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Static solutions for Choquard equations with Coulomb potential and upper critical growth

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Abstract

This paper focuses on static solutions for the following Choquard equation with zero mass and Coulomb potential

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \mu|u|^{p-2}u + (I_\alpha * |u|^{\alpha+3})|u|^{\alpha+1}u, \quad x \in \mathbb{R}^3,$$

where $\mu > 0$, $\frac{18}{7} < p \le 6$, $\alpha \in (0, 3)$, $\alpha + 3$ is the upper critical exponent in the sense of the Hardy–Littlewood–Sobolev inequality, $I_\alpha : \mathbb{R}^3 \to \mathbb{R}$ is the Riesz potential, and $\frac{1}{4\pi|x|}$ is the Coulomb potential. By carefully analyzing the intricate interplay between the power and Coulomb terms, we establish three types of variational geometries of the problem and prove the following existence results based on the behavior of p:

(1) the existence of two solutions, one being a local minimizer and the other of mountain-pass type, for an explicit range $0 < \mu <$ Const. when $\frac{18}{7} < p < 3$;
(2) the existence of a positive solution if μ takes some particular value when $p = 3$;
(3) the existence of a ground state solution for all $\mu > 0$ when $4 < p < 6$, and for two explicit ranges $\mu >$ Const. when $3 < p < 4$ and $p = 4$.

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Furthermore, we obtain a non-existence result for the case $p = 6$. Particularly, we identify different compactness thresholds for above three cases, and introduce three types of test functions to control the corresponding minimax levels to be less than prescribed thresholds, thereby overcoming the loss of compactness arising from the nonlocal critical term. The derivation of these strict inequalities is a novel contribution and constitutes one of the noteworthy highlights of this work, which is available and new for the limiting Sobolev critical problem as $\alpha \to 0$. We believe that the underlying ideas have potential for future development and can be applied to a broader range of variational problems with critical growth.

Mathematics Subject Classification 35J20 · 35J62 · 35Q55

1 Introduction

In this paper, we consider the following upper critical Choquard equation with zero mass and Coulomb potential:

$$- \Delta u + \left(\frac{1}{4\pi |x|} * u^2 \right) u = \mu |u|^{p-2} u + (I_\alpha * |u|^{\alpha+3}) |u|^{\alpha+1} u, \quad x \in \mathbb{R}^3, \tag{1.1}$$

where $\mu > 0$, $\frac{18}{7} < p \le 6$, $\alpha \in (0, 3)$, $I_\alpha \colon \mathbb{R}^3 \to \mathbb{R}$ is the Riesz potential defined by

$$I_\alpha(x) = \frac{\Gamma\left(\frac{3-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) 2^\alpha \pi^{\frac{3}{2}} |x|^{3-\alpha}} := \frac{\mathcal{K}_\alpha}{|x|^{3-\alpha}}, \quad x \in \mathbb{R}^3 \backslash \{0\}, \tag{1.2}$$

and $\frac{1}{4\pi|x|}$ is the *Coulomb potential*, which coincides with the Riesz potential I_2. Given the fact that the Coulomb potential is the fundamental solution of the operator $-\Delta$, it follows that solutions of (1.1) correspond to solutions (u, ϕ) of the nonlinear system

$$\begin{cases} -\Delta u + \phi u = \mu |u|^{p-2} u + (I_\alpha * |u|^{\alpha+3}) |u|^{\alpha+1} u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3. \end{cases}$$

A notable feature of this problem is that the local linearized operator at zero involves only the Laplacian operator. Following the pioneering work [4] by Berestycki and Lions, we can also say that this is a *zero mass problem*, whose solutions are called *static solutions*. Here, $\alpha + 3$ is called the *upper critical exponent* in the sense of the Hardy–Littlewood–Sobolev inequality, due to the following estimate:

$$\begin{aligned} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x)|^{\alpha+3} |v(y)|^{\alpha+3}}{|x-y|^{3-\alpha}} \mathrm{d}x \mathrm{d}y &\le 4^{\frac{\alpha}{3}} \pi^{\frac{9-4\alpha}{6}} \frac{\Gamma\left(\frac{\alpha}{2}\right)}{\Gamma\left(\frac{3+\alpha}{2}\right)} \|u\|_6^{\alpha+3} \|v\|_6^{\alpha+3} \\ &:= \mathcal{L}_\alpha \|u\|_6^{\alpha+3} \|v\|_6^{\alpha+3} < +\infty, \quad \forall\, u, v \in \mathcal{D}^{1,2}(\mathbb{R}^3). \end{aligned} \tag{1.3}$$

1.1 Research motivation and main difficulty

The study of (1.1) stems from the following Brezis–Nirenberg type problem for the Choquard equation with upper critical exponent:

$$-\Delta u+\omega u=\mu|u|^{p-2}u+(I_\alpha*|u|^{\alpha+3})|u|^{\alpha+1}u,\quad x\in\mathbb{R}^3, \tag{1.4}$$

where ω corresponds to the phase of the standing wave for the time-dependent equation, if $\omega=0$, its solutions correspond to *static solutions* (not periodic ones). Choquard equations arise in various fields of mathematical physics, such as the description of the quantum theory of a polaron at rest by Pekar [27] in 1954 and the modelling of an electron trapped in its own hole in 1976 in the work of Choquard [21]. It was also treated as a certain approximation to Hartree–Fock theory of one-component plasma. Mathematically, the study of Choquard equations goes back to the seminal work of Lieb [21] and Lions [23], which established the first existence and symmetry results of solutions to (1.4) with $\mu=0$ and replacing $(I_\alpha*|u|^{\alpha+3})|u|^{\alpha+1}u$ by $(I_2*u^2)u$. Over the past decades, a great deal of mathematical effort has been devoted to studying the existence, multiplicity and properties of solutions to Choquard equations. In 2018, Gao and Yang [10] first considered Brezis–Nirenberg type problems for Choquard equations on a bounded domain of $\mathbb{R}^N$ ($N\geq 3$). To overcome the possible loss of compactness caused by the critical growth, Gao and Yang [10] proved that the best constant $\mathcal{S}_\alpha$ of the Hardy–Littlewood–Sobolev inequality (defined in the three-dimensional case by [1.16]) can be attained, and used the extremal function of $\mathcal{S}_\alpha$ as a test function to ensure that the associated minimax level is strictly less than the compactness threshold under which the (PS) condition holds. This played a similar role to the Aubin–Talenti bubble, which is the optimal function of the best Sobolev constant $\mathcal{S}$ for the continuous embedding $\mathcal{D}^{1,2}(\mathbb{R}^N)\hookrightarrow L^{\frac{2N}{N-2}}(\mathbb{R}^N)$ for $N\geq 3$ in the study of the well-known Brezis–Nirenberg problems [5]. Since then, the extremal function of $\mathcal{S}_\alpha$ has become a standard tool to study various types of upper critical Choquard problems, considering different subcritical perturbations. Specifically, Alves et al. [1] dealt with singularly perturbed critical Choquard problems with the nonlocal subcritical perturbation, and extended the above results of [10] obtained in bounded domains to the whole space $\mathbb{R}^3$. Moreover, they showed that the Choquard equation (1.4) has no nontrivial solution for $\mu=0$ and $\omega\neq 0$. Instead of the nonlocal subcritical perturbation, Li and Ma [18] considered the power subcritical perturbation case of form (1.4), and proved the existence of a positive ground state solution if $4<p<6$ and $\mu>0$; or $2<p\leq 4$ and $\mu>0$ large enough. Moreover, they also considered higher dimensions $N>3$. Guo et al. [13] studied the linear perturbation case of form (1.4) with $\mu=0$ and replacing the positive number ω by the non-negative continuous function $\omega(x)$, and established the existence of a positive solution if $\|\omega\|_{3/2}>0$ is sufficiently small. For further details and important advances on this subject, we refer the reader to [6, 14, 26, 29, 38]. However, to the best of our knowledge, the existing results on upper critical Choquard problems were obtained exclusively under the positive potential or the nonnegative case where $\omega(x)>0$ at least on a set of positive measure. It seems open what happens for the zero mass case $\omega=0$, which is one of the reasons that motivates the present research.

Another motivation in this paper comes from recent studies on the static solutions of the following Schrödinger–Poisson–Slater equation:

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \mu|u|^{p-2}u + u^5, \quad x \in \mathbb{R}^3, \tag{1.5}$$

which can be seen as the limiting equation of (1.1) as $\alpha \to 0$. This is because the nonlocal upper critical term $(I_\alpha * |u|^{\alpha+3})|u|^{\alpha+1}u$ formally degenerates to the local Sobolev critical term u^5 as $\alpha \to 0$. This equation is also called as the Schrödinger–Newton equation as introduced by Penrose [28]. It arises in quantum mechanics as a Slater approximation of the exchange term in the Hartree–Fock model, as discussed in Slater [31]. In [31], without the critical term u^5, $p = 8/3$ and μ is called the Slater constant (up to renormalization). Other exponents have been used in different approximations, and we refer to [3, 22, 24] for more information on the relevance of these models and their derivation.

From a variational perspective, the absence of a phase term, i.e., the zero mass $\omega = 0$, means that the standard Sobolev space $H^1(\mathbb{R}^3)$ is not the appropriate framework for the problem. To overcome this, Ruiz [30] introduced the following *Coulomb-Sobolev space*:

$$E = \left\{u \in \mathcal{D}^{1,2}(\mathbb{R}^3)\colon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\mathrm{d}x\mathrm{d}y < \infty\right\} \tag{1.6}$$

with the norm

$$\|u\|_E := \left[\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\right]^{\frac{1}{2}}, \tag{1.7}$$

where the double integral expression is the so-called Coulomb energy of the wave. Ruiz proved that $(E, \|\cdot\|_E)$ is a uniformly convex Banach space, and that $E \hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in [3, 6]$, and $E_r \hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in \left(\frac{18}{7}, 6\right]$, where

$$E_r := \{u \in E\colon u \text{ is a radial function}\}. \tag{1.8}$$

In this framework, the following subcritical problem

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \mu|u|^{p-2}u, \quad x \in \mathbb{R}^3 \tag{1.9}$$

was studied by Ruiz [30] for $\frac{18}{7} < p < 3$ and by Ianni and Ruiz [15] for $3 \le p < 6$. Specifically, (1.9) admits a radial positive solution for $\frac{18}{7} < p < 3$ [30, Theorem 1.3], and a positive ground state solution for $3 < p < 6$ [15, Theorem 1.2]. A new critical phenomenon appears in the study of (1.9), that is *Coulomb–Sobolev critical case* $p = 3$. This case presents a certain scaling invariance, that is, given a solution u of (1.9) and a parameter $l \in \mathbb{R}$, the family of functions $l^2u(lx)$ is also a solution.

Table 1 Results in [25]

p	μ	Conclusion
$\left(\frac{18}{7}, 3\right)$	$0 < \mu < \hat{\mu}$ ($\exists\, \hat{\mu} > 0$)	(1.5) has a positive solution in E_r being a local minimizer of negative energy
3	$\mu > 0$ sufficiently large	(1.11) has a couple solution $(\bar{u}, \lambda_{\bar{u}}) \in E_r \times \mathbb{R}^+$
$(3, 4]$	$\mu > 0$ sufficiently large	(1.5) has a ground state solution in E
$(4, 6)$	$\mu > 0$	

Furthermore, $p = 3$ turns out to be the threshold exponent determining whether the associated energy functional has a mountain pass geometry on E or E_r (see [15, Remark 5.2]), leading to distinct research directions for $p \neq 3$ and $p = 3$. Specifically, in contrast with the cases $\frac{18}{7} < p < 3$ and $3 < p < 6$, for the Coulomb–Sobolev critical case $p = 3$, (1.9) was interpreted as an eigenvalue problem, and the following result was established in [15]:

Theorem [IR] ([15, Theorem 1.3]) *There exists an increasing sequence $\mu_k > 0$, $\mu_k \to +\infty$ such that the Coulomb–Sobolev critical problem*

$$-\Delta u + \left(u^2 * \frac{1}{4\pi|x|}\right) u = \mu_k |u| u \tag{1.10}$$

has a radial solution $u_k \in E_r$. Here μ_k is the Lagrange multiplier which is not priori.

In 2019, Liu et al. [25] extended these results on the Sobolev subcritical problem (1.9) and the Coulomb–Sobolev critical problem (1.10) to the Sobolev critical problem (1.5) and the following double-critical problem with a Lagrange multiplier λ:

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \lambda\mu|u|u + u^5, \quad x \in \mathbb{R}^3. \tag{1.11}$$

In that paper, the related results are summarized in Table 1.

Note that the case $p \in \left(\frac{18}{7}, 3\right)$ is special, as the increasing rate of the local power term is lower than that of the non-local convolution term. This allows the creation of a geometry of local minima for small values of $\mu > 0$. The presence of such a structure of local minima had already been observed in several related situations, see, for example, [2, 9, 11, 32] for L^2-constrained problems, and its presence suggests the possibility to search for another solution lying at a mountain pass level, besides the existence of one solution being a local minimum. However, compared with these works, due to the presence of the Coulomb term $\left(\frac{1}{4\pi|x|} * u^2\right) u$, the compactness analyses in the Coulomb–Sobolev space E or E_r is more difficult than that in the usual Sobolev space. Based on these observations, Liu et al. [25] were only able to find a negative energy solution which is a local minimizer in the case $p \in \left(\frac{18}{7}, 3\right)$, as shown in Table 1. Specifically, they first constructed a truncation functional (containing a non-local perturbed term with a sufficiently small coefficient) which is bounded below and its infimum on the whole space E_r is negative, then obtained a local (PS) condition to the truncation functional at the negative energy level based on very involved arguments relying on a measure representation concentration-compactness of Lions, finally returning to the original problem. In the cases $3 < p < 6$ and $p = 3$, to overcome the loss of compactness caused by the Sobolev critical term, Liu et al. [25] proved that the associated energy level is strictly less than the compactness threshold of the problem, specifically:

$$c < \begin{cases} \frac{1}{3}\mathcal{S}^{\frac{3}{2}}, & \text{if } 4 < p < 6 \text{ and } \mu > 0; \text{ or } 3 < p \leq 4 \text{ and} \\ & \quad \mu > 0 \text{ sufficiently large in (1.5)}, \\ \frac{\sqrt[3]{6}}{2}\mathcal{S}, & \text{if } \mu > 0 \text{ sufficiently large in (1.11)}, \end{cases} \tag{1.12}$$

below which the (PS) condition holds, see also [16, 17, 37] and see [12] for recent improvements from μ large enough to larger than some explicit lower bounds. However, it is worth pointing out that the effectiveness of their method for the case $p = 3$ remains to be further verified, as there appears to be a flaw in the proof of Lemma 4.2 in [25], where the claim $G(u_0) = 1$ (page 5933, line 8 from bottom) seems to be impossible to establish conclusively.

The study in [25] presents the different compactness thresholds of the problem for $p \in (3, 6)$ and $p = 3$, but leaves a gap for $p \in \left(\frac{18}{7}, 3\right)$. In fact, as pointed out in [25], it is very challenging to *find a concrete critical threshold and precisely control the associated energy level*, since the energy functional does not have the standard geometric properties of Mountain Pass type. To the best of our knowledge, nothing is known in the existing literature regarding this gap.

Inspired by the aforementioned work, especially critical problems (1.4), (1.5) and (1.11), in this paper, we focus on the existence and non-existence of static solutions to the upper critical Choquard problem (1.1) with Coulomb potential. Particularly, we give a complete analysis of the power exponent $p \in \left(\frac{18}{7}, 6\right]$, which is supposed to be the maximum range that allows us to use variational methods to study (1.1) in E or E_r, based on the conjecture in [30, Remark 4.1] that E_r is not included in $L^{\frac{18}{7}}(\mathbb{R}^3)$. Let

$$\begin{aligned} \Phi_\mu(u) := & \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y - \frac{\mu}{p}\int_{\mathbb{R}^3} |u|^p \mathrm{d}x \\ & - \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x. \end{aligned} \tag{1.13}$$

From (1.3) and the continuity of the embeddings $E \hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in [3, 6]$, and $E_r \hookrightarrow L^s(\mathbb{R}^3)$ for all $s \in \left(\frac{18}{7}, 6\right]$, it follows that the functional Φ_μ is well defined and $\mathcal{C}^1$ in E for $p \in [3, 6]$, the functional Φ_μ is well defined and $\mathcal{C}^1$ in E_r for $p \in \left(\frac{18}{7}, 3\right)$. Following the work of [30], solutions to (1.1) can be obtained as critical points of Φ_μ in E and E_r for $p \in (3, 6]$ and $p \in \left(\frac{18}{7}, 3\right)$, respectively. In the Sobolev critical case $p = 6$, we will prove that (1.1) has no nontrivial solution for any $\mu > 0$. In the case $p \in \left(\frac{18}{7}, 6\right)$, we are particularly interested in ground state solutions to (1.1). We recall a solution $\bar{u}$ to be a *ground state solution* if $\bar{u}$ minimizes the functional Φ_μ among all nontrivial solutions to (1.1), specifically,

$$\Phi_\mu(\bar{u}) = \inf_{u\in\mathcal{K}_\mu} \Phi_\mu(u) \;\text{ with } \mathcal{K}_\mu := \begin{cases} \{u \in E\backslash\{0\}: \Phi'_\mu(u) = 0\} \;\text{ for }\; p \in (3, 6); \\ \{u \in E_r\backslash\{0\}: \Phi'_\mu(u) = 0\} \;\text{ for }\; p \in \left(\frac{18}{7}, 3\right). \end{cases} \tag{1.14}$$

In what follows, we always assume that $\Phi_\mu: E \to \mathbb{R}$ for $p \in (3, 6]$ and $\Phi_\mu: E_r \to \mathbb{R}$ for $p \in \left(\frac{18}{7}, 3\right)$.

Compared to the previous work, the study of (1.1) with zero mass is much more challenging, due to the combined effect of the Coulomb potential and the upper critical growth of Choquard-type nonlinearity. For example,

(i) In the zero mass context, the presence of the Coulomb term necessitates studying the problem in the Coulomb–Sobolev space E or E_r by variational methods, rather than the standard Sobolev space $H^1(\mathbb{R}^3)$. The interplay between the Coulomb term and the nonlinear terms, especially the strong competition with the power function, not only significantly affects the geometric structure of Φ_μ, but also increases the complexity in identifying critical points of Φ_μ.

(ii) As is well known, the crucial step in dealing with critical problems is through the use of test functions to obtain a good energy estimate of minimax levels, such that the compactness of minimizing sequences or (PS) sequences at that energy level holds. This has been achieved for the upper critical Choquard problem (1.4) with $\omega > 0$ and $2 < p < 6$. Specifically, inspired by Gao and Yang [10], the following strict upper bound estimate has been derived by Li and Ma [18]:

$$c < \frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}} \quad \begin{cases} \text{for } 4 < p < 6 \text{ and } \mu > 0; \\ \text{for } 2 < p \le 4 \text{ and } \mu > 0 \text{ large enough.} \end{cases} \tag{1.15}$$

In the zero mass case $\omega = 0$, there is also a need to establish a similar inequality. However, extra efforts are always required to balance the competing effects between the Coulomb term and the power term, especially for the case $p \in (\frac{18}{7}, 3)$, in which the power term dominates the Coulomb term for Φ_μ near zero. It is natural to expect that the domination of the power term could help to lower the energy value, and this paper will confirm this expectation, as discussed in Remark 1.6 (iii) below. As mentioned in [25], there do not seem to be any relevant results in the existing literature even for the limit problem (1.5).

