Moreover, we have $bH^q \in \mathcal{RV}(\delta + \beta\vartheta)$ and

$$\delta + \beta\vartheta + 1 = \alpha\vartheta - \alpha + \gamma = \frac{\alpha}{\alpha - \beta}\left(1 + \beta + \delta - \frac{\beta\gamma}{\alpha}\right) > 0$$

from the first inequality of (1.11). From condition (1.10), there exists $z_0 > 0$ such that $z \mapsto G(t, z)$ is nondecreasing for $z \geq z_0$. Hence, we can take $\widetilde{G}(t, z) = |z|^{q(t)} L_{\widetilde{G}}(|z|)$ and $0 < N_1 \leq N_2 < 1$ such that

$$N_1 \widetilde{G}(t, z) \leq G(t, z) \leq N_2 \widetilde{G}(t, z) \tag{2.2}$$

for $t \geq t_0$, $z \in [c, \infty)$, where $c > 0$ can be taken arbitrarily close to 0 and $z \mapsto \widetilde{G}(t, z)$ is nondecreasing for $z \geq c$. Here we can set $c = \inf_{t \in [t_0, \infty)} K_1 H(t)$, where $K_1$ is specified later. Note that on $[c, z_0]$, $G$ can be changed at will, and the result is not affected. In fact, $L_{\widetilde{G}}$ can be such that $L_G(z) \sim L_{\widetilde{G}}(z)$ as $z \to \infty$. From (1.9), we get

$$L_G(t^{\rho} L(t)) \sim L_G(t^{\rho}) \tag{2.3}$$

as $t \to \infty$ for all $L \in \mathcal{SV}$ and $\rho > 0$ by [16, Lemma 5.1].

Applying the Karamata integration theorem (see [2, Theorem 1.5.11]) and l'Hôpital rule, we obtain

$$\begin{aligned}
&\int_{t_0}^{t} \left(\frac{1}{a(s)} \int_{t_0}^{s} b(\tau) G(\tau, H(\tau))\, d\tau\right)^{\frac{1}{p(s)}} ds \\
&\quad = \int_{t_0}^{t} \left(\frac{1}{a(s)} \int_{t_0}^{s} b(\tau)(H(\tau))^{q(\tau)} L_G(H(\tau))\, d\tau\right)^{\frac{1}{p(s)}} ds \\
&\quad \sim \int_{t_0}^{t} \left(\frac{1}{\delta + \beta\vartheta + 1} \cdot \frac{b(s)}{a(s)} s (H(s))^{q(s)} L_G(H(s))\right)^{\frac{1}{p(s)}} ds \\
&\quad \sim \left(\frac{1}{\alpha\vartheta - \alpha + \gamma}\right)^{\frac{1}{\alpha}} \int_{t_0}^{t} \left(\frac{b(s)}{a(s)} s (H(s))^{q(s)} L_G(H(s))\right)^{\frac{1}{p(s)}} ds \\
&\quad \sim \left(\frac{1}{\alpha\vartheta - \alpha + \gamma}\right)^{\frac{1}{\alpha}} \frac{1}{(1 - \gamma + \delta + \beta\vartheta)/\alpha + 1} \cdot t \left(\frac{b(t)}{a(t)} t (H(t))^{q(t)} L_G(H(t))\right)^{\frac{1}{p(t)}} \\
&\quad = \left(\frac{1}{\alpha\vartheta - \alpha + \gamma}\right)^{\frac{1}{\alpha}} \frac{1}{\vartheta} \left(\frac{b(t)}{a(t)} t^{p(t)+1} (H(t))^{q(t)} L_G(H(t))\right)^{\frac{1}{p(t)}} \\
&\quad = \left(\frac{1}{\alpha\vartheta - \alpha + \gamma}\right)^{\frac{1}{\alpha}} \frac{1}{\vartheta} \left(\frac{b(t)}{a(t)} t^{p(t)+1} (H(t))^{q(t)} L_G(t^{\vartheta} L_H(t))\right)^{\frac{1}{p(t)}}
\end{aligned}$$

$$\begin{aligned}
&\sim \left(\frac{1}{\alpha\vartheta-\alpha+\gamma}\right)^{\frac{1}{\alpha}}\frac{1}{\vartheta}\left(\frac{b(t)}{a(t)}t^{p(t)+1}(H(t))^{q(t)}L_G(t^{\vartheta})\right)^{\frac{1}{p(t)}}\\
&= \left(\frac{1}{\alpha\vartheta-\alpha+\gamma}\right)^{\frac{1}{\alpha}}\frac{1}{\vartheta}\left((H(t))^{p(t)-q(t)}(H(t))^{q(t)}\right)^{\frac{1}{p(t)}}\\
&= \left(\frac{1}{\alpha\vartheta-\alpha+\gamma}\right)^{\frac{1}{\alpha}}\frac{1}{\vartheta}\cdot H(t) \qquad (2.4)
\end{aligned}$$

as $t\to\infty$. Let $A$ be a positive number. In view of (2.4), $L_G(z)\sim L_{\widetilde{G}}(z)$ as $z\to\infty$, positivity of $H$ on $[t_0,\infty)$, and the fact that $H\in\mathcal{RV}(\vartheta)$ with $\vartheta>0$, we have $H(t)\to\infty$ as $t\to\infty$ and we can find $M_1$ and $M_2$ such that $0<M_1\le 1\le M_2$ and

$$M_1H(t)\le A+\int_{t_0}^{t}\left(\frac{1}{a(s)}\int_{t_0}^{s}b(\tau)\widetilde{G}(\tau,H(\tau))\,d\tau\right)^{\frac{1}{p(s)}}ds\le M_2H(t) \qquad (2.5)$$

for $t\ge t_0$. Denote $\underline{\alpha}=\inf_{t\in[t_0,\infty)}p(t)$ and $\overline{\beta}=\sup_{t\in[t_0,\infty)}q(t)$. Then, from (1.7), we have $0<\overline{\beta}/\underline{\alpha}<1$. We put $K_i=\left(N_i^{\frac{1}{\alpha}}M_i\right)^{1/(1-\overline{\beta}/\underline{\alpha})}$ $(i=1,2)$, and note that we can take $M_2$ so large that $K_1\le 1\le K_2$ and so

$$K_1^{q(t)}\ge K_1^{\overline{\beta}},\quad \left(K_1^{\overline{\beta}}\right)^{\frac{1}{p(t)}}\ge K_1^{\frac{\overline{\beta}}{\alpha}},\quad K_2^{q(t)}\le K_2^{\overline{\beta}},\quad \left(K_2^{\overline{\beta}}\right)^{\frac{1}{p(t)}}\le K_2^{\frac{\overline{\beta}}{\alpha}}.$$

Let $\mathcal{X}$ be the Fréchet space of all continuous functions defined for any $t\ge t_0$ endowed with the topology of uniform convergence on compact subintervals of $[t_0,\infty)$, and we put

$$\Omega=\left\{u\in\mathcal{X}\mid K_1H(t)\le u(t)\le K_2H(t),t\ge t_0\right\}.$$