(iii) The case where $p = 3$ appears to be the most delicate. As observed in [15] for the study of (1.9), this is viewed as the Coulomb–Sobolev critical case, as this problem presents scaling invariance under the transformation $t^2u(tx)$. In this case, the Coulomb term and the power term are in balance, leading to a subtle interplay that requires the introduction of a Lagrange multiplier λ in front of $\mu|u|^{p-2}u$ to establish the appropriate variational characterization of the problem. As one would naturally expect, this dual critical nature further complicates the variational study of the problem.

1.2 Statement of the main results

To obtain the sharp energy estimates, following [10, Lemma 1.2] dealing with the Brezis–Nirenberg problem of Choquard type, we define the best constant $\mathcal{S}_\alpha$ of the Hardy–Littlewood–Sobolev inequality:

$$\mathcal{S}_\alpha := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\backslash\{0\}} \frac{\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x}{\left[\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\right]^{\frac{1}{\alpha+3}}}. \tag{1.16}$$

Define the following important constant:

$$\mathcal{T}_\alpha := \int_{\mathbb{R}^3}\left(I_\alpha * e^{-(\alpha+3)|\cdot|}\right)e^{-(\alpha+3)|x|}\mathrm{d}x, \tag{1.17}$$

which will be required in the cases $p=3$, and $p\in(3,4)$. Setting

$$U(x) := \frac{\sqrt[4]{3}}{\sqrt{1+|x|^2}}, \tag{1.18}$$

then we have the following equation:

$$(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3(\alpha+2)}{2(\alpha+3)}}\mathcal{S}_\alpha^{\frac{\alpha}{2}}\int_{\mathbb{R}^3}|\nabla U|^2\mathrm{d}x = \int_{\mathbb{R}^3}\left(I_\alpha * |U|^{\alpha+3}\right)|U|^{\alpha+3}\mathrm{d}x = (\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha+3}{2}}, \tag{1.19}$$

where the constants $\mathcal{K}_\alpha$ and $\mathcal{L}_\alpha$ are defined by Eqs. (1.2) and (1.3), respectively. Combining (1.16) and (1.19), we see that $U(x)$ and the extremal function of $\mathcal{S}_\alpha$ differ only by a constant coefficient.

Letting

$$\begin{aligned} J_\mu(u) &:= \frac{\mathrm{d}}{\mathrm{d}t}\Phi_\mu(t^2u_t)\big|_{t=1} \\ &= \frac{3}{2}\|\nabla u\|_2^2 + \frac{3}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y - \frac{(2p-3)\mu}{p}\|u\|_p^p \\ &\quad - \frac{3}{2}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x, \end{aligned} \tag{1.20}$$

we define the following set:

$$\mathcal{M}_\mu := \begin{cases} \{u\in E\backslash\{0\}: J_\mu(u)=0\} \text{ for } p\in(3,6); \\ \{u\in E_r\backslash\{0\}: J_\mu(u)=0\} \text{ for } p\in\left(\frac{18}{7},3\right). \end{cases} \tag{1.21}$$

From [15, Page 9], we know that any critical point of Φ_μ stays in $\mathcal{M}_\mu$.

As mentioned previously, the strong interplay between the Coulomb term and the power term causes the geometry of Φ_μ to change according to the behavior of p. In the following, we will separately address the three cases: $p \in \left(\frac{18}{7}, 3\right)$, $p = 3$, and $p \in (3, 6)$, based on the observations provided earlier.

Case I: $\frac{18}{7} < p < 3$. For any $\frac{18}{7} < p < 3$, let us introduce the embedding constant $\mathcal{C}_s > 0$ ([15, Lemma 3.1]), which only depends on s, given by

$$\int_{\mathbb{R}^3} |u|^s \mathrm{d}x \le \mathcal{C}_s \left[\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{8\pi |x-y|} \mathrm{d}x\mathrm{d}y \right]^{\frac{2s-3}{3}}, \quad \forall\, u \in E_r. \tag{1.22}$$

By introducing an auxiliary function [see (3.1) below] and performing careful energy estimates, we manage to find an explicit value $\mu_0 = \mu_0(p)$, defined by

$$\mu_0 := \frac{3(\alpha+2)p\left[4(\alpha+3)(3-p)\mathcal{S}_\alpha^{\alpha+3}\right]^{\frac{2(3-p)}{3(\alpha+2)}}}{\mathcal{C}_p[2(3\alpha+12-2p)]^{\frac{3\alpha+12-2p}{3(\alpha+2)}}}, \tag{1.23}$$

such that Φ_μ has a geometry of local minima:

$$\inf_{u \in A_{s_0}} \Phi_\mu(u) < 0 < \inf_{u \in \partial A_{s_0}} \Phi_\mu(u) \tag{1.24}$$

when $0 < \mu < \mu_0$, where

$$s_0 := \left[\frac{2(\alpha+3)(3-p)\mathcal{S}_\alpha^{\alpha+3}}{3\alpha+12-2p}\right]^{\frac{1}{\alpha+2}} \tag{1.25}$$

and

$$A_{s_0} := \left\{ u \in E_r \colon \|\nabla u\|_2^2 + \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi |x-y|} \mathrm{d}x\mathrm{d}y < s_0 \right\}. \tag{1.26}$$

Starting from the local minimizer involved in (1.24), we also construct a new min–max structure: the non-standard mountain pass geometry. On this basis, we establish the existence of two solutions—one being a local minimizer and one of mountain-pass type. Our first result is as follows.

Theorem 1.1 *Let $\frac{18}{7} < p < 3$. Then for any $\mu \in (0, \mu_0)$, the following statements hold:*

(i) (1.1) *has a positive radial solution $u_\mu \in E_r$ which is a minimizer of Φ_μ in the set A_{s_0} such that*

$$\Phi_\mu(u_\mu) = m_\mu := \inf_{u \in A_{s_0}} \Phi_\mu(u) < 0. \tag{1.27}$$

Moreover, any ground state solution to (1.1) *is a minimizer of* Φ_μ *on* A_{s_0}*, that is*

$$\tilde{u} \in \mathcal{K}_\mu \text{ and } \Phi_\mu(\tilde{u}) = \inf_{\mathcal{K}_\mu} \Phi_\mu \implies \tilde{u} \in A_{s_0} \text{ and } \Phi_\mu(\tilde{u}) = \inf_{A_{s_0}} \Phi_\mu = m_\mu.$$

(ii) (1.1) *has a second solution* (*mountain pass type*) $\bar{u} \in E_r$*, which satisfies*

$$0 < \Phi_\mu(\bar{u}) < m_\mu + \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}. \tag{1.28}$$

Case II: $p = 3$. As mentioned before, due to the scaling invariance under the transformation $t^2 u(tx)$, we need the introduction of a Lagrange multiplier λ, and consider the following problem:

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \lambda\mu|u|u + (I_\alpha * |u|^{\alpha+3})|u|^{\alpha+1}u, \quad x \in \mathbb{R}^3. \tag{1.29}$$

To find solutions to (1.29), we seek for critical points of the C^1-functional $I\colon E_r \to \mathbb{R}$ defined by

$$I(u) := \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|} \mathrm{d}x\mathrm{d}y \tag{1.30}$$

under the constraint

$$\tilde{\mathcal{M}}_\mu := \left\{ u \in E_r\colon G(u) := \frac{\mu}{3}\|u\|_3^3 + \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x = 1 \right\}. \tag{1.31}$$

We will consider the minimizing problem: $\tilde{m}_\mu = \inf_{u \in \tilde{\mathcal{M}}_\mu} I(u)$, and find an explicit lower bound μ_* of μ defined by

$$\mu_* := \frac{75\sqrt[6]{2000\pi}\,[2(\alpha+3)]^{\frac{-1}{\alpha+3}}}{16\pi\sqrt{4-\pi}\,\mathcal{S}_\alpha} \left[1 - \left(\frac{\mathcal{S}_\alpha}{4}\right)^{\alpha+3} \mathcal{T}_\alpha\right], \tag{1.32}$$

to ensure the attainability of $\tilde{m}_\mu$ when $\mu > \mu_*$. Our result is stated as follows.

Theorem 1.2 *Assume that* $p = 3$. *Then for any* $\mu > \mu_*$*, there exists* $(u, \lambda_\mu) \in E_r \times \mathbb{R}^+$ *such that the following equation holds*

$$-\Delta u + \left(\frac{1}{4\pi|x|} * u^2\right) u = \lambda_\mu \mu|u|u + (I_\alpha * |u|^{\alpha+3})|u|^{\alpha+1}u, \quad x \in \mathbb{R}^3.$$

Remark 1.3 Theorem 1.2 implies that, in a sense, (1.1) with $p = 3$ has at least one solution only when μ takes some particular value.

Case III: $p \in (3, 6)$. In this case, it is not difficult to prove that Φ_μ is bounded from below on $\mathcal{M}_\mu$ for any $\mu > 0$. By distinguishing the three subcases: $p \in (4, 6)$, $p = 4$ and $p \in (3, 4)$, we could specify explicit conditions on μ under which the infimum $\inf_{u \in \mathcal{M}_\mu} \Phi_\mu(u)$ is achieved, and the minimizer is a critical point of Φ_μ. Particularly, the case $p \in (3, 4)$ is the most involved, in which we define the number:

$$\mu^* := \frac{3p^4}{16(2p-3)} \left[\frac{4(p-3)(\alpha+3)\pi}{(2p-3)(\alpha+2)} \right]^{\frac{2(p-3)}{3}} \times \left[\frac{3}{\sqrt[3]{2\pi}} \left(\frac{6}{5}\right)^5 \left(\frac{\mathcal{T}_\alpha}{2^{\alpha+2}} \right)^{\frac{1}{\alpha+3}} \right]^{\frac{p-6}{6}} \mathcal{S}_\alpha^{\frac{24+6\alpha-3p\alpha-10p}{6(\alpha+2)}}. \tag{1.33}$$

In this direction, our result reads as follows.

Theorem 1.4 *Assume that one of the following conditions holds:*

(i) $p \in (4, 6)$ *and* $\mu > 0$*;*
(ii) $p = 4$ *and* $\mu > \frac{7\sqrt{3}}{\pi} (\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{1}{\alpha+3}} \mathcal{S}_\alpha^{\frac{\alpha}{3(\alpha+2)}}$*;*
(iii) $p \in (3, 4)$ *and* $\mu > \mu^*$*.*

Then (1.1) *has a ground state solution* $\bar{u} \in E$ *such that* $\Phi_\mu(\bar{u}) = \inf_{\mathcal{M}_\mu} \Phi_\mu > 0$.

Finally, by means of a Pohozaev type identity, we could prove the following nonexistence result.

Theorem 1.5 *Assume that* $p = 6$*. Then for any* $\mu > 0$*,* (1.1) *has no nontrivial solution.*

To highlight the significant impact of the different power perturbations, let us summarize the results of our theorems in Table 2 as follows.

Remark 1.6 (i) Compared to the upper critical Choquard problem (1.4) in the non-static case where $\omega \neq 0$, the presence of the Coulomb potential gives rise to new phenomena in the static case where $\omega = 0$, occurring at different ranges of the power p, as present in Table 2. This makes the structure of the solution set considerably richer.

(ii) The existence results for the cases $p \in (\frac{18}{7}, 3)$ and $p \in (3, 6)$ in (1.1) can be viewed as exhibiting certain parallels with the analysis of L^2-subcritical and L^2-supercritical perturbation cases, respectively, in the context of the Brezis–Nirenberg problem with prescribed norm. Despite the similarities in the existence results between the two problems, the essential difficulties in the problem at hand mentioned previously lead to the failure of many existing methods that have been successfully employed to study problems with analogous results in the standard Sobolev space. It forces the implementation of new ideas to catch static solutions to (1.1).

(iii) For the ranges $p \in (\frac{18}{7}, 3)$, $p = 3$, and $p \in (3, 6)$, we establish distinct positive minimax levels, and succeed in identifying the compactness thresholds for the corresponding (PS) sequences or minimizing sequences, respectively. These

Table 2 Our results

p	μ	Conclusion	Energy level
$(\frac{18}{7},3)$	$0<\mu<\mu_0$	(1.1) has a ground state solution being local minimizer	$m_\mu := \inf_{A_{s_0}} \Phi_\mu = \inf_{\mathcal{K}_\mu} \Phi_\mu < 0$
		(1.1) has a second solution of mountain pass type	$< m_\mu + \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$
3	$\mu>\mu_*$	(1.29) has a couple solution $(u,\lambda_u) \in E_r \times \mathbb{R}^+$	$0 < \inf_{\tilde{\mathcal{M}}_\mu} I < \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}{2} \mathcal{S}_\alpha$
$(3,4)$	$\mu>\mu^*$	(1.1) has a ground state solution	$0 < \inf_{\mathcal{M}_\mu} \Phi_\mu < \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$
4	$\mu > \frac{7\sqrt{3}}{\pi}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{1}{\alpha+3}} \mathcal{S}_\alpha^{\frac{\alpha}{3(\alpha+2)}}$		
$(4,6)$	$\mu>0$		
6	$\mu>0$	(1.1) has no nontrivial solution	

compactness thresholds are presented in the "Energy Level" column of Table 2 and are highlighted in red. Through the careful selection of test functions, we provide rigorous energy estimates to ensure that the obtained minimax levels lie within the range where compactness holds. Precisely, we can derive the compactness of the obtained (PS) sequences and minimizing sequences provided that the corresponding energy level, denoted by $C(p)$, satisfies the following strict inequality:

$$C(p) < \begin{cases} m_\mu + \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}, & \text{if } p \in \left(\frac{18}{7},3\right), \\ \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}{2} \mathcal{S}_\alpha, & \text{if } p=3, \\ \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}, & \text{if } 3<p<6, \end{cases} \tag{1.34}$$

where $m_\mu = \inf_{A_{s_0}} \Phi_\mu < 0$. The derivation of these strict inequalities is a novel contribution and constitutes one of the noteworthy highlights of this work, see Lemmas 3.6, 4.2 and 5.8 for more details.

(iv) For the case $p \in \left(\frac{18}{7},3\right)$, the power term dominates the Coulomb term for Φ_μ near zero. This feature not only leads to a different geometric structure of Φ_μ from the one for the study of (1.4) in the non-static case where $\omega \neq 0$, but also lower the upper bound of the involved minimax level. Specifically, we develop a careful construction of the test functions, which can be viewed as the sum of a suitable truncated extremal function of $\mathcal{S}_\alpha$ and a local minimizer of $m_\mu < 0$. With refined energy estimates, we reduce the upper bound from $\frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$ for μ large enough, as given by (1.15), to $m_\mu + \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$ for $\mu \in (0,\mu_0)$.

(v) For the cases $p = 3$ and $3 < p < 6$, as $\alpha \to 0$, the inequality (1.34) formally reduces to the corresponding strict inequality (1.12) for the limiting problem (1.11). However, compared to the Sobolev critical term u^5, the nonlocal critical term $(I_\alpha * |u|^{\alpha+3})|u|^{\alpha+1}u$ leads to more mathematical difficulties, especially for the dual critical scenario when $p = 3$, where the Coulomb term and the power term exhibit the same growth rate, necessitating a more delicate analysis of the underlying variational geometry of the problem. Particularly, we introduce novel analytical techniques employing subtle test functions and paths (see (4.4) and (4.14)) to control the minimizing level $\tilde{m}_\mu = \inf_{u\in\tilde{\mathcal{M}_\mu}} I(u)$ to be less than a prescribed threshold, thereby overcoming the loss of compactness arising from the nonlocal critical term.

The paper is organized as follows. In Sect. 2 we present some preliminary results. In Sect. 2 we study the case when $\frac{18}{7} < p < 3$, and finish the proof of Theorem 1.1. In Sect. 4, we focus on the Coulomb–Sobolev critical case $p = 3$, and complete the proof of Theorem 1.2 In Sect. 5, we deal with the case when $3 < p < 6$, and complete the proof of Theorem 1.4. In Sect. 6, establish the non-existence result for the case when $p = 6$, and prove Theorem 1.5.

Throughout this paper, we let $u_t(x) := u(tx)$ for $t > 0$, and denote the norm of $L^s(\mathbb{R}^3)$ by $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s \mathrm{d}x\right)^{1/s}$ for $s \geq 2$, $B_r(x) = \{y \in \mathbb{R}^3 : |y - x| < r\}$, and positive constants possibly different in different places, by $C_1, C_2, \ldots$.

2 Preliminaries

In this section, we recall some properties of the working space E and E_r, and present some preliminary results, which will be of use throughout the paper.