Clearly $\Omega$ is bounded, closed, and convex. Let $\mathscr{T}\colon\Omega\to\mathcal{X}$ be an operator defined by

$$\mathscr{T}(u)(t)=A+\int_{t_0}^{t}\left(\frac{1}{a(s)}\int_{t_0}^{s}b(\tau)G(\tau,u(\tau))\,d\tau\right)^{\frac{1}{p(s)}}ds$$

for $t\ge t_0$. Then, $\mathscr{T}$ is well defined, and

$$\begin{aligned}
\mathscr{T}(u)(t)&\ge A+\int_{t_0}^{t}\left(\frac{N_1}{a(s)}\int_{t_0}^{s}b(\tau)\widetilde{G}(\tau,K_1H(\tau))\,d\tau\right)^{\frac{1}{p(s)}}ds\\
&\ge A+N_1^{\frac{1}{\alpha}}\int_{t_0}^{t}\left(\frac{K_1^{\overline{\beta}}}{a(s)}\int_{t_0}^{s}b(\tau)\widetilde{G}(\tau,H(\tau))\,d\tau\right)^{\frac{1}{p(s)}}ds\\
&\ge A+N_1^{\frac{1}{\alpha}}K_1^{\frac{\overline{\beta}}{\alpha}}\int_{t_0}^{t}\left(\frac{1}{a(s)}\int_{t_0}^{s}b(\tau)\widetilde{G}(\tau,H(\tau))\,d\tau\right)^{\frac{1}{p(s)}}ds\\
&\ge N_1^{\frac{1}{\alpha}}K_1^{\frac{\overline{\beta}}{\alpha}}\left\{A+\int_{t_0}^{t}\left(\frac{1}{a(s)}\int_{t_0}^{s}b(\tau)\widetilde{G}(\tau,H(\tau))\,d\tau\right)^{\frac{1}{p(s)}}ds\right\}\\
&\ge N_1^{\frac{1}{\alpha}}K_1^{\frac{\overline{\beta}}{\alpha}}M_1H(t)=K_1H(t)
\end{aligned}$$

holds for $t \geq t_0$. Similarly, we have $\mathscr{T}(u)(t) \leq K_2 H(t)$ for $t \geq t_0$. Hence, we see that $\mathscr{T}(\Omega) \subset \Omega$.

We prove that $\mathscr{T}(\Omega)$ is relatively compact. Thanks to the topology of $\mathcal{X}$, we can use the classical Ascoli–Arzelà theorem. Let $u \in \Omega$. Then, we have

$$|\mathscr{T}(u)(t) - \mathscr{T}(u)(\tilde{t})| = \left| \int_{\tilde{t}}^{t} \left( \frac{1}{a(s)} \int_{t_0}^{s} b(\tau) G(\tau, u(\tau))\, d\tau \right)^{\frac{1}{p(s)}} ds \right|$$
$$\leq \left| N_2^{\frac{1}{\alpha}} K_2^{\frac{\overline{\beta}}{\alpha}} \int_{\tilde{t}}^{t} \left( \frac{1}{a(s)} \int_{t_0}^{s} b(\tau) \widetilde{G}(\tau, H(\tau))\, d\tau \right)^{\frac{1}{p(s)}} ds \right|$$

for any $t \geq t_0$ and $\tilde{t} \geq t_0$. For any $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that $|\mathscr{T}(u)(t) - \mathscr{T}(u)(\tilde{t})| < \varepsilon$ if $|t - \tilde{t}| < \delta_\varepsilon$ for all $u \in \Omega$, that is, $\mathscr{T}(\Omega)$ is equicontinuous. In addition, it is obviously uniformly bounded because of $\mathscr{T}(\Omega) \subset \Omega$. From the Ascoli–Arzelà theorem, we see that $\mathscr{T}(\Omega)$ is relatively compact.

We next give the proof of the continuity of $\mathscr{T}$ in $\Omega$. Let $\{u_n\}$ $(n \in \mathbb{N})$ be a sequence in $\Omega$ which uniformly converges on every compact subinterval of $[t_0, \infty)$ to $\bar{u} \in \Omega$. Since $\mathscr{T}(\Omega)$ is relatively compact, the sequence $\{\mathscr{T}(u_n)\}$ admits a subsequence which converges to $\bar{z}_u \in \overline{\mathscr{T}(\Omega)}$ in the topology of $\mathcal{X}$. For simplicity, let $\{u_n\}$ be such a sequence and let

$$z_n(t) = \left( \frac{1}{a(t)} \int_{t_0}^{t} b(s) G(s, u_n(s))\, ds \right)^{\frac{1}{p(t)}}.$$

Since

$$\int_{t_0}^{t} b(s) G(s, u_n(s))\, ds < \infty$$

for any fixed $t \geq t_0$, we can apply the Lebesgue dominated convergence theorem and we obtain

$$\lim_{n \to \infty} z_n(t) = \left( \frac{1}{a(t)} \int_{t_0}^{t} b(s) G(s, \bar{u}(s))\, ds \right)^{\frac{1}{p(t)}}.$$

Using the Lebesgue dominated convergence theorem again, we see that $\{\mathscr{T}(u_n)\}$ pointwise converges to $\mathscr{T}(\bar{u})(t)$. Since $\mathscr{T}(\bar{u}) = \bar{z}_u$ is the only one cluster point of the sequence $\{\mathscr{T}(u_n)\}$, $\mathscr{T}$ is continuous in the topology of $\mathcal{X}$.

From the Schauder–Tychonoff fixed point theorem, there exists $y \in \Omega$ such that $\mathscr{T}(y) = y$. It is clear that $y \in \mathcal{IS}$. Moreover, we have $y(t) \geq K_1 H(t) \to \infty$ as $t \to \infty$ because of $H \in \mathcal{RV}(\vartheta)$ with $\vartheta > 0$, and so

$$y^{[1]}(t) = \int_{t_0}^{t} b(s) G(s, y(s))\, ds \geq N_1 K_1^{\overline{\beta}} \int_{t_0}^{t} b(s) \widetilde{G}(s, H(s))\, ds \to \infty$$

as $t \to \infty$ because of $bH^q \in \mathcal{RV}(\delta + \beta\vartheta)$ with $\delta + \beta\vartheta > -1$. Thus, we see that $y \in \mathcal{SIS}$.

Next, we show that $y \in \mathcal{RV}(\vartheta)$. Let $\lambda > 0$ and

$$m_* = \liminf_{t \to \infty} \frac{y(\lambda t)}{y(t)}, \quad m^* = \limsup_{t \to \infty} \frac{y(\lambda t)}{y(t)}. \tag{2.6}$$

Then, we have $m_*, m^* \in (0,\infty)$ because $y \in \Omega$ and $H \in \mathcal{RV}(\vartheta)$. Since $y(\lambda t)/y(t) \in [m_*/2, 2m^*]$ for all $t$ and $y(t) \to \infty$ as $t \to \infty$, thanks to the uniform convergence theorem (see [2, Theorem 1.2.1]), we have

$$\left|\frac{L_G(y(\lambda t))}{L_G(y(t))} - 1\right| = \left|\frac{L_G\left(\frac{y(\lambda t)}{y(t)} y(t)\right)}{L_G(y(t))} - 1\right| \le \sup_{\xi \in [m_*/2, 2m^*]} \left|\frac{L_G(\xi y(t))}{L_G(y(t))} - 1\right| \to 0 \tag{2.7}$$

as $t \to \infty$. Moreover, since $y(t) \asymp H(t) \in \mathcal{RV}(\vartheta)$, we have

$$y^{[1]}(t) \asymp K(t) := \int^t b(s) G(s, H(s))\, ds \in \mathcal{RV}(1 + \delta + \beta\vartheta)$$

as $t \to \infty$, and so

$$y'(t) \asymp \left(\frac{1}{a(t)} K(t)\right)^{\frac{1}{p(t)}} \in \mathcal{RV}\left(\frac{1 + \delta + \beta\vartheta - \gamma}{\alpha}\right). \tag{2.8}$$