Set

$$\mathcal{N}[u] := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \quad \text{and} \quad \mathcal{Q}[u] := \|\nabla u\|_2^2 + \frac{1}{2}\mathcal{N}[u]. \tag{2.1}$$

By (1.7) and (2.1), we have

$$\|u\|_E = \left[\|\nabla u\|_2^2 + \sqrt{\mathcal{N}[u]}\right]^{1/2}. \tag{2.2}$$

Lemma 2.1 [30] *$\|\cdot\|_E$ is a norm, and $(E, \|\cdot\|_E)$ is a uniformly convex Banach space. Moreover, $\mathcal{C}_0^\infty(\mathbb{R}^3)$ is dense in E, and $E \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $p \in [3, 6]$.*

Lemma 2.2 [30] *$E_r \hookrightarrow L^s(\mathbb{R}^3)$ is continuous for $p \in (\frac{18}{7}, 6]$, and the inclusion is compact for $p \in (\frac{18}{7}, 6)$.*

Lemma 2.3 [15] *For any $s \in (\frac{18}{7}, 6]$, there exists $\mathcal{C}_s > 0$ such that*

$$\|u\|_s^s \leq \mathcal{C}_s(\mathcal{Q}[u])^{(2s-3)/3}, \quad \forall\, u \in E_r. \tag{2.3}$$

Lemma 2.4 [34] *Assume that $a, b > 0$. Then there holds*

$$a\|\nabla u\|_2^2 + b\mathcal{N}[u] \geq 2\sqrt{ab}\|u\|_3^3, \quad \forall\, u \in E. \tag{2.4}$$

Let us define

$$\phi_u(x) := \frac{1}{4\pi|x|} * u^2 = \int_{\mathbb{R}^3} \frac{u^2(y)}{4\pi|x-y|}\mathrm{d}y, \quad \forall\, x \in \mathbb{R}^3, \tag{2.5}$$

then, $u \in E$ if and only if both $u, \phi_u \in \mathcal{D}^{1,2}(\mathbb{R}^3)$. In such a case, $-\Delta\phi_u = u^2$ in a weak sense, and

$$\int_{\mathbb{R}^3} \nabla\phi_u \cdot \nabla v \mathrm{d}x = \int_{\mathbb{R}^3} u^2 v \mathrm{d}x, \quad \forall\, v \in E, \tag{2.6}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} \phi_u(x)u^2\mathrm{d}x. \tag{2.7}$$

Moreover, $\phi_u(x) > 0$ when $u \neq 0$. By using Hardy–Littlewood–Sobolev inequality (see [19] or [20, page 98]), we have the following inequality:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)v(y)|}{|x-y|}\mathrm{d}x\mathrm{d}y \leq \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\|u\|_{6/5}\|v\|_{6/5}, \quad u, v \in L^{6/5}(\mathbb{R}^3). \tag{2.8}$$

Lemma 2.5 [30] *Suppose that $\{u_n\} \subset E$. Then*

(i) *$u_n \to \bar{u}$ in E if and only if $u_n \to \bar{u}$ and $\phi_{u_n} \to \phi_{\bar{u}}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$;*
(ii) *$u_n \rightharpoonup \bar{u}$ in E if and only if $u_n \rightharpoonup \bar{u}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$ and $\sup\mathcal{N}[u_n] < +\infty$. In such case, $\phi_{u_n} \rightharpoonup \phi_{\bar{u}}$ in $\mathcal{D}^{1,2}(\mathbb{R}^3)$.*

As in [15, 30], we define

$$T\colon E^4 \to \mathbb{R},\ T(u, v, w, z) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)v(x)w(y)z(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \tag{2.9}$$

and

$$D\colon E^2 \to \mathbb{R},\ D(u, v) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u(x)v(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y. \tag{2.10}$$

Lemma 2.6 [15] *Suppose that $\{u_n\}, \{v_n\}, \{w_n\} \subset E$, $z \in E$. If $u_n \rightharpoonup \bar{u}$, $v_n \rightharpoonup \bar{v}$, $w_n \rightharpoonup \bar{w}$ in E, then*

$$T(u_n, v_n, w_n, z) \to T(\bar{u}, \bar{v}, \bar{w}, z).$$

Lemma 2.7 *Assume that $u_n \rightharpoonup \bar{u}$ in E. Then*

$$\mathcal{N}[u_n] = \mathcal{N}[\bar{u}] + \mathcal{N}[u_n - \bar{u}] + o(1). \tag{2.11}$$

Proof Let $v_n = u_n - \bar{u}$. Then $u_n \rightharpoonup \bar{u}$ and $v_n \rightharpoonup 0$ in E. From (2.7), (2.9), (2.10) and Lemma 2.6, we have

$$\begin{aligned}
\mathcal{N}[u_n] &= D((\bar{u}+v_n)^2, (\bar{u}+v_n)^2) \\
&= D(\bar{u}^2, \bar{u}^2) + D(v_n^2, v_n^2) + 4D(\bar{u}^2, \bar{u}v_n) + 4D(v_n^2, \bar{u}v_n) \\
&\quad + 4D(\bar{u}v_n, \bar{u}v_n) + 2D(\bar{u}^2, v_n^2) \\
&= D(\bar{u}^2, \bar{u}^2) + D(v_n^2, v_n^2) + o(1) = \mathcal{N}[\bar{u}] + \mathcal{N}[v_n] + o(1).
\end{aligned}$$

□

Lemma 2.8 [20, Page 107:(6) and (9)] *For any $q > \frac{3}{3-\alpha}$, there exists a constant $\mathcal{C}(\alpha, q) > 0$ such that*

$$\|I_\alpha * |u|\|_q \le \mathcal{C}(\alpha, q)\|u\|_{\frac{3q}{3+\alpha q}}, \quad \forall\, u \in L^{\frac{3q}{3+\alpha q}}(\mathbb{R}^3). \tag{2.12}$$

In order to prove a Brezis–Lieb lemma for the functional $\int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x$, we state an easy variant of the classical Brezis–Lieb lemma [36, Theorem 4.2.7].

Lemma 2.9 [36] *Let $\Omega \subseteq \mathbb{R}^N$ be a domain, $q \in [1, \infty)$ and $\{u_n\}$ be a bounded sequence in $L^r(\Omega)$. If $u_n \to \bar{u}$ a.e. $x \in \Omega$, then for every $q \in [1, r]$*

$$\lim_{n\to\infty} \int_\Omega \left(|u_n|^q - |u_n - \bar{u}|^q - |\bar{u}|^q\right)^{\frac{r}{q}} \mathrm{d}x = 0. \tag{2.13}$$

Lemma 2.10 *Assume that $u_n \to \bar{u}$ a.e. $x \in \mathbb{R}^3$ and $\sup_{n\in\mathbb{N}} \|u_n\|_6 < +\infty$. Then*

$$\begin{aligned}
&\lim_{n\to\infty}\left[\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^{\alpha+3}\right)|u_n|^{\alpha+3}\mathrm{d}x - \int_{\mathbb{R}^3}\left(I_\alpha * |u_n-\bar{u}|^{\alpha+3}\right)|u_n-\bar{u}|^{\alpha+3}\mathrm{d}x\right] \\
&= \int_{\mathbb{R}^3}\left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|\bar{u}|^{\alpha+3}\mathrm{d}x. \qquad (2.14)
\end{aligned}$$

Proof Set $v_n = u_n - \bar{u}$. Then $v_n \to 0$ a.e. $x \in \mathbb{R}^3$. Since $\sup_{n\in\mathbb{N}} \|v_n\|_6 < +\infty$, it follows that $|v_n|^{\alpha+3} \rightharpoonup 0$ in $L^{\frac{6}{\alpha+3}}(\mathbb{R}^3)$. By Lemma 2.8 and the Fatou's lemma, one has

$$\int_{\mathbb{R}^3} \left|I_\alpha * |\bar{u}|^{\alpha+3}\right|^{\frac{6}{3-\alpha}} \mathrm{d}x \le C\left(\int_{\mathbb{R}^3} |\bar{u}|^6 \mathrm{d}x\right)^{\frac{\alpha+3}{3-\alpha}} < \infty. \tag{2.15}$$

This shows that $I_\alpha * |\bar{u}|^{\alpha+3} \in L^{\frac{6}{3-\alpha}}(\mathbb{R}^3)$, it follows that

$$\int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|v_n|^{\alpha+3}\mathrm{d}x = o(1). \tag{2.16}$$

By (1.2), (1.3), (2.16) and Lemma 2.9 with $q = \alpha + 3$ and $r = 6$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left[\left(I_\alpha * |u_n|^{\alpha+3}\right)|u_n|^{\alpha+3} - \left(I_\alpha * |v_n|^{\alpha+3}\right)|v_n|^{\alpha+3} - \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|\bar{u}|^{\alpha+3}\right] \mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left[I_\alpha * \left(|u_n|^{\alpha+3} - |v_n|^{\alpha+3} - |\bar{u}|^{\alpha+3}\right)\right]\left(|u_n|^{\alpha+3} - |v_n|^{\alpha+3}\right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)\left(|u_n|^{\alpha+3} - |v_n|^{\alpha+3} - |\bar{u}|^{\alpha+3}\right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \left[I_\alpha * \left(|u_n|^{\alpha+3} - |v_n|^{\alpha+3} - |\bar{u}|^{\alpha+3}\right)\right]|v_n|^{\alpha+3} \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{\alpha+3}\right)\left(|u_n|^{\alpha+3} - |v_n|^{\alpha+3} - |\bar{u}|^{\alpha+3}\right) \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|v_n|^{\alpha+3} \mathrm{d}x + \int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{\alpha+3}\right)|\bar{u}|^{\alpha+3} \mathrm{d}x \\
&\le \mathcal{L}_\alpha \mathcal{K}_\alpha \left[\int_{\mathbb{R}^3} \left||u_n|^{\alpha+3} - |v_n|^{\alpha+3} - |\bar{u}|^{\alpha+3}\right|^{\frac{6}{\alpha+3}} \mathrm{d}x \int_{\mathbb{R}^3} \left||u_n|^{\alpha+3} - |v_n|^{\alpha+3}\right|^{\frac{6}{\alpha+3}} \mathrm{d}x\right]^{\frac{\alpha+3}{6}} \\
&\quad + \mathcal{L}_\alpha \mathcal{K}_\alpha \left[\|\bar{u}\|_6^{\alpha+3} \int_{\mathbb{R}^3} \left||u_n|^{\alpha+3} - |v_n|^{\alpha+3} - |\bar{u}|^{\alpha+3}\right|^{\frac{6}{\alpha+3}} \mathrm{d}x\right]^{\frac{\alpha+3}{6}} \\
&\quad + 2\mathcal{L}_\alpha \mathcal{K}_\alpha \left[\|v_n\|_6^{\alpha+3} \int_{\mathbb{R}^3} \left||u_n|^{\alpha+3} - |v_n|^{\alpha+3} - |\bar{u}|^{\alpha+3}\right|^{\frac{6}{\alpha+3}} \mathrm{d}x\right]^{\frac{\alpha+3}{6}} \\
&\quad + 2\int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|v_n|^{\alpha+3} \mathrm{d}x \\
&= o(1).
\end{aligned}$$

This shows (2.14) holds. □

Lemma 2.11 *Assume that* $u_n \to \bar{u}$ *a.e.* $x \in \mathbb{R}^3$ *and* $\sup_{n\in\mathbb{N}} \|u_n\|_6 < +\infty$. *Then for any* $v \in L^6(\mathbb{R}^3)$,

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right)|u_n|^{\alpha+1} u_n v \mathrm{d}x = \int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|\bar{u}|^{\alpha+1} \bar{u} v \mathrm{d}x. \tag{2.17}$$

Proof By (2.12) and the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^3} \left|\left(I_\alpha * |u_n|^{\alpha+3}\right)|u_n|^{\alpha+1} u_n\right|^{\frac{6}{5}} \mathrm{d}x \\
&\le \left(\int_{\mathbb{R}^3} \left|I_\alpha * |u_n|^{\alpha+3}\right|^{\frac{6}{3-\alpha}} \mathrm{d}x\right)^{\frac{3-\alpha}{5}} \left(\int_{\mathbb{R}^3} |u_n|^6 \mathrm{d}x\right)^{\frac{\alpha+2}{5}} \le C\|u_n\|_6^{\frac{6(2\alpha+5)}{5}}.
\end{aligned} \tag{2.18}$$

This shows that

$$\left(I_\alpha * |u_n|^{\alpha+3}\right)|u_n|^{\alpha+1} u_n \rightharpoonup \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|\bar{u}|^{\alpha+1} \bar{u} \ \text{ in } L^{\frac{6}{5}}(\mathbb{R}^3). \tag{2.19}$$

It follows that (2.17) holds. □

From Lemmas 2.1–2.6 and 2.11, we derive that the functional Φ_μ, defined by (1.13), is well defined and $\mathcal{C}^1$ in E for $p \in [3, 6]$, and is well defined and $\mathcal{C}^1$ in E_r for $p \in (18/7, 3)$, moreover, for any $u, v \in E$ if $p \in [3, 6]$, any $u, v \in E_r$ if $p \in (18/7, 3)$, there holds

$$\langle \Phi'_\mu(u), v\rangle = \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \mathrm{d}x + \int_{\mathbb{R}^3} \frac{u(x)v(x)u^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}x - \int_{\mathbb{R}^3} |u|^{p-2}uv\mathrm{d}x - \int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+1}uv\mathrm{d}x. \tag{2.20}$$

Therefore, solutions of (1.13) are critical points of Φ_μ in E and E_r for $p \in [3, 6]$ and $p \in (18/7, 3)$, respectively.

Lemma 2.12 [15] *If u is a weak solution of* (1.1) (i.e. $\Phi'_\mu(u) = 0$), *then* $J_\mu(u) = 0$, *where J is defined by* (1.20).

Lemma 2.13 [10, 15] *If u is a weak solution of* (1.1) (i.e. $\Phi'_\mu(u) = 0$), *then*

$$\frac{1}{2}\|\nabla u\|_2^2 + \frac{5}{4}\mathcal{N}[u] - \frac{3\mu}{p}\|u\|_p^p - \frac{1}{2}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x = 0. \tag{2.21}$$

3 Case $\frac{18}{7} < p < 3$

In this section, we study the case when $\frac{18}{7} < p < 3$, restricting ourselves to the radial subspace E_r, and provide the proof of Theorem 1.1. We will find the specific condition $0 < \mu < \mu_0$ to ensure that the functional Φ_μ has a geometry of local minima and a minimax structure on E_r, and prove the existence of two solutions—one being a local minimizer and one of mountain-pass type.

For the existence of a geometry of local minima, for any $\mu > 0$, let us define the function $g_\mu(s)$ on $s \in (0, +\infty)$ as follows:

$$g_\mu(s) := \frac{1}{2} - \frac{\mu\mathcal{C}_p}{p}s^{\frac{-2(3-p)}{3}} - \frac{\mathcal{S}_\alpha^{-(\alpha+3)}}{2(\alpha+3)}s^{\alpha+2}. \tag{3.1}$$

A straightforward calculation can lead to the following property on g_μ.

Lemma 3.1 *Let $\frac{18}{7} < p < 3$ and $0 < \mu < \mu_0$. Then the function $g_\mu(s)$ has a unique global maximum and the maximum value satisfies*

$$\max_{0<s<+\infty} g_\mu(s) = g_\mu(s_\mu) = \frac{1}{2} - \frac{3\alpha+12-2p}{\left[4(\alpha+3)(3-p)\mathcal{S}_\alpha^{\alpha+3}\right]^{\frac{2(3-p)}{3\alpha+12-2p}}}\left[\frac{\mu\mathcal{C}_p}{3(\alpha+2)p}\right]^{\frac{3(\alpha+2)}{3\alpha+12-2p}}$$

$$\begin{cases} > 0, \ \textit{if}\ \mu < \mu_0, \\ = 0, \ \textit{if}\ \mu = \mu_0, \\ < 0, \ \textit{if}\ \mu > \mu_0, \end{cases} \tag{3.2}$$

where

$$s_\mu := \left[\frac{4(\alpha+3)(3-p)\mu\mathcal{C}_p\mathcal{S}_\alpha^{\alpha+3}}{3(\alpha+2)p}\right]^{\frac{3}{3\alpha+12-2p}}. \tag{3.3}$$

In particular, we have $s_{\mu_0} = s_0$.

The function g_μ plays a role in the following lemma.

Lemma 3.2 *Let* $\frac{18}{7} < p < 3$ *and* $0 < \mu < \mu_0$. *Then*

$$\Phi_\mu(u) \geq \mathcal{Q}[u]\, g_\mu(\mathcal{Q}[u]), \quad \forall\, u \in E_r. \tag{3.4}$$

Proof From (1.13), (1.16), (2.1), (2.3) and (3.1), we have

$$\begin{aligned}\Phi_\mu(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\mathcal{N}[u] - \frac{\mu}{p}\|u\|_p^p - \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x \\ &\geq \frac{1}{2}\mathcal{Q}[u] - \frac{\mathcal{S}_\alpha^{-(\alpha+3)}}{2(\alpha+3)}(\mathcal{Q}[u])^{\alpha+3} - \frac{\mu\mathcal{C}_p}{p}(\mathcal{Q}[u])^{\frac{2p-3}{3}} \\ &= \mathcal{Q}[u]\, g_\mu(\mathcal{Q}[u]), \quad \forall\, u \in E_r.\end{aligned}$$

□

For any $u \in E_r$, we define

$$\begin{aligned}h_u(t) := \Phi_\mu(t^2u_t) &= \frac{t^3}{2}\|\nabla u\|_2^2 + \frac{t^3}{4}\mathcal{N}[u] - \frac{\mu t^{2p-3}}{p}\|u\|_p^p \\ &\quad - \frac{t^{3(\alpha+3)}}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x.\end{aligned} \tag{3.5}$$

Then

$$\begin{aligned}h_u'(t) = \frac{1}{t}&\left\{\frac{3t^3}{2}\|\nabla u\|_2^2 + \frac{3t^3}{4}\mathcal{N}[u] - \frac{(2p-3)\mu t^{2p-3}}{p}\|u\|_p^p\right. \\ &\left. - \frac{3t^{3(\alpha+3)}}{2}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\right\} = \frac{1}{t}J(t^2u_t).\end{aligned} \tag{3.6}$$

For $\rho > 0$, we set

$$A_\rho := \{u \in E_r\colon \mathcal{Q}[u] < \rho\}.$$

A geometry of local minima is established in the following lemma.

Lemma 3.3 *Let $\frac{18}{7} < p < 3$ and $0 < \mu < \mu_0$. Then the following properties hold:*

(i)

$$m_\mu = \inf_{u \in A_{s_0}} \Phi_\mu(u) < 0 < \inf_{u \in \partial A_{s_0}} \Phi_\mu(u). \tag{3.7}$$

(ii) $\inf_{\mathcal{M}_\mu} \Phi_\mu \geq m_\mu$, *where $\mathcal{M}_\mu$ is defined by* (1.21).

Proof (i) For any $u \in \partial A_{s_0}$, we have $\mathcal{Q}[u] = s_0$. Thus, by using Lemmas 3.1 and 3.2, we get

$$\Phi_\mu(u) \geq \mathcal{Q}[u]\, g_\mu(\mathcal{Q}[u]) = s_0 g_\mu(s_0) > s_0 g_{\mu_0}(s_0) = 0.$$

Now let $u \in A_{s_0}$ be arbitrary but fixed. From (1.13), we have

$$\Phi_\mu(t^2 u_t) = \frac{t^3}{2}\|\nabla u\|_2^2 + \frac{t^3}{4}\mathcal{N}[u] - \frac{\mu t^{2p-3}}{p}\|u\|_p^p - \frac{t^{3(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x, \quad \forall\, t > 0.$$

Since $\frac{18}{7} < p < 3$, it follows that $\lim_{t \to 0^+} \Phi_\mu(t^2 u_t) = 0^-$. Therefore, there exists $t_0 > 0$ small enough such that $\mathcal{Q}[t_0^2 u_{t_0}] = t_0^3 \mathcal{Q}[u] < s_0$ and $\Phi_\mu(t_0^2 u_{t_0}) < 0$. This implies that $m_\mu < 0$.

(ii) Let $\bar{u} \in \mathcal{M}_\mu$ be arbitrary but fixed. Then it follows from (3.6) that

$$\frac{h'_{\bar{u}}(t)}{t^2} = \frac{3}{2}\|\nabla \bar{u}\|_2^2 + \frac{3}{4}\mathcal{N}[\bar{u}] - \frac{(2p-3)\mu t^{2(p-3)}}{p}\|\bar{u}\|_p^p - \frac{3t^{3(\alpha+2)}}{2} \int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right) |\bar{u}|^{\alpha+3} \mathrm{d}x, \quad \forall\, t > 0, \tag{3.8}$$

which implies that

$$\frac{d}{dt}\left[\frac{h'_{\bar{u}}(t)}{t^2}\right] = \frac{2(2p-3)(3-p)\mu t^{2p-7}}{p}\|\bar{u}\|_p^p - \frac{9(\alpha+2)t^{3\alpha+5}}{2} \int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right) |\bar{u}|^{\alpha+3} \mathrm{d}x, \quad \forall\, t > 0. \tag{3.9}$$

Since $\frac{18}{7} < p < 3$, then $\frac{d}{dt}\left[\frac{h'_{\bar{u}}(t)}{t^2}\right] = 0$ has a unique solution, and so $\frac{h'_{\bar{u}}(t)}{t^2}$ has at most two zeros. Thus $h'_{\bar{u}}(t)$ has also at most two zeros.