Hence, there exist $c_1, c_2 > 0$ such that

$$c_1 \le \liminf_{t\to\infty} \frac{y'(\lambda t)}{y'(t)}, \quad \limsup_{t\to\infty} \frac{y'(\lambda t)}{y'(t)} \le c_2.$$

Thanks to the generalized l'Hôpital rule, (1.8), (2.6), and $y \in \mathcal{SIS}$, we have

$$m_* \ge \liminf_{t\to\infty} \frac{\lambda y'(\lambda t)}{y'(t)}, \quad \limsup_{t\to\infty} \frac{\lambda y'(\lambda t)}{y'(t)} \ge m^*. \tag{2.9}$$

Applying Lemma 2.1 (ii) to the functions $y(t) \asymp H(t)$ and (2.8) $(t \to \infty)$, together with (2.7) and (2.9), we obtain

$$\begin{aligned}
m_*^\alpha &\ge \liminf_{t\to\infty} \left(\frac{\lambda y'(\lambda t)}{y'(t)}\right)^{p(\lambda t)} = \liminf_{t\to\infty} \frac{\lambda^{p(\lambda t)} \left(y'(\lambda t)\right)^{p(\lambda t)}}{(y'(t))^{p(\lambda t)}} \\
&= \liminf_{t\to\infty} \lambda^\alpha \cdot \frac{(y'(\lambda t))^{p(\lambda t)}}{(y'(t))^{p(t)}} \cdot \frac{(y'(t))^{p(t)}}{(y'(t))^{p(\lambda t)}} \\
&= \liminf_{t\to\infty} \lambda^{\alpha-\gamma} \cdot \frac{a(\lambda t) \left(y'(\lambda t)\right)^{p(\lambda t)}}{a(t) \left(y'(t)\right)^{p(t)}} \left(y'(t)\right)^{p(t) - p(\lambda t)} \\
&\ge \liminf_{t\to\infty} \lambda^{1+\alpha-\gamma} \cdot \frac{\left(a(\lambda t) \left(y'(\lambda t)\right)^{p(\lambda t)}\right)'}{\left(a(t) \left(y'(t)\right)^{p(t)}\right)'} = \liminf_{t\to\infty} \lambda^{1+\alpha-\gamma} \cdot \frac{b(\lambda t) G(\lambda t, y(\lambda t))}{b(t) G(t, y(t))} \\
&= \lambda^{1+\alpha-\gamma+\delta} \liminf_{t\to\infty} \frac{(y(\lambda t))^{q(\lambda t)} L_G(y(\lambda t))}{(y(t))^{q(t)} L_G(y(t))} = \lambda^{1+\alpha-\gamma+\delta} \liminf_{t\to\infty} \frac{(y(\lambda t))^{q(\lambda t)}}{(y(t))^{q(t)}} \\
&= \lambda^{1+\alpha-\gamma+\delta} \liminf_{t\to\infty} \left(\frac{\lambda y'(\lambda t)}{y'(t)}\right)^{q(\lambda t)} (y(t))^{q(\lambda t) - q(t)} = \lambda^{1+\alpha-\gamma+\delta} m_*^\beta,
\end{aligned}$$

which implies $m_* \ge \lambda^\vartheta$. Similarly, we get $m^* \le \lambda^\vartheta$. Consequently, we have

$$\lim_{t\to\infty} \frac{y(\lambda t)}{y(t)} = \lambda^\vartheta,$$

i.e., $y \in \mathcal{RV}(\vartheta)$. □

**Remark 2.1.** The existence of $\widetilde{G}(t,z) = |z|^{q(t)} L_{\widetilde{G}}(|z|)$ satisfying (2.2) is guaranteed also when monotonicity condition (1.10) is not fulfilled. Indeed, according to [10, Proposition 1.7], if $f \in \mathcal{RV}(\rho)$ with $\rho > 0$, then there exists strictly increasing $g \in \mathcal{RV}(\rho)$ such that $f(t) \sim g(t)$ as $t \to \infty$, from which the claim follows.

Next, we establish the following asymptotic formula. Observe that monotonicity condition (2.2) is not required.

**Lemma 2.3.** *Suppose* (1.2) *and* (1.6)–(1.9) *hold, and that* $y \in \mathcal{RV}(\rho) \cap \mathcal{SIS}$, *where* $\rho$ *satisfies* (1.11) *with substituting* $\rho$ *into* $\vartheta$. *Then,* $\rho = \vartheta$. *In addition,*

$$y(t) \sim \left( \frac{1}{\alpha\vartheta - \alpha + \gamma} \cdot \frac{1}{\vartheta^{\alpha}} \cdot \frac{b(t)}{a(t)} t^{p(t)+1} L_G(t^{\vartheta}) \right)^{\frac{1}{p(t)-q(t)}}$$

*as* $t \to \infty$.

*Proof.* From equation (1.1), we have

$$y^{[1]}(t) = y^{[1]}(t_0) + \int_{t_0}^{t} b(s) G(s, y(s))\, ds \to \infty$$

as $t \to \infty$ because of $y \in \mathcal{SIS}$. Using Lemma 2.1, we have $by^q L_G(y) \in \mathcal{RV}(\delta + \rho\beta)$. Hence, applying the Karamata integration theorem, we obtain

$$y^{[1]}(t) \sim \frac{1}{\delta + \rho\beta + 1} t b(t) (y(t))^{q(t)} L_G(y(t)),$$

and so

$$y'(t) \sim \left( \frac{1}{\delta + \beta\rho + 1} \right)^{\frac{1}{\alpha}} \left( \frac{b(t)}{a(t)} t (y(t))^{q(t)} L_G(y(t)) \right)^{\frac{1}{p(t)}},$$

Similarly, we get

$$y(t) \sim \left( \frac{1}{\delta + \beta\rho + 1} \right)^{\frac{1}{\alpha}} \frac{1}{(1 - \gamma + \delta + \beta\rho)/\alpha + 1} \left( \frac{b(t)}{a(t)} t^{p(t)+1} L_G(y(t)) \right)^{\frac{1}{p(t)}} ((y(t))^{\frac{q(t)}{p(t)}}.$$

We recall that $\vartheta = (1 + \alpha - \gamma + \delta)/(\alpha - \beta)$ holds and (1.9) implies (2.3), therefore

$$\begin{aligned} y(t) &\sim \left( \left( \frac{1}{\delta + \beta\rho + 1} \right)^{\frac{1}{\alpha}} \frac{1}{(1 - \gamma + \delta + \beta\rho)/\alpha + 1} \left( \frac{b(t)}{a(t)} t^{p(t)+1} L_G(y(t)) \right)^{\frac{1}{p(t)}} \right)^{\frac{p(t)}{p(t)-q(t)}} \\ &\sim \left( \frac{1}{\delta + \beta\rho + 1} \left( \frac{1}{(1 - \gamma + \delta + \beta\rho)/\alpha + 1} \right)^{\alpha} \frac{b(t)}{a(t)} t^{p(t)+1} L_G(t^{\vartheta}) \right)^{\frac{1}{p(t)-q(t)}} \in \mathcal{RV}(\vartheta). \end{aligned}$$

Hence, we have $\rho = \vartheta$. From $\delta + \beta\vartheta + 1 = \alpha\vartheta - \alpha + \gamma$, the assertion holds. □

Let $u = y$ and $v = y^{[1]}$. Then, equation (1.1) is transformed into the equivalent system

$$u' = c(t)|v|^{r(t)} \operatorname{sgn} v, \quad v' = b(t) G(t, u), \tag{2.10}$$

where $c(t) = (1/a(t))^{1/p(t)}$ and $r(t) = 1/p(t)$. Here, we consider the auxiliary system

$$\hat{u}' = \hat{c}(t)|\hat{v}|^{r(t)} \operatorname{sgn} \hat{v}, \quad \hat{v}' = \hat{b}(t)G(t, \hat{u}), \tag{2.11}$$

where $\hat{c}(t)$ and $\hat{b}(t)$ are positive continuous functions. The following lemma is a kind of comparison theorem to systems (2.10) and (2.11).

**Lemma 2.4.** *Let $\hat{c}(t) \le c(t)$ and $\hat{b}(t) \le b(t)$ for $t \in [t_1, t_2]$ with $t_0 \le t_1 < t_2$. Suppose that*

$$z \mapsto G(\cdot, z) \text{ is monotone for } z > 0. \tag{2.12}$$

*Let $(u(t), v(t))$ and $(\hat{u}(t), \hat{v}(t))$ be solutions of systems (2.10) and (2.11), respectively, such that $u(t)$, $v(t)$, $\hat{u}(t)$, and $\hat{v}(t)$ are positive for $t \in (t_1, t_2]$. If $\hat{u}(t_1) \le u(t_1)$ and $\hat{v}(t_1) \le v(t_1)$, where at least one of these inequalities is strict, then $\hat{u}(t) < u(t)$ and $\hat{v}(t) < v(t)$ for $t \in (t_1, t_2]$.*

*Proof.* Since $v(t)$ and $\hat{v}(t)$ are positive, we see that $u'(t)$ and $\hat{u}'(t)$ are also positive because of the systems. Without loss of generality, we may assume $\hat{u}(t_1) < u(t_1)$.

We first show that $\hat{u}(t) < u(t)$ holds for $t \in (t_1, t_2]$. Suppose, toward a contradiction, that there exists $T \in (t_1, t_2]$ such that $\hat{u}(t) < u(t)$ for $t \in (t_1, T]$ and $\hat{u}(T) = u(T)$. From the first equation of system (2.10), we have

$$u(t) = u(t_1) + \int_{t_1}^{t} c(s) \left(v(s)\right)^{r(s)} \, ds$$

for $t \in (t_1, t_2]$. Similarly, from the first equation of system (2.11)

$$\hat{u}(t) = \hat{u}(t_1) + \int_{t_1}^{t} \hat{c}(s) \left(\hat{v}(s)\right)^{r(s)} \, ds$$

holds. In view of (2.10)–(2.12), $\hat{v}'(t) < v'(t)$ for $t \in [t_1, T)$, together with $\hat{v}(t_1) \le v(t_1)$, we have $\hat{v}(t) < v(t)$ for $t \in (t_1, T]$. Thus, we get

$$u(T) - \hat{u}(T) = (u(t_1) - \hat{u}(t_1)) + \int_{t_1}^{T} \left( c(s) \left(v(s)\right)^{r(s)} - \hat{c}(s) \left(\hat{v}(s)\right)^{r(s)} \right) ds > 0$$

which is a contradiction to $u(T) = \hat{u}(T)$.

We next show that $\hat{v}(t) < v(t)$ holds for $t \in (t_1, t_2]$. Suppose, toward a contradiction, that there exists $T \in (t_1, t_2]$ such that $\hat{v}(t) < v(t)$ for $t \in (t_1, T]$ and $\hat{v}(T) = v(T)$. In the same manner as the previous paragraph, the second equations of systems (2.10) and (2.11) lead

$$v(T) - \hat{v}(T) = (v(t_1) - \hat{v}(t_1)) + \int_{t_1}^{T} \left( b(s)G(s, u(s)) - \hat{b}(s)G(s, \hat{u}(s)) \right) ds > 0$$

because $\hat{u}(t) < u(t)$ holds for $t \in (t_1, T]$. This is a contradiction. □

In view of this lemma, the following lemma can be also proved.

**Lemma 2.5.** *Assume that (1.2), (1.7), and (1.10) hold. Let $(u(t), v(t))$ and $(\hat{u}(t), \hat{v}(t))$ be solutions of systems (2.10) and (2.11), respectively, such that $u(t)$, $v(t)$, $\hat{u}(t)$, and $\hat{v}(t)$ are positive for large $t$. Suppose that $u(t) \to \infty$ and $v(t) \to \infty$ as $t \to \infty$. If $\hat{c}(t) \asymp c(t)$ and $\hat{b}(t) \asymp b(t)$, then $\hat{u}(t) \asymp u(t)$ and $\hat{v}(t) \asymp v(t)$ as $t \to \infty$.*

*Proof.* Since $u, v, \hat{u}, \hat{v}$ are eventually positive, there exists $t_1 \geq t_0$ such that $u(t) > 0$, $v(t) > 0$, $\hat{u}(t) > 0$, $\hat{v}(t) > 0$ for $t \geq t_1$, and $0 < m_1 < m_2$ such that

$$m_1^{\frac{1}{\sqrt{\beta}}} u(t_1) \leq \hat{u}(t_1) \leq m_2^{\frac{1}{\sqrt{\beta}}} u(t_1), \quad m_1^{\sqrt{\alpha}} v(t_1) \leq \hat{v}(t_1) \leq m_2^{\sqrt{\alpha}} v(t_1). \tag{2.13}$$

Denote $\Gamma_1(t) := 1/\sqrt{\beta} - r(t)\sqrt{\alpha}$, $\Gamma_2(t) := \sqrt{\alpha} - q(t)/\sqrt{\beta}$. Then, thanks to the homogeneity condition,

$$\Gamma_1(t) = \frac{1}{\sqrt{\beta}}\left(1 - r(t)\sqrt{\alpha\beta}\right) \geq \frac{1}{2\sqrt{\beta}}\left(1 - \sqrt{\frac{\beta}{\alpha}}\right) > 0,$$

$$\Gamma_2(t) = \sqrt{\alpha}\left(1 - \frac{q(t)}{\sqrt{\alpha\beta}}\right) \geq \frac{\sqrt{\alpha}}{2}\left(1 - \sqrt{\frac{\beta}{\alpha}}\right) > 0$$

for large $t$, say again $t \geq t_1$. Since $L_G \in \mathcal{SV}$ and $u(t) \to \infty$ as $t \to \infty$, we obtain $L_G\left(m_i^{\frac{1}{\sqrt{\beta}}} u(t)\right)/L_G(u(t)) \to 1$ as $t \to \infty$, $i = 1, 2$. Hence, in view of $\hat{c}(t) \asymp c(t)$ and $\hat{b}(t) \asymp b(t)$ as $t \to \infty$, there exist $0 < k_1 < k_2$ and $t_2 \geq t_0$ such that

$$\begin{gathered} k_1^{\Gamma_1(t)} c(t) \leq \hat{c}(t) \leq k_2^{\Gamma_1(t)} c(t), \\ k_1^{\Gamma_2(t)} b(t) \frac{L_G\left(m_1^{\frac{1}{\sqrt{\beta}}} u(t)\right)}{L_G(u(t))} \leq \hat{b}(t) \leq k_2^{\Gamma_2(t)} b(t) \frac{L_G\left(m_2^{\frac{1}{\sqrt{\beta}}} u(t)\right)}{L_G(u(t))} \end{gathered} \tag{2.14}$$

for $t \geq t_2$. Without loss of generality, we can take $t_2 = t_1$ and $k_i = m_i$, $i = 1, 2$. Moreover, $t_1$ can be taken such that $z \mapsto G(\cdot, z)$ is increasing for $z \geq \inf_{t \geq t_1} m_1^{\frac{1}{\sqrt{\beta}}} u(t)$. It is easy to see that $(\tilde{u}, \tilde{v}) := \left(m_2^{\frac{1}{\sqrt{\beta}}} u, m_2^{\sqrt{\alpha}} v\right)$ is a solution the system

$$\tilde{u}' = m_2^{\Gamma_1(t)} c(t) \tilde{v}^{r(t)}, \quad \tilde{v}' = m_2^{\Gamma_2(t)} b(t) \frac{L_G(u)}{L_G(\tilde{u})} G(t, \tilde{u}).$$