To prove (ii), there are two possible cases.

Case (a). $\mathcal{Q}[\bar{u}] \leq s_0$. In this case, we have $\bar{u} \in \overline{A_{s_0}}$, it follows that $\Phi_\mu(\bar{u}) \geq m_\mu$.

Case (b). $\mathcal{Q}[\bar{u}] > s_0$. It follows from (3.6) that $h'_{\bar{u}}(1) = 0$. By (3.5) and i), we have

$$\lim_{t \to 0^+} h_{\bar{u}}(t) = 0^-, \quad h_{\bar{u}}\left(\sqrt[3]{s_0/\mathcal{Q}[\bar{u}]}\right) > 0, \quad \lim_{t \to +\infty} h_{\bar{u}}(t) = -\infty. \tag{3.10}$$

(3.10) shows that $h'_{\bar{u}}(t)$ has a first zero $t^- \in (0, \sqrt[3]{s_0/\mathcal{Q}[\bar{u}]})$ corresponding to a local minima such that $h'_{\bar{u}}(t^-) = 0$. Since $h'_{\bar{u}}(t)$ has at most two zeros, so $1 \in (\sqrt[3]{s_0/\mathcal{Q}[\bar{u}]}, +\infty)$ is the second zero of $h'_{\bar{u}}(t)$ corresponding to a unique local maximum of $h_{\bar{u}}(t)$. Thus, $\Phi_\mu(\bar{u}) = h_{\bar{u}}(1) > 0 > m_\mu$. □

Proof of (i) in Theorem 1.1 Let $\{u_n\} \subset A_{s_0}$ be a minimizing sequence for m_μ. Then $\{|u_n|\} \subset A_{s_0}$ be also a minimizing sequence for m_μ, so we may assume that $u_n \geq 0$. By Lemma 3.3, we have

$$\mathcal{Q}[u_n] < s_0, \quad \Phi_\mu(u_n) = m_\mu + o(1) < 0. \tag{3.11}$$

Since $\{\|u_n\|_E\}$ is bounded, then from Lemma 2.2, we may thus assume, passing to a subsequence if necessary, that

$$\begin{cases} u_n \rightharpoonup \tilde{u}, & \text{in } E_r; \\ u_n \to \tilde{u}, & \text{in } L^s(\mathbb{R}^3),\ \forall\, s \in \left(\frac{18}{7}, 6\right); \\ u_n \to \tilde{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{3.12}$$

To obtain a minimizer for m_μ, we split the proof into the following steps.
Step 1. We prove that $\tilde{u} \neq 0$. Otherwise, we assume that $\tilde{u} = 0$. Then (3.12) yields

$$\|u_n\|_p^p = o(1). \tag{3.13}$$

From (1.13), (1.16), (2.1), (3.1), (3.2), (3.11) and (3.13), we have

$$\begin{aligned} m_\mu + o(1) &= \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\mathcal{N}[u_n] - \frac{\mu}{p}\|u_n\|_p^p \\ &\quad - \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^{\alpha+3}\right)|u_n|^{\alpha+3}\mathrm{d}x \\ &\geq \frac{1}{2}\mathcal{Q}[u_n] - \frac{\mathcal{S}_\alpha^{-(\alpha+3)}}{2(\alpha+3)}(\mathcal{Q}[u_n])^{\alpha+3} + o(1) \\ &\geq \mathcal{Q}[u_n]\left[\frac{1}{2} - \frac{\mathcal{S}_\alpha^{-(\alpha+3)}}{2(\alpha+3)}s_0^{\alpha+2}\right] + o(1) \\ &= \mathcal{Q}[u_n]\left[g_\mu(s_0) + \frac{\mu\mathcal{C}_p}{p}s_0^{\frac{-2(3-p)}{3}}\right] + o(1) \geq o(1). \end{aligned}$$

This contradiction shows that $\tilde{u} \neq 0$ due to $m_\mu < 0$.
Step 2. Set $v_n := u_n - \tilde{u}$. By (3.12), we have

$$\|\nabla u_n\|_2^2 = \|\nabla\tilde{u}\|_2^2 + \|\nabla v_n\|_2^2 + o(1). \tag{3.14}$$

Then it follows from (1.13), (2.11), (3.14), the Brezis–Lieb lemma and Lemma 2.10 that

$$\mathcal{Q}[u_n] = \mathcal{Q}[\tilde{u}] + \mathcal{Q}[v_n] + o(1) \tag{3.15}$$

and

$$\Phi_\mu(u_n) = \Phi_\mu(\tilde{u}) + \Phi_\mu(v_n) + o(1). \tag{3.16}$$

Step 3. By the weakly lower semi-continuity for the norm and the Fatou's lemma, we have

$$\liminf_{n\to\infty} \mathcal{Q}[u_n] \geq \mathcal{Q}[\tilde{u}]. \tag{3.17}$$

This shows that $\tilde{u} \in \overline{A_{s_0}}$, and so $\Phi_\mu(\tilde{u}) \geq m_\mu$. Jointly with (1.13), (1.16), (3.2), (3.11), (3.12), (3.15), (3.16) and (3.17), we have

$$\begin{aligned}
m_\mu + o(1) &= \Phi_\mu(u_n) \\
&= \Phi_\mu(\tilde{u}) + \Phi_\mu(v_n) + o(1) \\
&= \frac{1}{2}\|\nabla v_n\|_2^2 + \frac{1}{4}\mathcal{N}[v_n] \\
&\quad - \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |v_n|^{\alpha+3}\right)|v_n|^{\alpha+3}\mathrm{d}x + \Phi_\mu(\tilde{u}) + o(1) \\
&\geq \frac{1}{2}\mathcal{Q}[v_n] - \frac{\mathcal{S}_\alpha^{-(\alpha+3)}}{2(\alpha+3)}(\mathcal{Q}[v_n])^{\alpha+3} + \Phi_\mu(\tilde{u}) + o(1) \\
&\geq \mathcal{Q}[v_n]\left[\frac{1}{2} - \frac{\mathcal{S}_\alpha^{-(\alpha+3)}}{2(\alpha+3)}s_0^{\alpha+2}\right] + m_\mu + o(1) \\
&= \mathcal{Q}[v_n]\left[g_\mu(s_0) + \frac{\mu\mathcal{C}_p}{p}s_0^{\frac{-2(3-p)}{3}}\right] + m_\mu + o(1),
\end{aligned} \tag{3.18}$$

which yields that $\mathcal{Q}[v_n] = o(1)$, and so $u_n \to \tilde{u}$ in E_r. From (3.18), we can also derive that

$$\mathcal{Q}[\tilde{u}] \leq s_0, \quad \Phi_\mu(\tilde{u}) = m_\mu,$$

which, together with Lemma 3.3, implies that $\mathcal{Q}[\tilde{u}] < s_0$. Therefore, we obtain that $\tilde{u} \geq 0$ and $\Phi'_\mu(\tilde{u}) = 0$. In view of the maximum principle, we have $\tilde{u} > 0$.

Step 4. By Lemma 2.12 and Step 3, we have $\tilde{u} \in \mathcal{K}_\mu \subset \mathcal{M}_\mu$. Then it follows from Lemma 3.3 ii) that $m_\mu = \Phi_\mu(\tilde{u}) \geq \inf_{\mathcal{K}_\mu}\Phi_\mu \geq \inf_{\mathcal{M}_\mu}\Phi_\mu \geq m_\mu$, which leads to $\Phi_\mu(\tilde{u}) = \inf_{\mathcal{K}_\mu}\Phi_\mu$. Therefore, $\tilde{u}$ is a ground state solution of (1.1) which is a minimizer of Φ_μ in the set A_{s_0}.

Finally, we prove that any ground state solution to (1.1) is a minimizer of Φ_μ on A_{s_0}. let $\bar{u}$ be any ground state solution of (1.1), i.e. $\bar{u} \in \mathcal{K}_\mu$ and $\Phi_\mu(\bar{u}) = \inf_{\mathcal{K}_\mu}\Phi_\mu$. Following the above arguments, we have $\inf_{\mathcal{K}_\mu}\Phi_\mu \geq \inf_{\mathcal{M}_\mu}\Phi_\mu \geq m_\mu \geq \inf_{\mathcal{K}_\mu}\Phi_\mu$. Hence, we obtain $\Phi_\mu(\bar{u}) = m_\mu$. By the proof of Lemma 3.3 (ii), we have $\mathcal{Q}[\bar{u}] < s_0$, and thus $\bar{u}$ is a minimizer of Φ_μ on A_{s_0}. This completes the proof. □

To establish the existence of the second solution to (1.1), being of mountain-pass type. Using the positive ground state solution $u_\mu \in E_r$ through the above process as a starting point, we will construct a new minimax structure: the mountain pass geometry, which reads as follows.

Lemma 3.4 *Let* $\frac{18}{7} < p < 3$ *and* $0 < \mu < \mu_0$. *Then there exists* $\kappa_\mu > 0$ *such that*

$$M_\mu := \inf_{\gamma\in\Gamma_\mu} \max_{t\in[0,1]} \Phi_\mu(\gamma(t)) \geq \kappa_\mu > \sup_{\gamma\in\Gamma_\mu} \max\left\{\Phi_\mu(\gamma(0)), \Phi_\mu(\gamma(1))\right\}, \tag{3.19}$$

where

$$\Gamma_\mu = \left\{\gamma \in \mathcal{C}([0,1], E_r) \colon \gamma(0) = u_\mu, \Phi_\mu(\gamma(1)) < 2m_\mu\right\} \tag{3.20}$$

and $u_\mu \in E_r$ *is the positive ground state solution of* (1.1) *obtained in* (i) *of Theorem* 1.1.

Proof Setting $\kappa_\mu := \inf_{u\in\partial(A_{s_0})} \Phi_\mu(u)$, we have $\kappa_\mu > 0$ due to (3.7). Let $\gamma \in \Gamma_\mu$ be arbitrary. Since $\gamma(0) = u_\mu \in A_{s_0}$ and $\Phi_\mu(\gamma(1)) < 2m_\mu < m_\mu$, it follows from (3.7) that $\gamma(1) \notin \overline{A_{s_0}}$. From the continuity of $\gamma(t)$ on $[0,1]$, we derive that there exists a $t_0 \in (0,1)$ such that $\gamma(t_0) \in \partial A_{s_0}$, and so $\max_{t\in[0,1]} \Phi_\mu(\gamma(t)) \geq \kappa_\mu$. This shows that (3.19) holds. □

In view of the Mountain pass theorem and Lemma 3.4, we can derive the following lemma.

Lemma 3.5 *Let* $\frac{18}{7} < p < 3$ *and* $0 < \mu < \mu_0$. *Then there exists a sequence* $\{u_n\} \subset E_r$ *such that*

$$\Phi_\mu(u_n) \to M_\mu > 0, \quad \textit{and} \quad \Phi'_\mu(u_n) \to 0. \tag{3.21}$$

To ensure that the above (PS) sequence lies within the range where the (PS) condition holds, we will provide a precise estimate for M_μ, which is one of the key highlights of the present paper. Before proceeding, we will first introduce some necessary notations and provide new integral estimates.

In view of [10, Lemma 1.2] and [35, Theorem 1.4.2], we have

$$(\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{1}{\alpha+3}} \mathcal{S}_\alpha = \mathcal{S} := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_6^2} = \left(\frac{3\sqrt{3}\pi^2}{4}\right)^{\frac{2}{3}}. \tag{3.22}$$

As in [8], let us define functions $U_n(x) := \Theta_n(|x|)$, where

$$\Theta_n(r) = \sqrt[4]{3} \begin{cases} \sqrt{\frac{n}{1+n^2r^2}}, & 0 \leq r < 1; \\ \sqrt{\frac{n}{1+n^2}}(2-r), & 1 \leq r < 2; \\ 0, & r \geq 2. \end{cases} \tag{3.23}$$

Using (1.2), (1.3), (1.18), (1.19), (3.22), (3.23) and detailed calculations, we can deduce

$$
\begin{aligned}
\|\nabla U_n\|_2^2 &= \int_{\mathbb{R}^3} |\nabla U_n|^2 \mathrm{d}x = 4\pi \int_0^{+\infty} r^2 |\Theta_n'(r)|^2 \mathrm{d}r \\
&= 4\sqrt{3}\pi \left[\int_0^1 \frac{n^5 r^4}{(1+n^2r^2)^3} \mathrm{d}r + \frac{n}{1+n^2} \int_1^2 r^2 \mathrm{d}r \right] \\
&= 4\sqrt{3}\pi \left[\int_0^n \frac{s^4}{(1+s^2)^3} \mathrm{d}s + \frac{7n}{3(1+n^2)} \right] \\
&= \mathcal{S}^{\frac{3}{2}} + 4\sqrt{3}\pi \left[-\int_n^{+\infty} \frac{s^4}{(1+s^2)^3} \mathrm{d}s + \frac{7n}{3(1+n^2)} \right] \\
&< \mathcal{S}^{\frac{3}{2}} + \frac{28\sqrt{3}\pi n}{3(1+n^2)} = (\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} \mathcal{S}_\alpha^{\frac{3}{2}} + \frac{28\sqrt{3}\pi n}{3(1+n^2)},
\end{aligned} \tag{3.24}
$$

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left(I_\alpha * |U_n|^{\alpha+3} \right) |U_n|^{\alpha+3} \mathrm{d}x \\
&= \mathcal{K}_\alpha \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|U_n(x)|^{\alpha+3} |U_n(y)|^{\alpha+3}}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \\
&\geq \mathcal{K}_\alpha \int_{B_1} \int_{B_1} \frac{|U_n(x)|^{\alpha+3} |U_n(y)|^{\alpha+3}}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \\
&= 3^{\frac{\alpha+3}{2}} \mathcal{K}_\alpha \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{\alpha+3}{2}} \left(\frac{n}{1+n^2|y|^2}\right)^{\frac{\alpha+3}{2}}}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \\
&\quad - 2 \cdot 3^{\frac{\alpha+3}{2}} \mathcal{K}_\alpha \int_{\mathbb{R}^3 \setminus B_1} \int_{B_1} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{\alpha+3}{2}} \left(\frac{n}{1+n^2|y|^2}\right)^{\frac{\alpha+3}{2}}}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \\
&\quad - 3^{\frac{\alpha+3}{2}} \mathcal{K}_\alpha \int_{\mathbb{R}^3 \setminus B_1} \int_{\mathbb{R}^3 \setminus B_1} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{\alpha+3}{2}} \left(\frac{n}{1+n^2|y|^2}\right)^{\frac{\alpha+3}{2}}}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \\
&:= (\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{3}{2}} \mathcal{S}_\alpha^{\frac{\alpha+3}{2}} - 2 \cdot 3^{\frac{\alpha+3}{2}} \mathcal{K}_\alpha D_1 - 3^{\frac{\alpha+3}{2}} \mathcal{K}_\alpha D_2,
\end{aligned} \tag{3.25}
$$

$$
\begin{aligned}
D_1 &= \int_{\mathbb{R}^3 \setminus B_1} \int_{B_1} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{\alpha+3}{2}} \left(\frac{n}{1+n^2|y|^2}\right)^{\frac{\alpha+3}{2}}}{|x-y|^{3-\alpha}} \mathrm{d}x\mathrm{d}y \\
&\leq \mathcal{L}_\alpha \left[\int_{\mathbb{R}^3 \setminus B_1} \left(\frac{n}{1+n^2|x|^2} \right)^3 \mathrm{d}x \right]^{\frac{\alpha+3}{6}} \left[\int_{B_1} \left(\frac{n}{1+n^2|y|^2} \right)^3 \mathrm{d}y \right]^{\frac{\alpha+3}{6}} \\
&= \mathcal{L}_\alpha \left[4\pi \int_1^{+\infty} \frac{n^3 r^2}{(1+n^2r^2)^3} \mathrm{d}r \right]^{\frac{\alpha+3}{6}} \left[4\pi \int_0^1 \frac{n^3 r^2}{(1+n^2r^2)^3} \mathrm{d}r \right]^{\frac{\alpha+3}{6}} \\
&= \mathcal{L}_\alpha \left[16\pi^2 \int_n^{+\infty} \frac{s^2}{(1+s^2)^3} \mathrm{d}s \int_0^n \frac{s^2}{(1+s^2)^3} \mathrm{d}s \right]^{\frac{\alpha+3}{6}} \\
&= O\left(\frac{1}{n^{(\alpha+3)/2}} \right), \quad n \to \infty,
\end{aligned} \tag{3.26}
$$

$$
\begin{aligned}
D_2 &= \int_{\mathbb{R}^3\setminus B_1}\int_{\mathbb{R}^3\setminus B_1} \frac{\left(\frac{n}{1+n^2|x|^2}\right)^{\frac{\alpha+3}{2}}\left(\frac{n}{1+n^2|y|^2}\right)^{\frac{\alpha+3}{2}}}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
&\le \mathcal{L}_\alpha\left[\int_{\mathbb{R}^3\setminus B_1}\left(\frac{n}{1+n^2|x|^2}\right)^3\mathrm{d}x\right]^{\frac{\alpha+3}{6}}\left[\int_{\mathbb{R}^3\setminus B_1}\left(\frac{n}{1+n^2|y|^2}\right)^3\mathrm{d}y\right]^{\frac{\alpha+3}{6}} \\
&= \mathcal{L}_\alpha\left[4\pi\int_1^{+\infty}\frac{n^3r^2}{(1+n^2r^2)^3}\mathrm{d}r\right]^{\frac{\alpha+3}{6}}\left[4\pi\int_1^{+\infty}\frac{n^3r^2}{(1+n^2r^2)^3}\mathrm{d}r\right]^{\frac{\alpha+3}{6}} \\
&= \mathcal{L}_\alpha\left[16\pi^2\int_n^{+\infty}\frac{s^2}{(1+s^2)^3}\mathrm{d}s\int_n^{+\infty}\frac{s^2}{(1+s^2)^3}\mathrm{d}s\right]^{\frac{\alpha+3}{6}} \\
&= O\left(\frac{1}{n^{\alpha+3}}\right),\quad n\to\infty, \qquad (3.27)\\
\|U_n\|_q^q &= \int_{\mathbb{R}^3}|U_n|^q\mathrm{d}x = 4\pi\int_0^{+\infty} r^2|\Theta_n(r)|^q\mathrm{d}r \\
&= 4(\sqrt[4]{3})^q\pi\left[\int_0^1\frac{n^{q/2}r^2}{(1+n^2r^2)^{q/2}}\mathrm{d}r + \left(\frac{n}{1+n^2}\right)^{q/2}\int_1^2 r^2(2-r)^q\mathrm{d}r\right] \\
&= 4(\sqrt[4]{3})^q\pi\left[\frac{1}{n^{(6-q)/2}}\int_0^n\frac{s^2}{(1+s^2)^{q/2}}\mathrm{d}s + \left(\frac{n}{1+n^2}\right)^{q/2}\int_0^1 s^q(2-s)^2\mathrm{d}s\right] \\
&= 4(\sqrt[4]{3})^q\pi\left[\frac{1}{n^{(6-q)/2}}\int_0^n\frac{s^2\mathrm{d}s}{(1+s^2)^{q/2}} + \frac{q^2+7q+14}{(q+1)(q+2)(q+3)}\left(\frac{n}{1+n^2}\right)^{\frac{q}{2}}\right] \qquad (3.28)
\end{aligned}
$$

and

$$
\|U_n\|_{12/5}^{12/5} = 4(\sqrt[4]{3})^{12/5}\pi\left[\frac{1}{n^{9/5}}\int_0^n\frac{s^2}{(1+s^2)^{6/5}}\mathrm{d}s + \frac{2285}{5049}\left(\frac{n}{1+n^2}\right)^{\frac{6}{5}}\right]. \qquad (3.29)
$$

The combination of (2.8), (3.24) and (3.29) yields that $U_n \in E_r$ for all $n \in \mathbb{N}$. Using the above estimates, we will prove the following lemma.