Since $\tilde{u}(t_1) \geq \hat{u}(t_1)$ and $\tilde{v}(t_1) \geq \hat{v}(t_1)$ by (2.13), in view of (2.14), using Lemma 2.4, we get $\tilde{u}(t) \geq \hat{u}(t)$ and $\tilde{v}(t) \geq \hat{v}(t)$ for $t \geq t_1$. Similarly we obtain $m_1^{\frac{1}{\sqrt{\beta}}} u(t) \leq \hat{u}(t)$ and $m_1^{\sqrt{\alpha}} v(t) \leq \hat{v}(t)$ for $t \geq t_1$. Consequently, $u(t) \asymp \hat{u}(t)$ and $v(t) \asymp \hat{v}(t)$ as $t \to \infty$. □

**Remark 2.2.** The conditions that $u(t) \to \infty$ and $v(t) \to \infty$ as $t \to \infty$ imply the corresponding solution $y(t)$ of (1.1) belongs to $\mathcal{SIS}$.

According to Lemma 2.5, we can obtain the following lemma immediately for the pair of equations (1.1) and

$$(\hat{a}(t) F(t, \hat{y}'))' = \hat{b}(t) G(t, \hat{y}), \tag{2.15}$$

where $\hat{a}(t)$ is a positive continuous function.

**Lemma 2.6.** *Assume that* (1.2), (1.7), *and* (1.10) *hold. Let* $y \in \mathcal{SIS}$ *and* $\hat{y}(t)$ *be a solution of equation* (2.15) *satisfying* $\hat{y}(t) > 0$ *and* $\hat{y}^{[1]}(t) > 0$ *for large* $t$. *If* $\hat{a}(t) \asymp a(t)$ *and* $\hat{b}(t) \asymp b(t)$, *then* $\hat{y}(t) \asymp y(t)$ *and* $\hat{y}^{[1]}(t) \asymp y^{[1]}(t)$ *as* $t \to \infty$.

Hence, we can show the following lemma.

**Lemma 2.7.** *Assume that* (1.2), (1.7), *and* (1.10) *hold. Let* $y \in \mathcal{SIS}$ *and* $\hat{y}(t)$ *be a solution of equation* (2.15) *satisfying* $\hat{y}(t) > 0$ *and* $\hat{y}^{[1]}(t) > 0$ *for large* $t$. *If* $a(t) \sim \hat{a}(t)$ *and* $b(t) \sim \hat{b}(t)$, *then* $y(t) \sim \hat{y}(t)$ *and* $y^{[1]}(t) \sim \hat{y}^{[1]}(t)$ *as* $t \to \infty$.

*Proof.* Since $a(t) \sim \hat{a}(t)$ and $b(t) \sim \hat{b}(t)$, it is easy to show that $\hat{a}(t) \asymp a(t)$ and $\hat{b}(t) \asymp b(t)$ as $t \to \infty$. In view of Lemma 2.6, we have $\hat{y}(t) \asymp y(t)$, $\hat{y}^{[1]}(t) \asymp y^{[1]}(t)$, and $\hat{y}'(t) \asymp y'(t)$ as $t \to \infty$. Consequently, we put

$$L_* = \liminf_{t\to\infty} \frac{\hat{y}(t)}{y(t)} \in (0,\infty), \quad L^* = \limsup_{t\to\infty} \frac{\hat{y}(t)}{y(t)} \in (0,\infty).$$

Similarly as (2.7), thanks to the uniform convergence theorem, we have

$$\left|\frac{L_G(\hat{y}(t))}{L_G(y(t))} - 1\right| = \left|\frac{L_G\left(\frac{\hat{y}(t)}{y(t)}y(t)\right)}{L_G(y(t))} - 1\right| \le \sup_{\xi\in[L_*/2,2L^*]}\left|\frac{L_G\left(\xi y(t)\right)}{L_G(y(t))} - 1\right| \to 0$$

as $t \to \infty$.

By the l'Hôpital rule, since $y \in \mathcal{SIS}$, we have

$$L_* \ge \liminf_{t\to\infty} \frac{\hat{y}'(t)}{y'(t)}.$$

Using the l'Hôpital rule again, together with $a(t) \sim \hat{a}(t)$ and $b(t) \sim \hat{b}(t)$ as $t \to \infty$, we see that

$$\begin{aligned} L_*^{\alpha} &\ge \liminf_{t\to\infty} \frac{\hat{a}(t)(\hat{y}'(t))^{p(t)}}{a(t)(y'(t))^{p(t)}} \ge \liminf_{t\to\infty} \frac{(\hat{a}(t)F(t,\hat{y}'(t)))'}{(a(t)F(t,y'(t)))'} \\ &= \liminf_{t\to\infty} \frac{\hat{b}(t)G(t,\hat{y}(t))}{b(t)G(t,y(t))} = \liminf_{t\to\infty} \frac{\hat{b}(t)(\hat{y}(t))^{q(t)}}{b(t)(y(t))^{q(t)}} \cdot \frac{L_G(\hat{y}(t))}{L_G(y(t))} \\ &= \liminf_{t\to\infty} \frac{\hat{b}(t)(\hat{y}(t))^{q(t)}}{b(t)(y(t))^{q(t)}} = L_*^{\beta}. \end{aligned}$$

Since $\alpha > \beta$, this implies $L_* \ge 1$. Similarly, we obtain $L^* \le 1$. Hence, we have

$$\lim_{t\to\infty} \frac{\hat{y}(t)}{y(t)} = 1,$$

that is, $y(t) \sim \hat{y}(t)$ as $t \to \infty$. From the previous computations, it is easy to see that

$$\liminf_{t\to\infty} \frac{(\hat{a}(t)F(t,\hat{y}'(t))'}{(a(t)F(t,y'(t)))'} = 1 = \limsup_{t\to\infty} \frac{(\hat{a}(t)F(t,\hat{y}'(t))'}{(a(t)F(t,y'(t)))'},$$

that is to say, $y^{[1]}(t) \sim \hat{y}^{[1]}(t)$ as $t \to \infty$. □

**Remark 2.3.** From Lemma 2.7, if $y \in \mathcal{IS}$ and $x \in \mathcal{SIS}$, then $y(t) \sim x(t)$ and $y^{[1]}(t) \sim x^{[1]}(t)$ as $t \to \infty$. Hence, in particular, $\mathcal{SIS} \neq \emptyset$ implies $\mathcal{IS} = \mathcal{SIS}$.

We show the nonexistence of singular solutions of second kind.