Lemma 3.6 *Let $\frac{18}{7} < p < 3$ and $0 < \mu < \mu_0$. Then there holds:*

$$
M_\mu < m_\mu + \frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}. \qquad (3.30)
$$

Proof Let $u_\mu \in E_r$ be given in i) of Theorem 1.1. Then by (i) of Theorem 1.1, we have

$$
\Phi(u_\mu) = m_\mu,\quad u_\mu \in L^s(\mathbb{R}^3),\quad \forall\, s\in\left(\frac{18}{7}, 6\right],\quad u_\mu(x) > 0,\quad \forall\, x\in\mathbb{R}^3 \qquad (3.31)
$$

and

$$\begin{aligned}&\int_{\mathbb{R}^3} \nabla u_\mu \cdot \nabla U_n \mathrm{d}x + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu^2(x)u_\mu(y)U_n(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\ &= \mu \int_{\mathbb{R}^3} |u_\mu|^{p-2}u_\mu U_n \mathrm{d}x + \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^{\alpha+3}\right)|u_\mu|^{\alpha+1}u_\mu U_n \mathrm{d}x. \end{aligned} \tag{3.32}$$

By (2.8), (3.28), (3.29), (3.31), Lemma 2.8 with $\alpha = 2$, $q = 4$ and the Hölder inequality, we have

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu(x)U_n(x)u_\mu(y)U_n(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y\right| &\le C\|u_\mu U_n\|_{6/5}^2 \\ &\le C\|u_\mu\|_3^2\|U_n\|_2^2 = O\left(\frac{1}{n}\right), \quad n\to\infty, \end{aligned} \tag{3.33}$$

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu^2(x)U_n^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y\right| &= \left|\int_{\mathbb{R}^3}\left(I_2 * U_n^2\right)u_\mu^2(x)\mathrm{d}x\right| \\ &\le \left\|I_2 * U_n^2\right\|_4 \|u_\mu\|_{8/3}^2 \\ &\le C\|u_\mu\|_{8/3}^2\|U_n\|_{24/11}^2 \\ &= O\left(\frac{1}{n}\right), \quad n\to\infty, \end{aligned} \tag{3.34}$$

$$\begin{aligned}\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu(x)U_n(x)U_n^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y\right| &\le C\|u_\mu U_n\|_{6/5}\|U_n\|_{12/5}^2 \\ &\le C\|u_\mu\|_3\|U_n\|_2\|U_n\|_{12/5}^2 \\ &= O\left(\frac{1}{n\sqrt{n}}\right), \quad n\to\infty \end{aligned} \tag{3.35}$$

and

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{U_n^2(x)U_n^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y\right| \le C\|U_n\|_{12/5}^4 = O\left(\frac{1}{n^2}\right), \quad n\to\infty. \tag{3.36}$$

Setting $B := \inf_{|x|\le 1} u_\mu(x)$, we have $B > 0$. Then it follows from (3.23), (3.25), (3.26) and (3.27) that

$$\begin{aligned}&\int_{\mathbb{R}^3}\left[I_\alpha * \left(|u_\mu||U_n|^{\alpha+2}\right)\right]|U_n|^{\alpha+3}\mathrm{d}x \\ &= \mathcal{K}_\alpha \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|U_n(x)|^{\alpha+3}|u_\mu(y)||U_n(y)|^{\alpha+2}}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\ &\ge \mathcal{K}_\alpha \int_{B_1}\int_{B_1} \frac{|U_n(x)|^{\alpha+3}|u_\mu(y)||U_n(y)|^{\alpha+2}}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{\mathcal{K}_\alpha B}{\sqrt[4]{3}\sqrt{n}} \int_{B_1}\int_{B_1} \frac{|U_n(x)|^{\alpha+3}|U_n(y)|^{\alpha+3}}{|x-y|^{3-\alpha}}\mathrm{d}x\mathrm{d}y \\
&= \frac{\mathcal{K}_\alpha B(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha+3}{2}}}{\sqrt[4]{3}\sqrt{n}} - O\left(\frac{1}{n^{(\alpha+3)/2}}\right), \quad n\to\infty. \qquad (3.37)
\end{aligned}
$$

To obtain the suitable testing function for the proof of (3.30), let us define a sequence of functions as follows:

$$
W_{n,t}(x) := u_\mu(x) + tU_n(x). \qquad (3.38)
$$

It is easy to verify the following two inequalities

$$
(s+t)^p \geq s^p + ps^{p-1}t + t^p, \quad \forall\, s,t\geq 0 \qquad (3.39)
$$

and

$$
(s+t)^{\alpha+3} \geq s^{\alpha+3} + (\alpha+3)s^{\alpha+2}t + (\alpha+3)st^{\alpha+2} + t^{\alpha+3}, \quad \forall\, s,t\geq 0. \qquad (3.40)
$$

From (3.33)–(3.36) and (3.40), we can derive that

$$
\begin{aligned}
\mathcal{N}[W_{n,t}] &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{[u_\mu(x)+tU_n(x)]^2[u_\mu(y)+tU_n(y)]^2}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\
&= \mathcal{N}[u_\mu] + t^4\mathcal{N}[U_n] + 4t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu^2(x)u_\mu(y)U_n(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\
&\quad + 4t^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu(x)U_n(x)u_\mu(y)U_n(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\
&\quad + 2t^2\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu^2(x)U_n^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y + 4t^3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu(x)U_n(x)U_n^2(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\
&= \mathcal{N}[u_\mu] + 4t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu^2(x)u_\mu(y)U_n(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\
&\quad + t^2\left[O\left(\frac{1}{n}\right)\right] + t^3\left[O\left(\frac{1}{n\sqrt{n}}\right)\right] + t^4\left[O\left(\frac{1}{n^2}\right)\right], \quad n\to\infty \qquad (3.41)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \left(I_\alpha * |W_{n,t}|^{\alpha+3}\right)|W_{n,t}|^{\alpha+3}\mathrm{d}x \\
&= \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu + tU_n|^{\alpha+3}\right)|u_\mu + tU_n|^{\alpha+3}\mathrm{d}x \\
&\geq \int_{\mathbb{R}^3} \left(I_\alpha * \left[|u_\mu|^{\alpha+3} + (\alpha+3)t|u_\mu|^{\alpha+2}U_n + (\alpha+3)t^{\alpha+2}u_\mu|U_n|^{\alpha+2} + t^{\alpha+3}|U_n|^{\alpha+3}\right]\right) \\
&\quad \times \left[|u_\mu|^{\alpha+3} + (\alpha+3)t|u_\mu|^{\alpha+2}U_n + (\alpha+3)t^{\alpha+2}u_\mu|U_n|^{\alpha+2} + t^{\alpha+3}|U_n|^{\alpha+3}\right]\mathrm{d}x
\end{aligned}
$$

$$\geq \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^{\alpha+3}\right) |u_\mu|^{\alpha+3}\mathrm{d}x + t^{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |U_n|^{\alpha+3}\right) |U_n|^{\alpha+3}\mathrm{d}x$$
$$+ 2(\alpha+3)t \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^{\alpha+3}\right) |u_\mu|^{\alpha+2} U_n \mathrm{d}x$$
$$+ 2(\alpha+3)t^{2\alpha+5} \int_{\mathbb{R}^3} \left[I_\alpha * \left(|u_\mu||U_n|^{\alpha+2}\right)\right] |U_n|^{\alpha+3}\mathrm{d}x. \tag{3.42}$$

From (1.13), (3.24)–(3.29), (3.31), (3.32), (3.37), (3.38), (3.39), (3.41) and (3.42), we have

$$\begin{aligned}
&\Phi_\mu(W_{n,t}) \\
&= \frac{1}{2}\|\nabla W_{n,t}\|_2^2 + \frac{1}{4}\mathcal{N}[W_{n,t}] - \frac{1}{2(\alpha+3)} \\
&\quad \times \int_{\mathbb{R}^3} \left(I_\alpha * |W_{n,t}|^{\alpha+3}\right) |W_{n,t}|^{\alpha+3}\mathrm{d}x - \frac{\mu}{p}\|W_{n,t}\|_p^p \\
&\leq \frac{1}{2}\|\nabla u_\mu\|_2^2 + \frac{1}{4}\mathcal{N}[u_\mu] - \frac{1}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^{\alpha+3}\right) |u_\mu|^{\alpha+3}\mathrm{d}x - \frac{\mu}{p}\|u_\mu\|_p^p \\
&\quad + \frac{t^2}{2}\|\nabla U_n\|_2^2 - \frac{t^{2(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |U_n|^{\alpha+3}\right) |U_n|^{\alpha+3}\mathrm{d}x + t\int_{\mathbb{R}^3} \nabla u_\mu \cdot \nabla U_n \mathrm{d}x \\
&\quad + t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u_\mu^2(x) u_\mu(y) U_n(y)}{4\pi|x-y|}\mathrm{d}x\mathrm{d}y \\
&\quad - t\int_{\mathbb{R}^3} \left(I_\alpha * |u_\mu|^{\alpha+3}\right) |u_\mu|^{\alpha+2} U_n \mathrm{d}x - \mu t \int_{\mathbb{R}^3} u_\mu^{p-1} U_n \mathrm{d}x \\
&\quad - t^{2\alpha+5} \int_{\mathbb{R}^3} \left[I_\alpha * \left(|u_\mu||U_n|^{\alpha+2}\right)\right] |U_n|^{\alpha+3}\mathrm{d}x + \left(t^2+t^4\right)\left[O\left(\frac{1}{n}\right)\right] \\
&= \Phi_\mu(u_\mu) + \frac{t^2}{2}\|\nabla U_n\|_2^2 - \frac{t^{2(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |U_n|^{\alpha+3}\right) |U_n|^{\alpha+3}\mathrm{d}x \\
&\quad - t^{2\alpha+5} \int_{\mathbb{R}^3} \left(I_\alpha * (|u_\mu||U_n|^{\alpha+2})\right) |U_n|^{\alpha+3}\mathrm{d}x + \left(t^2+t^4\right)\left[O\left(\frac{1}{n}\right)\right] \\
&= m_\mu + \frac{t^2}{2}\|\nabla U_n\|_2^2 - \frac{t^{2(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |U_n|^{\alpha+3}\right) |U_n|^{\alpha+3}\mathrm{d}x \\
&\quad - t^{2\alpha+5} \int_{\mathbb{R}^3} \left(I_\alpha * (|u_\mu||U_n|^{\alpha+2})\right) |U_n|^{\alpha+3}\mathrm{d}x + \left(t^2+t^4\right)\left[O\left(\frac{1}{n}\right)\right] \\
&< m_\mu + \frac{t^2}{2}\left[(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}}\mathcal{S}_\alpha^{\frac{3}{2}} + \frac{28\sqrt{3}\pi n}{3(1+n^2)}\right] \\
&\quad - \frac{t^{2(\alpha+3)}}{2(\alpha+3)}\left[(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha+3}{2}} - O\left(\frac{1}{n^{(\alpha+3)/2}}\right)\right] \\
&\quad - \frac{\mathcal{K}_\alpha B(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha+3}{2}} t^{2\alpha+5}}{\sqrt[4]{3}\sqrt{n}} + t^{2\alpha+5}\left[O\left(\frac{1}{n^{(\alpha+3)/2}}\right)\right] + \left(t^2+t^4\right)\left[O\left(\frac{1}{n}\right)\right] \\
&< m_\mu + \left[\frac{t^2}{2}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} - \frac{t^{2(\alpha+3)}}{2(\alpha+3)}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha}{2}}\right]\mathcal{S}_\alpha^{\frac{3}{2}}
\end{aligned}$$

$$-\frac{\mathcal{K}_\alpha B(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha+3}{2}}t^{2\alpha+5}}{\sqrt[4]{3}\sqrt{n}}+t^{2(\alpha+3)}\left[O\left(\frac{1}{n^{(\alpha+3)/2}}\right)\right]+\left(t^2+t^4\right)\left[O\left(\frac{1}{n}\right)\right] \tag{3.43}$$

$$\leq m_\mu+\frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}-O\left(\frac{1}{\sqrt{n}}\right),\quad \forall\, t>0. \tag{3.44}$$

This shows that there exists $\bar{n}\in\mathbb{N}$ such that

$$\sup_{t>0}\Phi_\mu(W_{\bar{n},t})<m_\mu+\frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}. \tag{3.45}$$

From (3.38) and (3.43), we derive that $W_{\bar{n},0}=u_\mu$ and $\Phi_\mu(W_{\bar{n},t})<2m_\mu$ for large $t>0$. Thus, there exists $\bar{t}>0$ such that

$$\Phi_\mu(W_{\bar{n},\bar{t}})<2m_\mu. \tag{3.46}$$

Let $\gamma_{\bar{n}}(t):=W_{\bar{n},t\bar{t}}$. Then $\gamma_{\bar{n}}\in\Gamma_\mu$, where Γ_μ is defined by (3.20). Hence, (3.30) follows from (3.19) and (3.45). □

Proof of (ii) in Theorem 1.1 In view of Lemmas 3.5 and 3.6, there exists $\{u_n\}\subset E_r$ such that

$$\Phi_\mu(u_n)\to M_\mu\in\left(0,m_\mu+\frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}\right),\quad \Phi'_\mu(u_n)\to 0. \tag{3.47}$$

By (1.13), (2.1) and (3.47), we have

$$\begin{aligned} M_\mu+o(1)&=\frac{1}{2}\|\nabla u_n\|_2^2+\frac{1}{4}\mathcal{N}[u_n]\\ &\quad-\frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha*|u_n|^{\alpha+3}\right)|u_n|^{\alpha+3}\mathrm{d}x-\frac{\mu}{p}\|u_n\|_p^p \end{aligned} \tag{3.48}$$

and

$$o(1)\|u_n\|_E=\|\nabla u_n\|_2^2+\mathcal{N}[u_n]-\int_{\mathbb{R}^3}\left(I_\alpha*|u_n|^{\alpha+3}\right)|u_n|^{\alpha+3}\mathrm{d}x-\mu\|u_n\|_p^p. \tag{3.49}$$

Combining (2.3), (3.48) and (3.49), we obtain

$$\begin{aligned} M_\mu+o(1)\|u_n\|_E&=\frac{\alpha+2}{2(\alpha+3)}\|\nabla u_n\|_2^2+\frac{\alpha+1}{4(\alpha+3)}\mathcal{N}[u_n]-\frac{(2\alpha+6-p)\mu}{2p(\alpha+3)}\|u_n\|_p^p\\ &\geq\frac{\alpha+1}{2(\alpha+3)}\mathcal{Q}[u_n]-\frac{(2\alpha+6-p)\mu\mathcal{C}_p}{2p(\alpha+3)}\left(\mathcal{Q}[u_n]\right)^{\frac{2p-3}{3}}, \end{aligned} \tag{3.50}$$

which, together with $\frac{18}{7} < p < 3$, shows that $\{\mathcal{Q}[u_n]\}$ is bounded, and so $\{\|u_n\|_E\}$ is bounded. Then by Lemma 2.2, we may thus assume, passing to a subsequence if necessary, that

$$\begin{cases} u_n \rightharpoonup \bar{u}, & \text{in } E_r; \\ u_n \to \bar{u}, & \text{in } L^s(\mathbb{R}^3), \ \forall\, s \in \left(\frac{18}{7}, 6\right); \\ u_n \to \bar{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{3.51}$$

Now, we claim that $\bar{u} \neq 0$. Otherwise, we assume that $\bar{u} = 0$. Then $\|u_n\|_p^p \to 0$, and so (3.49), together with $\sup_{n\in\mathbb{N}} \|u_n\|_E < \infty$, implies that

$$o(1) = \|\nabla u_n\|_2^2 + \mathcal{N}[u_n] - \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x. \tag{3.52}$$

Up to a subsequence, we assume that

$$\|\nabla u_n\|_2^2 \to \hat{l}_1 \geq 0, \quad \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x \to \hat{l}_2 \geq 0. \tag{3.53}$$

From (1.16), (3.52) and (3.53), we obtain

$$\begin{aligned} \hat{l}_2 &= \lim_{n\to\infty} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x \\ &\leq \mathcal{S}_\alpha^{-(\alpha+3)} \lim_{n\to\infty} \|\nabla u_n\|_2^{2(\alpha+3)} = \mathcal{S}_\alpha^{-(\alpha+3)} \hat{l}_1^{\alpha+3} \leq \mathcal{S}_\alpha^{-(\alpha+3)} \hat{l}_2^{\alpha+3}. \end{aligned} \tag{3.54}$$

We next derive a contradiction by distinguishing the two cases: $\hat{l}_2 > 0$ and $\hat{l}_2 = 0$. If $\hat{l}_2 > 0$, then (3.54) implies that $\hat{l}_2 \geq \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$ and $\hat{l}_1 \geq \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$. This, together with (3.48) and (3.52), implies that

$$\begin{aligned} M_\mu + o(1) &= \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\mathcal{N}[u_n] \\ &\quad - \frac{1}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x - \frac{\mu}{p}\|u_n\|_p^p \\ &= \frac{\alpha+2}{2(\alpha+3)}\|\nabla u_n\|_2^2 + \frac{\alpha+1}{4(\alpha+3)}\mathcal{N}[u_n] + o(1) \\ &\geq \frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}} + o(1). \end{aligned}$$

This contradicts with (3.47) due to $m_\mu < 0$. If $\hat{l}_2 = 0$, then (3.52) implies that $\|\nabla u_n\|_2^2 + \mathcal{N}[u_n] \to 0$. This, together with (3.48) and (3.52), implies that

$$\begin{aligned} M_\mu + o(1) = \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\mathcal{N}[u_n] - \frac{1}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x \\ - \frac{\mu}{p}\|u_n\|_p^p = o(1). \end{aligned}$$

This contradicts with (3.47). The above argument shows that $\bar{u} \neq 0$. By Lemmas 2.6, 2.11 and a standard argument, we have $\Phi'_\mu(\bar{u}) = 0$. Hence, Lemmas 2.12 and 3.3 show that $\Phi(\bar{u}) \geq m_\mu$.

Finally, we prove that $\|u_n - \bar{u}\|_E \to 0$. Let $v_n := u_n - \bar{u}$. Then $v_n \rightharpoonup 0$ in E_r and $v_n \to 0$ in $L^s(\mathbb{R}^3)$ for all $s \in \left(\frac{18}{7}, 6\right)$. Using (3.51), the Brezis–Lieb lemma and Lemma 2.10, we have

$$\begin{cases} \|\nabla u_n\|_2^2 = \|\nabla \bar{u}\|_2^2 + \|\nabla v_n\|_2^2 + o(1); \\ \|v_n\|_p^p = \|u_n\|_p^p - \|\bar{u}\|_p^p + o(1) = o(1); \\ \int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{\alpha+3}\right) |v_n|^{\alpha+3} \mathrm{d}x \\ = \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x - \int_{\mathbb{R}^3} \left(I_\alpha * |\bar{u}|^{\alpha+3}\right) |\bar{u}|^{\alpha+3} \mathrm{d}x + o(1). \end{cases} \tag{3.55}$$

From (2.10) and Lemma 2.6, we deduce

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n(y) v_n(y)}{4\pi |x-y|} \mathrm{d}x \mathrm{d}y = D(u_n^2, u_n v_n) \\ &\quad = D(v_n^2, v_n^2) + 2D(\bar{u} u_n, v_n^2) - D(\bar{u}^2, v_n^2) + D(u_n^2, \bar{u} v_n) + o(1) \\ &\quad = \mathcal{N}[v_n] + o(1). \end{aligned} \tag{3.56}$$

It follows from (1.13), (3.47), (3.51), (3.55), (3.56) and Lemma 2.11 that

$$\begin{aligned} o(1) &= \langle \Phi'_\mu(u_n), v_n \rangle \\ &= \int_{\mathbb{R}^3} \nabla u_n \cdot \nabla v_n \mathrm{d}x + \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{u_n^2(x) u_n(y) v_n(y)}{4\pi |x-y|} \mathrm{d}x \mathrm{d}y \\ &\quad - \mu \int_{\mathbb{R}^3} |u_n|^{p-2} u_n v_n \mathrm{d}x - \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+1} u_n v_n \mathrm{d}x \\ &= \|\nabla v_n\|_2^2 + \mathcal{N}[v_n] - \int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{\alpha+3}\right) |v_n|^{\alpha+3} \mathrm{d}x + o(1). \end{aligned} \tag{3.57}$$

Up to a subsequence, we assume that

$$\|\nabla v_n\|_2^2 \to \tilde{l}_1 \geq 0, \quad \int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{\alpha+3}\right) |v_n|^{\alpha+3} \mathrm{d}x \to \tilde{l}_2 \geq 0. \tag{3.58}$$

From (1.16) and (3.57), we obtain

$$\begin{aligned} \tilde{l}_2 &= \lim_{n\to\infty} \int_{\mathbb{R}^3} \left(I_\alpha * |v_n|^{\alpha+3}\right) |v_n|^{\alpha+3} \mathrm{d}x \\ &\leq \mathcal{S}_\alpha^{-(\alpha+3)} \lim_{n\to\infty} \|\nabla v_n\|_2^{2(\alpha+3)} = \mathcal{S}_\alpha^{-(\alpha+3)} \tilde{l}_1^{\alpha+3} \leq \mathcal{S}_\alpha^{-(\alpha+3)} \tilde{l}_2^{\alpha+3}. \end{aligned} \tag{3.59}$$

If $\tilde{l}_2 > 0$, then (3.59) yields that $\tilde{l}_2 \geq \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$ and $\tilde{l}_1 \geq \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$. This, together with (1.13), (3.48), (3.55) and (3.57), implies that

$$
\begin{aligned}
M_\mu + o(1) &= \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\mathcal{N}[u_n] \\
&\quad - \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |u_n|^{\alpha+3}\right)|u_n|^{\alpha+3}\mathrm{d}x - \frac{\mu}{p}\|u_n\|_p^p \\
&= \frac{1}{2}\|\nabla v_n\|_2^2 + \frac{1}{4}\mathcal{N}[v_n] \\
&\quad - \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |v_n|^{\alpha+3}\right)|v_n|^{\alpha+3}\mathrm{d}x + \Phi_\mu(\bar{u}) + o(1) \\
&= \frac{\alpha+2}{2(\alpha+3)}\|\nabla v_n\|_2^2 + \frac{\alpha+1}{4(\alpha+3)}\mathcal{N}[v_n] + \Phi_\mu(\bar{u}) + o(1) \\
&\geq \frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}} + m_\mu + o(1).
\end{aligned}
$$

Thus, $\tilde{l}_2 = 0$. It follows from (3.57) that $\|u_n - \bar{u}\|_E \to 0$. Using (1.13), (3.48) and (3.55), it is easy to deduce that

$$
\Phi_\mu(\bar{u}) = M_\mu, \quad \Phi'_\mu(\bar{u}) = 0.
$$

□

4 Case $p = 3$

In this section, based on the Lagrange multipliers theorem, we establish the existence of solutions to (1.29) by looking for critical points of the following $\mathcal{C}^1$-functional:

$$
I(u) = \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\mathcal{N}[u], \quad \forall\, u \in E_r \tag{4.1}
$$

constrained on $\tilde{\mathcal{M}}_\mu$, and complete the proof of Theorem 1.2. Here, $\mathcal{N}[u]$ and $\tilde{\mathcal{M}}_\mu$ are given by (2.1) and (1.31), respectively. For this, we will deal with the minimizing problem: $\tilde{m}_\mu = \inf_{u\in\tilde{\mathcal{M}}_\mu} I(u)$, and find the specific condition $\mu > \mu_*$ to prove the attainability of $\tilde{m}_\mu$.

We now begin by the following lemma.

Lemma 4.1 *Assume that $\mu > 0$. Then*

$$
\tilde{m}_\mu = \inf_{u\in\tilde{\mathcal{M}}_\mu} I(u) > 0. \tag{4.2}
$$

Proof By (1.31), one has

$$
\frac{\mu}{3}\|u\|_3^3 + \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x = 1, \ \ \forall\, u \in \tilde{\mathcal{M}}_\mu. \tag{4.3}
$$

Hence, it follows from (1.16), (2.4), (4.1) and (4.3) that

$$\begin{aligned} I(u) &= \frac{1}{2}\|\nabla u\|_2^2 + \frac{1}{4}\mathcal{N}[u] \\ &\geq \frac{\mathcal{S}_\alpha}{4}\left[\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\right]^{\frac{1}{\alpha+3}} + \frac{1}{2}\|u\|_3^3 \\ &\geq \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}\mathcal{S}_\alpha}{4}\left[\frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\right] + \frac{1}{2}\|u\|_3^3 \\ &\geq \min\left\{\frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}\mathcal{S}_\alpha}{4}, \frac{3}{2\mu}\right\} \\ &\quad \times \left(\frac{\mu}{3}\|u\|_3^3 + \frac{1}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\right) \\ &= \min\left\{\frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}\mathcal{S}_\alpha}{4}, \frac{3}{2\mu}\right\}, \quad \forall\, u \in \tilde{\mathcal{M}}_\mu. \end{aligned}$$

This shows that (4.2) holds. □

We will proceed from the minimizing sequence of $\tilde{m}_\mu$ to prove that $\tilde{m}_\mu$ is attained. In order to overcome the lack of compactness caused by the upper critical exponent, we need to make precise estimates on $\tilde{m}_\mu$ to ensure that it is less than the compactness threshold. To this end, for any fixed $\kappa > 0$, we consider the following function:

$$w(x) := \kappa e^{-|x|}, \quad \forall\, x \in \mathbb{R}^3. \tag{4.4}$$

Straightforward calculations yield that $w \in H^1(\mathbb{R}^3)$, moreover,

$$\|\nabla w\|_2^2 = \int_{\mathbb{R}^3}|\nabla w|^2\mathrm{d}x = 4\pi\kappa^2\int_0^{+\infty} r^2 e^{-2r}\mathrm{d}r = \pi\kappa^2, \tag{4.5}$$

$$\|w\|_s^s = \int_{\mathbb{R}^3}|w|^s\mathrm{d}x = 4\pi\kappa^s\int_0^{+\infty} r^2 e^{-sr}\mathrm{d}r = \frac{8\pi\kappa^s}{s^3}, \quad \forall\, s \in [2,6] \tag{4.6}$$

and

$$\|w\|_{12/5}^4 = \left(\int_{\mathbb{R}^3}|w|^{12/5}\mathrm{d}x\right)^{\frac{5}{3}} = \left[8\pi\kappa^{\frac{12}{5}}\left(\frac{5}{12}\right)^3\right]^{\frac{5}{3}} = \left(\frac{5}{6}\right)^5\pi\sqrt[3]{\pi^2}\kappa^4. \tag{4.7}$$

By (1.16), (1.17) and (4.5), we have

$$\mathcal{S}_\alpha \leq \frac{\|\nabla w\|_2^2}{\left[\int_{\mathbb{R}^3}\left(I_\alpha * |w|^{\alpha+3}\right)|w|^{\alpha+3}\mathrm{d}x\right]^{\frac{1}{\alpha+3}}} = \frac{\pi}{\mathcal{T}_\alpha^{\frac{1}{\alpha+3}}}. \tag{4.8}$$

Setting

$$\kappa_1 = \left[\frac{16\pi(\alpha+3)}{81} \left(\frac{\mathcal{S}_\alpha}{4} \right)^{\alpha+3} \mu \right]^{\frac{1}{2\alpha+3}} \left[1 - \left(\frac{\mathcal{S}_\alpha}{4} \right)^{\alpha+3} \mathcal{T}_\alpha \right]^{-\frac{1}{2\alpha+3}}, \tag{4.9}$$

then (4.8) leads to $\kappa_1 > 0$. By means of the function $w(x)$ with $\kappa = \kappa_1$, we obtain the sharp estimate of $\tilde{m}_\mu$ in the following lemma.

Lemma 4.2 *Assume that $\mu > \mu_*$. Then*

$$\tilde{m}_\mu < \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}{2} \mathcal{S}_\alpha. \tag{4.10}$$

Proof By (2.8) and (4.7), we have

$$\mathcal{N}[w] = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{w^2(x) w^2(y)}{4\pi |x-y|} \mathrm{d}x\mathrm{d}x \le \frac{2\sqrt[3]{2}}{3\pi\sqrt[3]{\pi}} \|w\|_{12/5}^4 = \frac{2\sqrt[3]{2\pi}}{3} \left(\frac{5}{6} \right)^5 \kappa_1^4. \tag{4.11}$$

Using (4.6), we can choose $t_0 > 0$ such that

$$\begin{aligned} t_0^{\frac{3(\alpha+3)}{2\alpha+3}} &:= \left[\frac{\mu}{3} \|w\|_3^3 + \frac{1}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |w|^{\alpha+3} \right) |w|^{\alpha+3} \mathrm{d}x \right]^{-1} \\ &= \left[\frac{162(\alpha+3)}{16\pi(\alpha+3)\mu + 81\mathcal{T}_\alpha \kappa_1^{2\alpha+3}} \right] \kappa_1^{-3}. \end{aligned} \tag{4.12}$$

By (4.9), one has

$$81\kappa_1^{2\alpha+3} = \left[16\pi(\alpha+3)\mu + 81\mathcal{T}_\alpha \kappa_1^{2\alpha+3} \right] \left(\frac{\mathcal{S}_\alpha}{4} \right)^{\alpha+3}. \tag{4.13}$$

Setting $\tilde{w}(x) = t_0^{\frac{-\alpha}{2\alpha+3}} w(x/t_0)$, we have $\tilde{w} \in \tilde{\mathcal{M}}_\mu$ due to (4.12). Then it follows from (1.32), (4.5), (4.9), (4.11), (4.12) and (4.13), that

$$\begin{aligned} I(\tilde{w}) &= \frac{1}{2} \|\nabla \tilde{w}\|_2^2 + \frac{1}{4} \mathcal{N}[\tilde{w}] \\ &= \frac{1}{2} \|\nabla w\|_2^2 t_0^{\frac{3}{2\alpha+3}} + \frac{1}{4} \mathcal{N}[w] t_0^{\frac{3(2\alpha+5)}{2\alpha+3}} \\ &\le \frac{\pi\kappa_1^2}{2} t_0^{\frac{3}{2\alpha+3}} + \frac{\sqrt[3]{2\pi}}{6} \left(\frac{5}{6} \right)^5 \kappa_1^4 t_0^{\frac{3(2\alpha+5)}{2\alpha+3}} \\ &= \left[\frac{\pi}{2} + \frac{5^5 \sqrt[3]{2\pi}}{6^6} \left[\frac{162(\alpha+3)\kappa_1^{2\alpha+3}}{16\pi(\alpha+3)\mu + 81\mathcal{T}_\alpha \kappa^{2\alpha+3}} \right]^{\frac{2(\alpha+2)}{\alpha+3}} \kappa_1^{-2(2\alpha+3)} \right] \end{aligned}$$

$$
\begin{aligned}
&\times \left[\frac{162(\alpha+3)\kappa_1^{2\alpha+3}}{16\pi(\alpha+3)\mu + 81\mathcal{T}_\alpha \kappa_1^{2\alpha+3}} \right]^{\frac{1}{\alpha+3}} \\
&= \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}{4} \mathcal{S}_\alpha \left\{ \frac{\pi}{2} + \frac{28125\sqrt[3]{2\pi}[2(\alpha+3)]^{\frac{-2}{\alpha+3}}}{256\pi^2\mu^2\mathcal{S}_\alpha^2} \left[1 - \left(\frac{\mathcal{S}_\alpha}{4} \right)^{\alpha+3} \mathcal{T}_\alpha \right]^2 \right\} \\
&< \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}{2} \mathcal{S}_\alpha, \quad \forall\, \mu > \mu_*. \qquad (4.14)
\end{aligned}
$$

This, together with (4.2), shows that (4.10) holds. □

Next, we prove that $\tilde{m}_\mu$ can be attained.

Lemma 4.3 *Assume that the conditions in Theorem 1.2 hold. Then there exists $\bar{u} \in \tilde{\mathcal{M}}_\mu$ such that $I(\bar{u}) = \tilde{m}_\mu$.*

Proof Let $\{u_n\} \subset \tilde{\mathcal{M}}_\mu$ be such that $I(u_n) \to \tilde{m}_\mu$. Since $G(u_n) = 1$, then it follows from (1.31) and (4.1) that

$$
\tilde{m}_\mu + o(1) = I(u_n) = \frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\mathcal{N}[u_n] \qquad (4.15)
$$

and

$$
G(u_n) = \frac{\mu}{3}\|u_n\|_3^3 + \frac{1}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3} \right) |u_n|^{\alpha+3} \mathrm{d}x = 1. \qquad (4.16)
$$

(4.15) shows that $\{u_n\}$ is bounded in E_r. Therefore, from Lemma 2.2, there exists $\bar{u} \in E_r$ such that, passing to a subsequence,

$$
\begin{cases}
u_n \rightharpoonup \bar{u}, & \text{in } E_r; \\
u_n \to \bar{u}, & \text{in } L^s(\mathbb{R}^3),\ \forall\, s \in \left(\frac{18}{7}, 6\right); \\
u_n \to \bar{u}, & \text{a.e. on } \mathbb{R}^3.
\end{cases} \qquad (4.17)
$$

We claim that $\bar{u} \neq 0$. Indeed, suppose that $\bar{u} = 0$. Then by (4.16) and (4.17), we have

$$
\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3} \right) |u_n|^{\alpha+3} \mathrm{d}x \to 2(\alpha+3). \qquad (4.18)
$$

Then it follows from (1.16), (4.15) and (4.18) that

$$
\begin{aligned}
\tilde{m}_\mu &= \lim_{n\to\infty} \left(\frac{1}{2}\|\nabla u_n\|_2^2 + \frac{1}{4}\mathcal{N}[u_n] \right) \geq \frac{1}{2} \liminf_{n\to\infty} \|\nabla u_n\|_2^2 \\
&\geq \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}{2} \liminf_{n\to\infty} \frac{\|\nabla u_n\|_2^2}{\left[\int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3} \right) |u_n|^{\alpha+3} \mathrm{d}x \right]^{\frac{1}{\alpha+3}}}
\end{aligned}
$$

$$\geq \frac{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}{2}\mathcal{S}_\alpha,$$

which contradicts with (4.10). Therefore, $\bar{u} \neq 0$.

Let $w_n = u_n - \bar{u}$. Up to a subsequence, we assume that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\left(I_\alpha * |w_n|^{\alpha+3}\right)|w_n|^{\alpha+3}\mathrm{d}x := A^{\alpha+3}. \tag{4.19}$$

By (4.15), (4.16), (4.17), the Brezis–Lieb lemma, Lemmas 2.7 and 2.10, we have

$$\tilde{m}_\mu = \lim_{n\to\infty} I(u_n) = I(\bar{u}) + \lim_{n\to\infty} I(w_n) \tag{4.20}$$

and

$$\begin{aligned}1 &= G(\bar{u}) + \lim_{n\to\infty} G(w_n)\\ &= G(\bar{u}) + \frac{1}{2(\alpha+3)}\lim_{n\to\infty}\int_{\mathbb{R}^3}\left(I_\alpha * |w_n|^{\alpha+3}\right)|w_n|^{\alpha+3}\mathrm{d}x\\ &= G(\bar{u}) + \frac{A^{\alpha+3}}{2(\alpha+3)}.\end{aligned} \tag{4.21}$$

To derive the conclusion of Lemma 4.3, we distinguish two cases on A as follows.

Case (1). $A > 0$. Using (4.21), we can choose $t_n, \bar{t} \in [1, +\infty)$ such that

$$\frac{\mu t_n^3}{3}\|w_n\|_3^3 + \frac{t_n^{3(\alpha+3)}}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |w_n|^{\alpha+3}\right)|w_n|^{\alpha+3}\mathrm{d}x = 1 \tag{4.22}$$

and

$$\frac{\mu \bar{t}^3}{3}\|\bar{u}\|_3^3 + \frac{\bar{t}^{3(\alpha+3)}}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|\bar{u}|^{\alpha+3}\mathrm{d}x = 1. \tag{4.23}$$

Then it follows from (1.31), (4.17), (4.19), (4.22) and (4.23) that

$$\lim_{n\to\infty} t_n^{3(\alpha+3)} = \frac{2(\alpha+3)}{A^{\alpha+3}}, \tag{4.24}$$

$$G(t_n^2(w_n)_{t_n}) = G(\bar{t}^2\bar{u}_{\bar{t}}) = 1 \tag{4.25}$$

and

$$1 = \frac{\mu \bar{t}^3}{3}\|\bar{u}\|_3^3 + \frac{\bar{t}^{3(\alpha+3)}}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|\bar{u}|^{\alpha+3}\mathrm{d}x \geq \bar{t}^3 G(\bar{u}). \tag{4.26}$$

Combining (4.2), (4.20), (4.21), (4.24), (4.25) and (4.26), we have

$$\tilde{m}_\mu - I(\bar{u}) = \lim_{n\to\infty} I(w_n) = \lim_{n\to\infty}\left[t_n^{-3} I\left(t_n^2 (w_n)_{t_n}\right)\right] \\ \geq \frac{A}{[2(\alpha+3)]^{\frac{1}{\alpha+3}}}\tilde{m}_\mu = [1-G(\bar{u})]^{\frac{1}{\alpha+3}}\tilde{m}_\mu \tag{4.27}$$

and

$$\tilde{m}_\mu \leq I(\bar{t}^2\bar{u}_{\bar{t}}) = \bar{t}^3 I(\bar{u}) \leq \frac{I(\bar{u})}{G(\bar{u})}. \tag{4.28}$$

From (4.27) and (4.28), we derive

$$G(\bar{u}) \leq \frac{I(\bar{u})}{\tilde{m}_\mu} \leq 1 - [1-G(\bar{u})]^{\frac{1}{\alpha+3}}, \tag{4.29}$$

which yields that

$$G(\bar{u}) + [1-G(\bar{u})]^{\frac{1}{\alpha+3}} \leq 1.$$

This shows that $G(\bar{u}) = 1$, and so (4.21) implies that $A = 0$, a contradiction.