**Lemma 2.8.** *Assume that* (1.2) *and* (1.7) *hold. Then, equation* (1.1) *has no singular solutions of the second kind.*

*Proof.* Write equation (1.1) as system (2.10). Integrating from $t_0$ to $t$, we obtain

$$\begin{aligned} u(t) &= u(t_0) + \int_{t_0}^{t} c(s)|v(s)|^{r(s)} \operatorname{sgn} v(s)\, ds, \\ v(t) &= v(t_0) + \int_{t_0}^{t} b(s)G(s, u(s))\, ds. \end{aligned} \tag{2.16}$$

Let $(u, v)$ be a solution of (2.10) (and thus of (2.16)) such that its maximal interval of existence is $[t_0, \tau)$. If $\tau < \infty$, then $|u|, |v|$ are unbounded on $[t_0, \tau)$. Indeed, if $|v|$ is bounded, then $|u|$ is bounded as well since

$$|u(t)| \le |u(t_0)| + \int_{t_0}^{t} c(s)|v(s)|^{r(s)}\, ds \le |u(t_0)| + \sup_{s\in[t_0,\tau]} |v(s)| \int_{t_0}^{\tau} c(s)\, ds.$$

Moreover, from (2.16), we get that $u$ is continuous on $[t_0, \tau]$, which is a contradiction with the fact that $[t_0, \tau)$ is the maximal interval of existence. A similar conclusion can be obtained when we assume boundedness of $v$. Therefore, $|u|, |v|$ are unbounded on $[t_0, \tau)$ when $\tau < \infty$. Since we focus on eventually positive solutions and in view of the positivity of the coefficients in (2.10), which imply eventual monotonicity of solutions, we can take $\lim_{t\to\tau-} u(t) = \lim_{t\to\tau-} v(t) = \infty$. Recall that if $h_i \in \mathcal{RV}(\vartheta_i)$, $\vartheta_1 < \vartheta_2$, then $\lim_{t\to\infty} h_2(t)/h_1(t) = \infty$ and, in particular, $h_1(t) \le h_2(t)$ for large $t$. Hence, in view of (1.7), there exists $t_1 \in [t_0, \tau)$ and $\varepsilon > 0$ such that $G(t, u(t)) \le (u(t))^{\tilde{q}}$ for $t \in [t_1, \tau)$, where

$$\underline{\alpha} := \inf_{t\in[t_0,\infty)} p(t) > \varepsilon + \sup_{t\in[t_0,\infty)} q(t) =: \tilde{q}. \tag{2.17}$$

At the same time, we can take $A \ge 1$ and $t_1$ such that $u(t), v(t) \ge 1$, $A(v(t))^{\frac{1}{\underline{\alpha}}} \ge u(t_1)$, and $A^{\tilde{q}}(v(t))^{\frac{\tilde{q}}{\underline{\alpha}}} \ge v(t_1)$ for $t \ge t_1$. Consequently,

$$\begin{aligned} v(t) &\le v(t_1) + \int_{t_1}^{t} b(s) \left( u(t_1) + \int_{t_1}^{s} c(\sigma)(v(\sigma))^{r(\sigma)}\, d\sigma \right)^{\tilde{q}} ds \\ &\le v(t_1) + \int_{t_1}^{t} b(s) \left( u(t_1) + (v(s))^{\frac{1}{\underline{\alpha}}} \int_{t_1}^{s} c(\sigma)\, d\sigma \right)^{\tilde{q}} ds \\ &\le (v(t))^{\frac{\tilde{q}}{\underline{\alpha}}} \left( 1 + \int_{t_1}^{t} b(s) \left( 1 + \int_{t_1}^{s} c(\sigma)\, d\sigma \right)^{\tilde{q}} ds \right) \end{aligned}$$

for $t \ge t_1$. Hence,

$$\begin{aligned} (v(t))^{1-\frac{\tilde{q}}{\underline{\alpha}}} &\le 1 + \int_{t_1}^{t} b(s) \left( 1 + \int_{t_1}^{s} c(\sigma)\, d\sigma \right)^{\tilde{q}} ds \\ &\le 1 + \int_{t_1}^{T} b(s) \left( 1 + \int_{t_1}^{s} c(\sigma)\, d\sigma \right)^{\tilde{q}} ds < \infty, \end{aligned}$$

which is a contradiction. □

We finish this section with the proof of the main theorem, i.e., Theorem 1.1.

*Proof of Theorem 1.1.* From Lemma 2.2, we can find $x \in \mathcal{SIS} \cap \mathcal{RV}(\vartheta)$. Let $y \in \mathcal{IS}$. Then, in view of Lemma 2.7 and Remark 2.3, we have $y \in \mathcal{SIS}$ and $y(t) \sim x(t)$ as $t \to \infty$. Since $x \in \mathcal{RV}(\vartheta)$ and $y(t) \sim x(t)$ as $t \to \infty$, we see that $y \in \mathcal{RV}(\vartheta)$. From Lemma 2.3, we get that $y(t)$ satisfies

$$y(t) \sim \left( \frac{1}{\alpha\vartheta - \alpha + \gamma} \cdot \frac{1}{\vartheta^\alpha} \cdot \frac{b(t)}{a(t)} t^{p(t)+1} L_G(t^\vartheta) \right)^{\frac{1}{p(t)-q(t)}}$$

as $t \to \infty$. □

## 3. Necessity

In this section, we provide some necessary conditions for the existence of strongly increasing solutions.

**Theorem 3.1.** *Assume that* (1.2) *holds. If* $\mathcal{SIS} \neq \emptyset$, *then*

$$\int_{t_0}^{\infty} \left( \frac{1}{a(t)} \int_{t_0}^{s} b(\tau)\, d\tau \right)^{\frac{1}{p(s)}} ds = \infty. \tag{3.1}$$

*Proof.* Let $y \in \mathcal{SIS}$. Then, there exists $T \geq t_0$ such that $y(t) > 0$ and $y'(t) > 0$ for $t \geq T$. From equation (1.3), we get

$$y^{[1]}(t) = y^{[1]}(T) + \int_T^t b(s) G(s, y(s))\, ds \tag{3.2}$$

and

$$y(t) = y(T) + \int_T^t \left( \frac{y^{[1]}(T)}{a(s)} + \frac{1}{a(s)} \int_T^s b(\tau) G(\tau, y(\tau))\, d\tau \right)^{\frac{1}{p(s)}} ds \tag{3.3}$$

for $t \geq T$. Since $y^{[1]}(t) \to \infty$ as $t \to \infty$ because of $y \in \mathcal{SIS}$, together with (3.2), we have

$$\int_T^\infty b(s) G(s, y(s))\, ds = \infty.$$

Taking into account also that $y(t) \to \infty$ as $t \to \infty$ and utilizing (3.3), we can find $A > 0$ such that

$$y(t) \leq A \int_T^t \left( \frac{1}{a(s)} \int_T^s b(\tau) G(\tau, y(\tau))\, d\tau \right)^{\frac{1}{p(s)}} ds$$

for $t \geq 2T$. Without loss of generality, we take $T$ so large that $G(t, y(t)) \leq (y(t))^{\tilde{q}}$ and $y(t) \geq 1$ for $t \geq T$, where $\tilde{q}$ is as in (2.17). Therefore, in view of monotonicity of $y$, we have

$$\begin{aligned} y(t) &\leq A \int_T^t \left( \frac{1}{a(s)} \int_T^s b(\tau) (y(\tau))^{\tilde{q}}\, d\tau \right)^{\frac{1}{p(s)}} ds \\ &\leq A (y(t))^{\frac{\tilde{q}}{\alpha}} \int_T^t \left( \frac{1}{a(s)} \int_T^s b(\tau)\, d\tau \right)^{\frac{1}{p(s)}} ds \end{aligned}$$

for $t \geq 2T$. Therefore

$$A\int_T^t \left(\frac{1}{a(s)}\int_T^s b(\tau)\,d\tau\right)^{\frac{1}{p(s)}} ds \geq (y(t))^{1-\frac{\tilde{q}}{\alpha}} \to \infty$$

as $t \to \infty$. Hence, we have (3.1). □

Let $y(t)$ be a solution of equation (1.3), and let $u(t) := y^{[1]}(t)$. Then, $u(t)$ is a solution of the reciprocal equation

$$\left(\left(\frac{1}{b(t)}\right)^{\frac{1}{q(t)}} |u'|^{\frac{1}{q(t)}} \operatorname{sgn} u'\right)' = \left(\frac{1}{a(t)}\right)^{\frac{1}{p(t)}} |u|^{\frac{1}{p(t)}} \operatorname{sgn} u. \tag{3.4}$$