Case (2). $A = 0$. Then (4.21) yields that

$$1 = G(\bar{u}) + \lim_{n\to\infty} G(w_n) = G(\bar{u}). \tag{4.30}$$

By (4.2), (4.20) and (4.30), we have

$$\tilde{m}_\mu = \lim_{n\to\infty} I(u_n) = I(\bar{u}) + \lim_{n\to\infty} I(w_n) \geq \tilde{m}_\mu + \lim_{n\to\infty} I(w_n), \tag{4.31}$$

which implies that $u_n \to \bar{u}$ in E_r, and so $G(\bar{u}) = 1$ and $I(\bar{u}) = \tilde{m}_\mu$. □

Proof of Theorem 1.2 From Lemma 4.3, we know that $\bar{u}$ is a radially symmetric non-negative minimizer of I constrained on $\tilde{\mathcal{M}}_\mu$. By Lagrange Multipliers theorem there exists a multiplier $\bar{\lambda} > 0$ such that $\bar{u}$ satisfies the following equation

$$-\Delta\bar{u} + \left(\frac{1}{4\pi|x|} * \bar{u}^2\right)\bar{u} = \bar{\lambda}\left[\mu|\bar{u}|\bar{u} + \left(I_\alpha * |\bar{u}|^{\alpha+3}\right)|\bar{u}|^{\alpha+1}\bar{u}\right], \quad x \in \mathbb{R}^3. \tag{4.32}$$

Let $\tilde{u}(x) := \bar{\lambda}^{\frac{2}{3(\alpha+2)}}\bar{u}\left(\bar{\lambda}^{\frac{1}{3(\alpha+2)}}x\right)$, then $\tilde{u}$ satisfies the following equation

$$-\Delta\tilde{u} + \left(\frac{1}{4\pi|x|} * \tilde{u}^2\right)\tilde{u} = \lambda_\mu\mu|\tilde{u}|\tilde{u} + \left(I_\alpha * |\tilde{u}|^{\alpha+3}\right)|\tilde{u}|^{\alpha+1}\tilde{u}, \quad x \in \mathbb{R}^3. \tag{4.33}$$

Here, $\lambda_\mu = \bar{\lambda}$ depends on μ. The proof is completed. □

5 Case $3 < p < 6$

In this section, working on the whole space E instead of E_r used in the previous two sections, we establish the existence of ground state solutions to (1.1) with $3 < p < 6$, and provide the proof of Theorem 1.4. We will first show that Φ_μ is bounded from below on $\mathcal{M}_\mu$. By distinguishing the three subcases: $p \in (4, 6)$, $p = 4$, and $p \in (3, 4)$, we will control the minima $\inf_{u\in\mathcal{M}_\mu} \Phi_\mu(u)$ from above by the compactness threshold. We will then prove that the minimum $\inf_{u\in\mathcal{M}_\mu} \Phi_\mu(u)$ is achieved, and moreover, the minimizer is a critical point of Φ_μ, where $\mathcal{M}_\mu$ is defined by (1.21).

To do the first step, let us consider two functions as follows:

$$g(t) := \frac{2(p-3) - (2p-3)t^3 + 3t^{2p-3}}{3p}, \quad t > 0 \tag{5.1}$$

and

$$h(t) := \frac{\alpha + 2 - (\alpha+3)t^3 + t^{3(\alpha+3)}}{2(\alpha+3)}, \quad t > 0. \tag{5.2}$$

A simple computation can lead to the following lemma.

Lemma 5.1 *Assume that $p \in (3, 6)$ and $\mu > 0$. Then $g(t) > g(1) = 0$ and $h(t) > h(1) = 0$ for all $t \in (0, 1) \cup (1, +\infty)$.*

Lemma 5.2 *Assume that $p \in (3, 6)$ and $\mu > 0$. Then*

$$\Phi_\mu(u) \geq \Phi_\mu\left(t^2 u_t\right) + \frac{1-t^3}{3} J_\mu(u) + \frac{\alpha + 2 - (\alpha+3)t^3 + t^{3(\alpha+3)}}{2(\alpha+3)} \times \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x, \quad \forall\, u \in E, \quad t \geq 0. \tag{5.3}$$

Proof Note that

$$\Phi_\mu\left(t^2 u_t\right) = \frac{t^3}{2}\|\nabla u\|_2^2 + \frac{t^3}{4}\mathcal{N}[u] - \frac{\mu t^{2p-3}}{p}\|u\|_p^p - \frac{t^{3(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x. \tag{5.4}$$

Then by (1.13), (1.20), (5.1) and (5.4), we have

$$\Phi_\mu(u) - \Phi_\mu\left(t^2 u_t\right) = \frac{1-t^3}{2}\|\nabla u\|_2^2 + \frac{1-t^3}{4}\mathcal{N}[u] + \frac{\mu(t^{2p-3}-1)}{p}\|u\|_p^p + \frac{t^{3(\alpha+3)} - 1}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x$$

$$= \frac{1-t^3}{3} J_\mu(u) + \mu g(t)\|u\|_p^p$$
$$+ h(t) \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x.$$

This, together with Lemma 5.1, shows that (5.3) holds. □

From Lemma 5.2, we have the following corollary.

Corollary 5.3 *Assume that $p \in (3,6)$ and $\mu > 0$. Then for $u \in \mathcal{M}_\mu$,*

$$\Phi_\mu(u) = \max_{t \geq 0} \Phi_\mu(t^2 u_t). \tag{5.5}$$

Lemma 5.4 *Assume that $p \in (3,6)$ and $\mu > 0$. Then for any $u \in E\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u^2 u_{t_u} \in \mathcal{M}_\mu$.*

Proof Let $u \in E\backslash\{0\}$ be fixed and define a function $\zeta(t) := \Phi_\mu(t^2 u_t)$ on $[0,\infty)$. Clearly, by (5.4), we have

$$\zeta'(t) = 0 \Leftrightarrow \frac{3t^3}{2}\|\nabla u\|_2^2 + \frac{3t^3}{4}\mathcal{N}[u] - \frac{(2p-3)\mu t^{2p-3}}{p}\|u\|_p^p$$
$$- \frac{3t^{3(\alpha+3)}}{2} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x = 0$$
$$\Leftrightarrow \quad J_\mu(t^2 u_t) = 0 \quad \Leftrightarrow \quad t^2 u_t \in \mathcal{M}_\mu.$$

It is easy to verify that $\zeta(0) = 0$, $\zeta(t) > 0$ for $t > 0$ small and $\zeta(t) < 0$ for t large. Therefore $\max_{t \in [0,\infty)} \zeta(t)$ is achieved at a $t_0 = t_u > 0$ so that $\zeta'(t_0) = 0$ and $t_0^2 u_{t_0} \in \mathcal{M}_\mu$.

Next we claim that t_u is unique for any $u \in E\backslash\{0\}$. In fact, for any given $u \in E\backslash\{0\}$, let $t_1, t_2 > 0$ such that $\zeta'(t_1) = \zeta'(t_2) = 0$. Then $J_\mu(t_1^2 u_{t_1}) = J_\mu(t_2^2 u_{t_2}) = 0$. Jointly with (5.3), we have

$$\Phi_\mu\left(t_1^2 u_{t_1}\right) \geq \Phi_\mu\left(t_2^2 u_{t_2}\right) + \frac{t_1^3 - t_2^3}{3t_1^3} J_\mu\left(t_1^2 u_{t_1}\right)$$
$$+ \frac{(\alpha+2)t_1^{3(\alpha+3)} - (\alpha+3)t_1^{3(\alpha+2)} t_2^3 + t_2^{3(\alpha+3)}}{2(\alpha+3)t_1^{3(\alpha+3)}} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x$$
$$= \Phi_\mu\left(t_2^2 u_{t_2}\right)$$
$$+ \frac{(\alpha+2)t_1^{3(\alpha+3)} - (\alpha+3)t_1^{3(\alpha+2)} t_2^3 + t_2^{3(\alpha+3)}}{2(\alpha+3)t_1^{3(\alpha+3)}} \int_{\mathbb{R}^3} \left(I_\alpha * |u|^{\alpha+3}\right) |u|^{\alpha+3} \mathrm{d}x$$

and

$$\Phi_\mu\left(t_2^2 u_{t_2}\right) \geq \Phi_\mu\left(t_1^2 u_{t_1}\right) + \frac{t_2^3 - t_1^3}{3t_2^3} J_\mu\left(t_2^2 u_{t_2}\right)$$

$$
\begin{aligned}
&+\frac{(\alpha+2)t_2^{3(\alpha+3)}-(\alpha+3)t_1^3t_2^{3(\alpha+2)}+t_1^{3(\alpha+3)}}{2(\alpha+3)t_2^{3(\alpha+3)}}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\\
=&\ \Phi_\mu\left(t_1^2u_{t_1}\right)\\
&+\frac{(\alpha+2)t_2^{3(\alpha+3)}-(\alpha+3)t_1^3t_2^{3(\alpha+2)}+t_1^{3(\alpha+3)}}{2(\alpha+3)t_2^{3(\alpha+3)}}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x.
\end{aligned}
$$

The combination of the above two inequalities implies that $t_1 = t_2$. Therefore, $t_u > 0$ is unique for any $u \in E\backslash\{0\}$. □

From Corollary 5.3 and Lemma 5.4, we can obtain the following lemma.

Lemma 5.5 *Assume that* $p \in (3, 6)$ *and* $\mu > 0$. *Then*

$$
\inf_{u\in\mathcal{M}_\mu}\Phi_\mu(u) := \hat{m}_\mu = \inf_{u\in E\backslash\{0\}}\max_{t\geq 0}\Phi_\mu(t^2u_t).
$$

Lemma 5.6 *Assume that* $p \in (3, 6)$ *and* $\mu > 0$. *Then*

(i) *there exists* $\rho_0 > 0$ *such that* $\|\nabla u\|_2^2 \geq \rho_0,\ \forall\, u \in \mathcal{M}_\mu$;
(ii) $\hat{m}_\mu = \inf_{u\in\mathcal{M}_\mu}\Phi_\mu(u) > 0$.

Proof Since $J_\mu(u) = 0,\ \forall u \in \mathcal{M}_\mu$, by (1.16), (1.20), (2.4), the Sobolev inequality and the Young inequality, it has

$$
\begin{aligned}
\frac{3}{4}\|\nabla u\|_2^2+\frac{3}{2}\|u\|_3^3 &\leq \frac{3}{2}\|\nabla u\|_2^2+\frac{3}{4}\mathcal{N}[u]\\
&= \frac{(2p-3)\mu}{p}\|u\|_p^p+\frac{3}{2}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\\
&\leq \frac{3}{2}\|u\|_3^3+C_1\|u\|_6^6+\frac{3}{2}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x\\
&\leq \frac{3}{2}\|u\|_3^3+C_2\|\nabla u\|_2^6+\frac{3}{2\mathcal{S}_\alpha^{\alpha+3}}\|\nabla u\|_2^{2(\alpha+3)},
\end{aligned}
\tag{5.6}
$$

where C_1 and C_2 are positive constants. This implies there exists $\rho_0 > 0$ such that

$$
\|\nabla u\|_2^2 \geq \rho_0, \quad \forall\, u \in \mathcal{M}_\mu. \tag{5.7}
$$

From (1.13), (1.20) and (5.7), we have

$$
\begin{aligned}
\Phi_\mu(u) &= \Phi_\mu(u)-\frac{1}{2p-3}J_\mu(u)\\
&= \frac{p-3}{2p-3}\|\nabla u\|_2^2+\frac{p-3}{2(2p-3)}\mathcal{N}[u]\\
&\quad+\frac{3\alpha+2(6-p)}{2(2p-3)(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha*|u|^{\alpha+3}\right)|u|^{\alpha+3}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\geq \frac{p-3}{2p-3}\|\nabla u\|_2^2 \\
&\geq \frac{p-3}{2p-3}\rho_0, \quad \forall\, u \in \mathcal{M}_\mu.
\end{aligned}
$$

This shows that $\hat{m}_\mu = \inf_{u\in\mathcal{M}_\mu} \Phi_\mu(u) > 0$. □

Next, by distinguishing the three cases: $p \in (4,6)$, $p = 4$ and $p \in (3,4)$, we could find the specific conditions on μ to obtain the sharp estimate of $\hat{m}_\mu$. The following lemma deals with the first two cases.

Lemma 5.7 *Assume that condition* (i) *or* (ii) *in Theorem 1.4 holds. Then there exists a positive integer $\hat{n}$ such that*

$$
\hat{m}_\mu \leq \sup_{t>0} \Phi_\mu\left(t^2 (U_{\hat{n}})_t\right) < \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}, \tag{5.8}
$$

where the function $U_n(x) = \Theta_n(|x|)$ and $\Theta_n(r)$ is defined by (3.23).

Proof By (2.8), (3.24), (3.25), (3.28), (3.29) and (5.4), we have

$$
\begin{aligned}
&\Phi_\mu\left(t^2(U_n)_t\right) \\
&= \frac{t^3}{2}\|\nabla U_n\|_2^2 + \frac{t^3}{4}\mathcal{N}[U_n] - \frac{\mu t^{2p-3}}{p}\|U_n\|_p^p \\
&- \frac{t^{3(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |U_n|^{\alpha+3}\right)|U_n|^{\alpha+3} \mathrm{d}x \\
&< \frac{t^3}{2}\left[(\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} \mathcal{S}_\alpha^{\frac{3}{2}} + \frac{28\sqrt{3}\pi n}{3(1+n^2)}\right] \\
&+ 4\sqrt[3]{4\pi}\, t^3 \left[\frac{1}{n^{9/5}} \int_0^n \frac{s^2}{(1+s^2)^{6/5}} \mathrm{d}s + \frac{2285}{5049}\left(\frac{n}{1+n^2}\right)^{\frac{6}{5}}\right]^{\frac{5}{3}} \\
&- \frac{4(\sqrt[4]{3})^p \pi \mu t^{2p-3}}{p n^{(6-p)/2}} \int_0^n \frac{s^2}{(1+s^2)^{p/2}} \mathrm{d}s \\
&- \frac{t^{3(\alpha+3)}}{2(\alpha+3)}\left[(\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{3}{2}} \mathcal{S}_\alpha^{\frac{\alpha+3}{2}} - O\left(\frac{1}{n^{(\alpha+3)/2}}\right)\right] \\
&< \left[\frac{t^3}{2}(\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} - \frac{t^{3(\alpha+3)}}{2(\alpha+3)}(\mathcal{L}_\alpha \mathcal{K}_\alpha)^{\frac{3}{2}} \mathcal{S}_\alpha^{\frac{\alpha}{2}}\right] \mathcal{S}_\alpha^{\frac{3}{2}} \\
&+ \frac{29\sqrt{3}\pi}{6n} t^3 + \left[O\left(\frac{1}{n^{(\alpha+3)/2}}\right)\right] t^{3(\alpha+3)} \\
&- \frac{4(\sqrt[4]{3})^p \pi \mu t^{2p-3}}{p n^{(6-p)/2}} \int_0^n \frac{s^2}{(1+s^2)^{p/2}} \mathrm{d}s, \qquad \forall\, n \geq 100.
\end{aligned} \tag{5.9}
$$

Under condition (i) or (ii) of Theorem 1.4, we distinguish the following three cases on t.

Case 1. $t \in \left[(\alpha+3)^{\frac{1}{3(\alpha+2)}}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{-1}{2(\alpha+3)}}\mathcal{S}_\alpha^{-\frac{\alpha}{6(\alpha+2)}}, +\infty\right)$, $p \in (3,6)$ and $\mu > 0$. It follows from (5.9) that

$$\begin{aligned}
\Phi_\mu\left(t^2(U_n)_t\right) &< \left[\frac{t^3}{2}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} - \frac{t^{3(\alpha+3)}}{2(\alpha+3)}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha}{2}}\right]\mathcal{S}_\alpha^{\frac{3}{2}} \\
&\quad + \frac{29\sqrt{3}\pi}{6n}t^3 + \left[O\left(\frac{1}{n^{(\alpha+3)/2}}\right)\right]t^{3(\alpha+3)} \\
&\le O\left(\frac{1}{n}\right), \quad n \to \infty. \qquad (5.10)
\end{aligned}$$

Case 2. $t \in \left(0, (\alpha+3)^{\frac{1}{3(\alpha+2)}}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{-1}{2(\alpha+3)}}\mathcal{S}_\alpha^{-\frac{\alpha}{6(\alpha+2)}}\right)$, $p \in (4,6)$ and $\mu > 0$. It follows from (5.9) that

$$\begin{aligned}
\Phi_\mu\left(t^2(U_n)_t\right) &< \left[\frac{t^3}{2}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} - \frac{t^{3(\alpha+3)}}{2(\alpha+3)}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha}{2}}\right]\mathcal{S}_\alpha^{\frac{3}{2}} \\
&\quad + O\left(\frac{1}{n}\right) - \frac{C_1\mu}{n^{(6-p)/2}}t^{2p-3} \\
&\le \frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}} - \frac{C_2\mu}{n^{(6-p)/2}}, \quad n \to \infty, \qquad (5.11)
\end{aligned}$$

where $C_1, C_2 > 0$.