Since $u(t) = y^{[1]}(t)$ and $u^{[1]}(t) = y(t)$, $y \in \mathcal{SIS}$ implies that $u(t)$ is a strongly increasing solution of equation (3.4). Hence, we can apply Theorem 3.1 to equation (3.4), and obtain the following corollary.

**Corollary 3.1.** *Assume that* (1.2) *holds. If* $\mathcal{SIS} \neq \emptyset$, *then*

$$\int_{t_0}^{\infty} b(s)\left(\int_{t_0}^{s}\left(\frac{1}{a(\tau)}\right)^{\frac{1}{p(\tau)}} d\tau\right)^{q(s)} ds = \infty. \tag{3.5}$$

Using Theorem 3.1 and Corollary 3.1, we can also show the following proposition. Note that part (i) in fact shows the sharpness of the second condition in (1.11), while part (iii) shows the sharpness of the first condition in (1.11).

**Proposition 3.1.** *Assume that* (1.2) *holds,* $\mathcal{SIS} \neq \emptyset$, $a \in \mathcal{RV}(\gamma)$, *and* $b \in \mathcal{RV}(\delta)$. *Then, the following statements hold.*

*(i) If* $\delta > -1$, *then* $1+\alpha-\gamma+\delta \geq 0$.
*(ii) If* $\delta \leq -1$, *then* $\alpha \geq \gamma$.
*(iii) If* $\alpha > \gamma$, *then* $\delta+\beta+1-\beta\gamma/\alpha \geq 0$.
*(iv) If* $\alpha \leq \gamma$, *then* $\delta \geq -1$.

*Proof.* (i) Let $\delta > -1$. Suppose, toward a contradiction, that $1+\alpha-\gamma+\delta < 0$. Then, we see that $(1-\gamma+\delta)/\alpha < -1$. Thanks to the Karamata integration theorem, we can find $K > 0$ such that

$$\int_{t_0}^t \left(\frac{1}{a(s)}\int_{t_0}^s b(\tau)\,d\tau\right)^{\frac{1}{p(s)}} ds \leq K\int_{t_0}^t \left(\frac{sb(s)}{a(s)}\right)^{\frac{1}{p(s)}} ds.$$

Since $(tb(t)/a(t))^{1/p(t)} \in \mathcal{RV}((1-\gamma+\delta)/\alpha)$, which index is less than $-1$. Hence, the integral on the right-hand side converges, which contradicts to (3.1) shown in Theorem 3.1.

(ii) Let $\delta \leq -1$. Then, we have $\int_{t_0}^t b(s)\,ds \in \mathcal{SV}$, and therefore,

$$\left(\frac{1}{a(t)}\int_{t_0}^t b(s)\,ds\right)^{\frac{1}{p(t)}} \in \mathcal{RV}\left(-\frac{\gamma}{\alpha}\right).$$

Suppose by a contradiction that $\alpha < \gamma$. Then, the integration converges, which is a contradiction.

(iii) Let $\alpha > \gamma$. Suppose, toward a contradiction, that $\delta+\beta+1-\beta\gamma/\alpha < 0$. Then, we see that $\delta + (1-\gamma/\alpha)\beta < -1$. In the same manner as (i), we can find $K > 0$ such that

$$\int_{t_0}^{t} b(s) \left( \int_{t_0}^{s} \left( \frac{1}{a(\tau)} \right)^{\frac{1}{p(\tau)}} d\tau \right)^{q(s)} ds \leq K \int_{t_0}^{t} b(s) \left( s \left( \frac{1}{a(s)} \right)^{\frac{1}{p(s)}} \right)^{q(s)} ds.$$

Since $b(t)(t(1/a(t))^{1/p(t)})^{q(t)} \in \mathcal{RV}(\delta+(1-\gamma/\alpha)\beta)$, where the index is less than $-1$. Hence, the integral on the right-hand side converges, which contradicts to (3.5) shown in Corollary 3.1.

(iv) Let $\alpha \leq \gamma$. Then, we have $\int_{t_0}^{t}(1/a(s))^{1/p(s)}\, ds \in \mathcal{SV}$, and therefore,

$$b(t) \left( \int_{t_0}^{t} \left( \frac{1}{a(s)} \right)^{\frac{1}{p(s)}} ds \right)^{q(t)} \in \mathcal{RV}\left(\delta\right).$$

Suppose by a contradiction that $\delta < -1$. Then, the integral converges, which is a contradiction. □

## 4. Examples and remarks

In this section, we first provide some examples of $\varphi_i(t)$ satisfying (1.6).

**Proposition 4.1.** *If $\varphi(t) = o(1/\log t)$ as $t \to \infty$, then $\varphi(t)$ satisfies (1.6) for any $\lambda > 0$.*

*Proof.* Since $\log \in \mathcal{SV}$, $\log t/\log \lambda t \to 1$ as $t \to \infty$. From the assumption, we have $\varphi(t)\log t \to 0$ and

$$\varphi(\lambda t)\log t = \varphi(\lambda t)\log \lambda t \cdot \frac{\log t}{\log \lambda t} \to 0$$

as $t \to \infty$, which imply $(\varphi(\lambda t) - \varphi(t))\log t \to 0$ as $t \to \infty$. □

*Example 4.1.* From the previous proposition, we see that, in particular, functions $\varphi \in \mathcal{RV}(\omega)$ with $\omega < 0$ or $\varphi$ which tends exponentially to 0 satisfy (1.6). Similarly, if $\varphi(t) = \pm 1/\log t$, then $\varphi(t)$ satisfies (1.6) for any $\lambda > 0$. Indeed, $\log \in \mathcal{SV}$ leads

$$\left( \frac{1}{\log \lambda t} - \frac{1}{\log t} \right) \log t = \frac{\log t}{\log \lambda t} - 1 \to 0$$

as $t \to \infty$.

In view of the following proposition, we can take $\varphi(t)$ which tends to 0 even more slowly than $1/\log t$.

**Proposition 4.2.** *Let*

$$\varphi(t) = \pm \int_t^\infty \frac{L(s)}{s}\, ds,$$

*where $L \in \mathcal{SV}$ and $L(t) = o(1/\log t)$. Then, $\varphi(t)$ satisfies (1.6) for any $\lambda > 0$.*

*Proof.* From the assumption, we have

$$\begin{aligned}(\varphi(\lambda t) - \varphi(t)) \log t &= \pm \log t \int_{\lambda t}^{t} \frac{L(s)}{s}\, ds = \pm \log t \int_\lambda^1 \frac{L(tx)}{tx} \cdot t\, dx \\ &= \pm L(t) \log t \int_\lambda^1 \frac{L(tx)}{L(t)} \cdot \frac{1}{x}\, dx.\end{aligned}$$

Here, according to the uniform convergence theorem, we see that $L \in \mathcal{SV}$ leads to

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1$$

uniformly on any compact subset of $(0, \infty)$. Hence, we have

$$\lim_{t\to\infty} \int_\lambda^1 \frac{L(tx)}{L(t)} \cdot \frac{1}{x}\, dx = \int_\lambda^1 \frac{1}{x}\, dx = -\log \lambda,$$

and therefore, we obtain $(\varphi(\lambda t) - \varphi(t)) \log t \to 0$ as $t \to \infty$. □

*Example 4.2.* In the case of $\varphi(t) = 1/\log \log t$, we put

$$L(t) = \frac{1}{(\log \log t)^2 \log t}.$$

Then, we have

$$\varphi'(t) = -\frac{1}{(\log \log t)^2 t \log t} = -\frac{L(t)}{t}$$

and $L \in \mathcal{SV}$. Moreover, since

$$(\log t) L(t) = \frac{1}{(\log \log t)^2} \to 0$$

as $t \to \infty$, we get $L(t) = o(1/\log t)$. From Proposition 4.2, we see that $\varphi(t)$ satisfies (1.6) for any $\lambda > 0$.

*Example 4.3.* In the case of $\varphi(t) = (\log t)^\eta$ $(\eta < 0)$, we put

$$L(t) = \eta (\log t)^{\eta - 1}.$$

Then, we have

$$\varphi'(t) = \frac{\eta (\log t)^{\eta-1}}{t} = \frac{L(t)}{t}$$

and $L \in \mathcal{SV}$. Moreover, since

$$(\log t) L(t) = \eta (\log t)^\eta \to 0$$

as $t \to \infty$, we get $L(t) = o(1/\log t)$. From Proposition 4.2, we see that $\varphi(t)$ satisfies (1.6) for any $\lambda > 0$.

A closer examination of the computations in Proposition 4.2 show the condition $L \in \mathcal{SV}$ can be relaxed to the condition that $L(t)$ is regularly bounded, i.e.,

$$\limsup_{t\to\infty} \frac{L(\lambda t)}{L(t)} < \infty$$

for any $\lambda > 0$. Note that the uniform convergence holds also for regularly bounded functions, see [2].

**Proposition 4.3.** *Let*

$$\varphi(t) = \pm \int_t^\infty \frac{L(s)}{s}\, ds,$$

*where $L(t)$ is regularly bounded and $L(t) = o(1/\log t)$. Then, $\varphi(t)$ satisfies (1.6) for any $\lambda > 0$.*

*Proof.* Proceeding in the same argument as the proof of Proposition 4.2, we have

$$(\varphi(\lambda t) - \varphi(t)) \log t = \pm L(t) \log t \int_\lambda^1 \frac{L(tx)}{L(t)} \cdot \frac{1}{x}\, dx.$$

From the uniform convergence theorem, we see that

$$\int_\lambda^1 \frac{L(tx)}{L(t)} \cdot \frac{1}{x}\, dx$$

is bounded. Since $L(t) = o(1/\log t)$, we obtain $(\varphi(\lambda t) - \varphi(t)) \log t \to 0$ as $t \to \infty$. □

We introduce an another class of functions, which corresponds to the de Haan theory. For $g \in \mathcal{SV}$, we define the class $\Pi_g$ be the set of measurable function $f$ such that

$$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{g(t)} = c \log \lambda$$

for any $\lambda > 0$ and for some $c$ called $g$-index of $f$. We note that $g$ is called the auxiliary function of $f$, and the de Haan class $\Pi$ is the set of $f$ for which there exists $g \in \mathcal{SV}$ such that $f \in \Pi_g$ with non-zero $g$-index. It is known that the class of positive functions $f \in \Pi$ forms a proper subset of $\mathcal{SV}$ (see [2]). We can easily show the following proposition.

**Proposition 4.4.** *Let $\varphi \in \Pi_g$ be a positive function with $g(t) = o(1/\log t)$. Then $\varphi(t)$ satisfies (1.6) for any $\lambda > 0$.*

We next consider the alternative formulation of the condition $y \in \mathcal{IS}$ in Theorem 1.1 by using the initial conditions.

**Proposition 4.5.** *Let $z \mapsto G(\cdot, z)$ be monotone for $z \geq y_0$. If $y(t)$ is a solution of equation (1.1) satisfying*

$$y(t_0) = y_0 \geq 0, \quad y'(t_0) = y_1 \geq 0, \quad \max\{y_0, y_1\} > 0, \tag{4.1}$$

*then $y \in \mathcal{IS}$.*

*Proof.* We first consider the case when $y_0 > 0$. Thanks to the continuity of $y(t)$ and equation (1.3), we can find $\varepsilon > 0$ such that $y(t) > 0$ for $t \in [t_0, t_0 + \varepsilon]$. Suppose, toward a contradiction, that there exists $T > t_0$ such that $y(t) > 0$ for $t \in [t_0, T)$ and $y(T) = 0$. Then, from equation (1.3), we have

$$y'(t) = \left( \frac{a(t_0) y_1^{p(t_0)}}{a(t)} + \frac{1}{a(t)} \int_{t_0}^{t} b(s) G(s, y(s))\, ds \right)^{\frac{1}{p(t)}} > 0 \tag{4.2}$$

for $t \in (t_0, T]$ and

$$y(t) = y_0 + \int_{t_0}^{t} \left( \frac{a(t_0) y_1^{p(t_0)}}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} b(\tau) G(s, y(s))\, d\tau \right)^{\frac{1}{p(s)}} ds \tag{4.3}$$

for $t \in [t_0, T]$. Thus, we obtain

$$0 = y(T) = y_0 + \int_{t_0}^{T} \left( \frac{a(t_0) y_1^{p(t_0)}}{a(s)} + \frac{1}{a(s)} \int_{t_0}^{s} b(\tau) G(s, y(s))\, d\tau \right)^{\frac{1}{p(s)}} ds > 0,$$

which is a contradiction. Hence, $y(t) > 0$ for $t \geq t_0$. Moreover, according to (4.2), we have $y'(t) > 0$ for $t > t_0$.

In the case when $y_0 = 0$, we have $y_1 > 0$, and therefore, we can find $\varepsilon > 0$ such that $y(t) > 0$ for $t \in (t_0, t_0 + \varepsilon]$. In the same way as the previous paragraph, we prove that $y(t) > 0$ for $t > t_0$ and $y'(t) > 0$ for $t \geq t_0$. □

From Proposition 4.5, we can change the condition $y \in \mathcal{IS}$ in Theorem 1.1 as follows: the solution $y(t)$ of equation (1.3) satisfying (4.1). Instead of (1.10) we assume (2.12). In view of this alternative formulation, once we have a solution given by the initial conditions (4.1), we can provide its precise asymptotic description.

We finally propose open problems: (i) In this research, we focused only on increasing solutions. It is an open problem to show the existence of (strongly) decreasing solutions and to give asymptotic formula for them. (ii) In this paper, we assume the subhomogeneity condition (1.7). Under this condition, we showed the nonexistence of singular solutions of the second kind and studied the asymptotic behavior of the solutions. It is an open problem if equation (1.3) has singular solutions of the first kind under the superhomogeneity condition

$$\inf_{t \in [t_0, \infty)} p(t) > \sup_{t \in [t_0, \infty)} q(t),$$

and to obtain asymptotic formula for solutions.

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