Case 3. $t \in \left(0, (\alpha+3)^{\frac{1}{3(\alpha+2)}}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{-1}{2(\alpha+3)}}\mathcal{S}_\alpha^{-\frac{\alpha}{6(\alpha+2)}}\right)$, $p = 4$ and $\mu > \frac{7\sqrt{3}}{\pi}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{1}{\alpha+3}}$ $\mathcal{S}_\alpha^{\frac{\alpha}{3(\alpha+2)}}$. It follows from (5.9) that

$$\begin{aligned}
\Phi_\mu\left(t^2(U_n)_t\right) &< \left[\frac{t^3}{2}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} - \frac{t^{3(\alpha+3)}}{2(\alpha+3)}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha}{2}}\right]\mathcal{S}_\alpha^{\frac{3}{2}} \\
&\quad + \frac{5\sqrt{3}\pi t^3}{n} - \frac{3\pi\mu t^5}{n}\int_0^n \frac{s^2}{(1+s^2)^2}\mathrm{d}s \\
&= \left[\frac{t^3}{2}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2(\alpha+3)}} - \frac{t^{3(\alpha+3)}}{2(\alpha+3)}(\mathcal{L}_\alpha\mathcal{K}_\alpha)^{\frac{3}{2}}\mathcal{S}_\alpha^{\frac{\alpha}{2}}\right]\mathcal{S}_\alpha^{\frac{3}{2}} \\
&\quad + \frac{5\sqrt{3}\pi}{n}t^3 - \frac{3\pi^2\mu}{4n}t^5 + O\left(\frac{1}{n^2}\right) \\
&\le \frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}} - O\left(\frac{1}{n}\right), \quad n \to \infty. \qquad (5.12)
\end{aligned}$$

Cases 1–3 imply that there exists a positive integer $\hat{n} > 100$ such that (5.8) holds. □

The following lemma deals with the case $p \in (3,4)$. Setting

$$\kappa_2^2 := \frac{3}{\sqrt[3]{2\pi}} \left(\frac{6}{5}\right)^5 \left(\frac{\mathcal{T}_\alpha}{2^{\alpha+2}}\right)^{\frac{1}{\alpha+3}} \mathcal{S}_\alpha, \tag{5.13}$$

we consider the function $w(x)$ with $\kappa = \kappa_2$, where the constant $\mathcal{T}_\alpha$ and the function $w(x)$ are defined by (1.17) and (4.4), respectively. With this, we establish the following sharp estimate of $\hat{m}_\mu$.

Lemma 5.8 *Assume that condition* (iii) *in Theorem* 1.4 *holds. Then*

$$\hat{m}_\mu \le \sup_{t>0} \Phi_\mu\left(t^2 w_t\right) < \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}. \tag{5.14}$$

Proof From (1.17), (1.33), (2.8), (4.5), (4.6), (4.11) by utilizing κ_2 instead of κ_1, (5.13) and condition (iii) in Theorem 1.4, we have

$$\begin{aligned}
\Phi_\mu(t^2 w_t) &= \frac{t^3}{2}\|\nabla w\|_2^2 + \frac{t^3}{4}\mathcal{N}[w] - \frac{\mu t^{2p-3}}{p}\|w\|_p^p \\
&\quad - \frac{t^{3(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |w|^{\alpha+3}\right)|w|^{\alpha+3} \mathrm{d}x \\
&\le \frac{\pi\kappa_2^2 t^3}{2} + \frac{\sqrt[3]{2\pi}\, t^3}{6}\left(\frac{5}{6}\right)^5 \kappa_2^4 - \frac{8\pi\kappa_2^p \mu t^{2p-3}}{p^4} - \frac{\mathcal{T}_\alpha \kappa_2^{2(\alpha+3)} t^{3(\alpha+3)}}{2(\alpha+3)} \\
&= \pi\kappa_2^2\left[\frac{t^3}{2} - \frac{8\kappa_2^{p-2}\mu t^{2p-3}}{p^4}\right] + \frac{\kappa_2^4}{2}\left[\frac{\sqrt[3]{2\pi}\, t^3}{3}\left(\frac{5}{6}\right)^5 - \frac{\mathcal{T}_\alpha \kappa_2^{2(\alpha+1)} t^{3(\alpha+3)}}{\alpha+3}\right] \\
&\le \frac{(p-3)\pi}{2p-3}\kappa_2^{\frac{p-6}{2(p-3)}}\left[\frac{3p^4}{16(2p-3)\mu}\right]^{\frac{3}{2(p-3)}} \\
&\quad + \frac{\alpha+2}{2(\alpha+3)}\left[\frac{\sqrt[3]{2\pi}}{3}\left(\frac{5}{6}\right)^5 \kappa_2^2\right]^{\frac{\alpha+3}{\alpha+2}} \mathcal{T}_\alpha^{-\frac{1}{\alpha+2}} \\
&= \frac{(p-3)\pi}{2p-3}\kappa_2^{\frac{p-6}{2(p-3)}}\left[\frac{3p^4}{16(2p-3)\mu}\right]^{\frac{3}{2(p-3)}} + \frac{\alpha+2}{4(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}} \\
&< \frac{\alpha+2}{2(\alpha+3)}\mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}.
\end{aligned} \tag{5.15}$$

This shows that (5.14) holds. □

In view of the Brezis–Lieb lemma, Lemmas 2.7 and 2.10, one can easily prove the following lemma.

Lemma 5.9 *Assume that* $p \in (3,6)$ *and* $\mu > 0$. *If* $u_n \rightharpoonup \bar{u}$ *in* E, *then*

$$\Phi_\mu(u_n) = \Phi_\mu(\bar{u}) + \Phi_\mu(u_n - \bar{u}) + o(1), \tag{5.16}$$

$$\langle \Phi'(u_n), u_n \rangle = \langle \Phi'(\bar{u}), \bar{u} \rangle + \langle \Phi'(u_n - \bar{u}), u_n - \bar{u} \rangle + o(1) \tag{5.17}$$

and

$$J_\mu(u_n) = J_\mu(\bar{u}) + J_\mu(u_n - \bar{u}) + o(1). \tag{5.18}$$

Following the idea of [33], we prove the attainable of $\hat{m}_\mu$, which reads as follows.

Lemma 5.10 *Assume that the conditions in Theorem 1.4 hold. Then $\hat{m}_\mu$ is achieved.*

Proof Let $\{u_n\} \subset \mathcal{M}_\mu$ be such that $\Phi_\mu(u_n) \to \hat{m}_\mu$. Since $J_\mu(u_n) = 0$, then it follows from (1.13) and (1.20) that

$$\hat{m}_\mu + o(1) = \frac{2(p-3)\mu}{3p} \|u_n\|_p^p + \frac{\alpha+2}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3} \right) |u_n|^{\alpha+3} \mathrm{d}x \tag{5.19}$$

and

$$\hat{m}_\mu + o(1) = \frac{\alpha+2}{2(\alpha+3)} \|\nabla u_n\|_2^2 + \frac{\alpha+2}{4(\alpha+3)} \mathcal{N}[u_n] - \frac{[3(\alpha+4)-2p]\mu}{3p(\alpha+3)} \|u_n\|_p^p. \tag{5.20}$$

By (1.20) and $J_\mu(u_n) = 0$, we have

$$\frac{3}{2} \|\nabla u_n\|_2^2 + \frac{3}{4} \mathcal{N}[u_n] = \frac{(2p-3)\mu}{p} \|u_n\|_p^p + \frac{3}{2} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3} \right) |u_n|^{\alpha+3} \mathrm{d}x. \tag{5.21}$$

The combination of (5.19) and (5.21) shows that $\{u_n\}$ is bounded in E. From (5.21), we have also

$$\|\nabla u_n\|_2^2 \leq \frac{2(2p-3)\mu}{3p} \|u_n\|_p^p + \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3} \right) |u_n|^{\alpha+3} \mathrm{d}x. \tag{5.22}$$

We claim that there exist a $\delta > 0$ and a sequence $\{y_n\} \subset \mathbb{R}^3$ such that

$$\liminf_{n\to\infty} \int_{B_1(y_n)} |u_n|^3 \mathrm{d}x > \delta. \tag{5.23}$$

Indeed, suppose that (5.23) does not hold. Then we have

$$\limsup_{n\to\infty} \sup_{y \in \mathbb{R}^3} \int_{B_1(y)} |u_n|^3 \mathrm{d}x = 0. \tag{5.24}$$

By [12, Lemma 2.5], we have

$$\|u_n\|_p^p \to 0. \tag{5.25}$$

Up to a subsequence, we assume that

$$\|\nabla u_n\|_2^2 \to l_1 \geq 0, \quad \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x \to l_2 \geq 0. \tag{5.26}$$

Then it follows from (1.16), (5.22), (5.25) and (5.26) that

$$\begin{aligned} l_1 &= \lim_{n\to\infty} \|\nabla u_n\|_2^2 \leq \lim_{n\to\infty} \int_{\mathbb{R}^3} \left(I_\alpha * |u_n|^{\alpha+3}\right) |u_n|^{\alpha+3} \mathrm{d}x \\ &\leq \mathcal{S}_\alpha^{-(\alpha+3)} \lim_{n\to\infty} \|\nabla u_n\|_2^{2(\alpha+3)} = \mathcal{S}_\alpha^{-(\alpha+3)} l_1^{\alpha+3}. \end{aligned} \tag{5.27}$$

If $l_1 > 0$, then (5.27) implies that $l_1 \geq \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}$, which, together with (5.20) and (5.25), implies that

$$\hat{m}_\mu \geq \frac{\alpha+2}{2(\alpha+3)} \mathcal{S}_\alpha^{\frac{\alpha+3}{\alpha+2}}.$$

This contradicts with (5.8) and (5.14). Therefore, (5.23) holds.

Letting $\hat{u}_n(x) = u_n(x + y_n)$, we have $\|\hat{u}_n\|_E = \|u_n\|_E$ and

$$J_\mu(\hat{u}_n) = 0, \quad \Phi_\mu(\hat{u}_n) \to \hat{m}_\mu, \quad \liminf_{n\to\infty} \int_{B_1(0)} |\hat{u}_n|^3 \mathrm{d}x > \delta. \tag{5.28}$$

Then there exists $\hat{u} \in E\backslash\{0\}$ such that, passing to a subsequence,

$$\begin{cases} \hat{u}_n \rightharpoonup \hat{u}, & \text{in } E; \\ \hat{u}_n \rightharpoonup \hat{u}, & \text{in } L^s_{\mathrm{loc}}(\mathbb{R}^3), \ \forall\, s \in [1, 6); \\ \hat{u}_n \to \hat{u}, & \text{a.e. on } \mathbb{R}^3. \end{cases} \tag{5.29}$$

Letting $w_n = \hat{u}_n - \hat{u}$, it follows from (5.29) and Lemma 5.9 that

$$\Phi_\mu(\hat{u}_n) = \Phi_\mu(\hat{u}) + \Phi_\mu(w_n) + o(1) \tag{5.30}$$

and

$$J_\mu(\hat{u}_n) = J_\mu(\hat{u}) + J_\mu(w_n) + o(1). \tag{5.31}$$

By (1.13), (1.20), (5.28), (5.30) and (5.31), we have

$$\frac{2(p-3)\mu}{3p} \|w_n\|_p^p + \frac{\alpha+2}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |w_n|^{\alpha+3}\right) |w_n|^{\alpha+3} \mathrm{d}x$$

$$= \hat{m}_\mu - \frac{2(p-3)\mu}{3p}\|\hat{u}\|_p^p - \frac{\alpha+2}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |\hat{u}|^{\alpha+3}\right)|\hat{u}|^{\alpha+3}\mathrm{d}x + o(1) \tag{5.32}$$

and

$$J_\mu(w_n) = -J_\mu(\hat{u}) + o(1). \tag{5.33}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then going to this subsequence, we have

$$\Phi_\mu(\hat{u}) = \hat{m}_\mu, \quad J_\mu(\hat{u}) = 0, \tag{5.34}$$

which implies the conclusion of Lemma 5.10 holds. Next, we assume that $w_n \neq 0$. In view of Lemma 5.4, there exists $t_n > 0$ such that $t_n^2(w_n)_{t_n} \in \mathcal{M}_\mu$. We claim that $J_\mu(\hat{u}) \leq 0$. Otherwise, if $J_\mu(\hat{u}) > 0$, then (5.33) implies that $J_\mu(w_n) < 0$ for large n. From (1.13), (1.20), (5.3) and (5.32), we obtain

$$\begin{aligned}
&\hat{m}_\mu - \frac{2(p-3)\mu}{3p}\|\hat{u}\|_p^p - \frac{\alpha+2}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |\hat{u}|^{\alpha+3}\right)|\hat{u}|^{\alpha+3}\mathrm{d}x + o(1) \\
&\quad = \frac{2(p-3)\mu}{3p}\|w_n\|_p^p + \frac{\alpha+2}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |w_n|^{\alpha+3}\right)|w_n|^{\alpha+3}\mathrm{d}x \\
&\quad = \Phi_\mu(w_n) - \frac{1}{3}J_\mu(w_n) \\
&\quad \geq \Phi_\mu\left(t_n^2(w_n)_{t_n}\right) - \frac{t_n^3}{3}J_\mu(w_n) \\
&\quad \geq \hat{m}_\mu - \frac{t_n^3}{3}J_\mu(w_n) \geq \hat{m}_\mu,
\end{aligned}$$

which implies $J_\mu(\hat{u}) \leq 0$ due to $\frac{2(p-3)\mu}{3p}\|\hat{u}\|_p^p + \frac{\alpha+2}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |\hat{u}|^{\alpha+3}\right)|\hat{u}|^{\alpha+3}\mathrm{d}x > 0$. Since $\hat{u} \in E\backslash\{0\}$, from Lemma 5.4, there exists $\hat{t} > 0$ such that $\hat{t}^2\hat{u}_{\hat{t}} \in \mathcal{M}_\mu$. From (1.13), (1.20), (5.3), (5.28) and Fatou's lemma, we derive

$$\begin{aligned}
\hat{m}_\mu &= \lim_{n\to\infty}\left[\Phi_\mu(\hat{u}_n) - \frac{1}{3}J_\mu(\hat{u}_n)\right] \\
&= \lim_{n\to\infty}\left[\frac{2(p-3)\mu}{3p}\|\hat{u}_n\|_p^p + \frac{\alpha+2}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |\hat{u}_n|^{\alpha+3}\right)|\hat{u}_n|^{\alpha+3}\mathrm{d}x\right] \\
&\geq \frac{2(p-3)\mu}{3p}\|\hat{u}\|_p^p + \frac{\alpha+2}{2(\alpha+3)}\int_{\mathbb{R}^3}\left(I_\alpha * |\hat{u}|^{\alpha+3}\right)|\hat{u}|^{\alpha+3}\mathrm{d}x \\
&= \Phi_\mu(\hat{u}) - \frac{1}{3}J_\mu(\hat{u}) \\
&\geq \Phi_\mu\left(\hat{t}^2\hat{u}_{\hat{t}}\right) - \frac{\hat{t}^3}{3}J_\mu(\hat{u})
\end{aligned}$$

$$\geq \hat{m}_\mu - \frac{\hat{t}^3}{3} J_\mu(\hat{u}) \geq \hat{m}_\mu,$$

which implies that (5.34) holds also. □

Following the idea of [7], we prove the following lemma.

Lemma 5.11 *Assume that the conditions in Theorem 1.4 hold. If $\hat{u} \in \mathcal{M}_\mu$ and $\Phi_\mu(\hat{u}) = \hat{m}_\mu$, then $\hat{u}$ is a critical point of Φ_μ.*

Proof Assume that $\Phi'_\mu(\hat{u}) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - \hat{u}\|_E \leq 3\delta \Rightarrow \|\Phi'_\mu(u)\| \geq \varrho. \tag{5.35}$$

Let $\{t_n\} \subset \mathbb{R}$ such that $t_n \to 1$. Since $t_n^2 \hat{u}_{t_n} \rightharpoonup \hat{u}$ in E, then it follows from (2.10) and Lemma 2.6 that

$$\begin{aligned} \left\|\nabla\left(t_n^2 \hat{u}_{t_n}\right) - \nabla \hat{u}\right\|_2^2 &= \int_{\mathbb{R}^3} \left|\nabla\left(t_n^2 \hat{u}_{t_n}\right) - \nabla \hat{u}\right|^2 \mathrm{d}x \\ &= (t_n^3 + 1)\int_{\mathbb{R}^3} |\nabla \hat{u}|^2 \mathrm{d}x - 2\int_{\mathbb{R}^3} \nabla\left(t_n^2 \hat{u}_{t_n}\right) \cdot \nabla \hat{u} \mathrm{d}x = o(1) \end{aligned} \tag{5.36}$$

and

$$\begin{aligned} &\mathcal{N}\left(t_n^2 \hat{u}_{t_n} - \hat{u}\right) \\ &= D\left((t_n^2 \hat{u}_{t_n} - \hat{u})^2, (t_n^2 \hat{u}_{t_n} - \hat{u})^2\right) \\ &= D\left((t_n^2 \hat{u}_{t_n})^2, (t_n^2 \hat{u}_{t_n})^2\right) + D\left(\hat{u}^2, \hat{u}^2\right) - 4D\left((t_n^2 \hat{u}_{t_n})^2, (t_n^2 \hat{u}_{t_n})\hat{u}\right) \\ &\quad - 4D\left(\hat{u}^2, (t_n^2 \hat{u}_{t_n})\hat{u}\right) + 4D\left((t_n^2 \hat{u}_{t_n})\hat{u}, (t_n^2 \hat{u}_{t_n})\hat{u}\right) + 2D\left((t_n^2 \hat{u}_{t_n})^2, \hat{u}^2\right) \\ &= D\left((t_n^2 \hat{u}_{t_n})^2, (t_n^2 \hat{u}_{t_n})^2\right) - D\left(\hat{u}^2, \hat{u}^2\right) + o(1) \\ &= (t_n^3 - 1) D\left(\hat{u}^2, \hat{u}^2\right) + o(1) = o(1). \end{aligned} \tag{5.37}$$

Combining (5.36) with (5.37), we have

$$\lim_{t \to 1} \left\|t^2 \hat{u}_t - \hat{u}\right\|_E = 0. \tag{5.38}$$

Thus, there exists $\delta_1 > 0$ such that

$$|t - 1| < \delta_1 \Rightarrow \|t^2 \hat{u}_t - \hat{u}\|_E < \delta. \tag{5.39}$$

From Lemma 5.1, we derive

$$\Phi_\mu\left(t^2 \hat{u}_t\right) \leq \Phi_\mu(\hat{u}) - \frac{\alpha + 2 - (\alpha+3)t^3 + t^{3(\alpha+3)}}{2(\alpha+3)} \int_{\mathbb{R}^3} \left(I_\alpha * |\hat{u}|^{\alpha+3}\right) |\hat{u}|^{\alpha+3} \mathrm{d}x$$

$$= \hat{m}_\mu - \frac{\alpha + 2 - (\alpha + 3)t^3 + t^{3(\alpha+3)}}{2(\alpha + 3)} \int_{\mathbb{R}^3} \left(I_\alpha * |\hat{u}|^{\alpha+3} \right) |\hat{u}|^{\alpha+3} \mathrm{d}x, \quad \forall\, t > 0. \tag{5.40}$$

Using (1.20), it is easy to check that there exist $T_1 \in (0, 1)$ and $T_2 \in (1, \infty)$ such that

$$J\left(T_1^2 \hat{u}_{T_1}\right) > 0, \quad J\left(T_2^2 \hat{u}_{T_2}\right) < 0. \tag{5.41}$$

Set $\Theta = \frac{1}{2(\alpha+3)} \min\{h(T_1), h(T_2)\} \int_{\mathbb{R}^3} \left(I_\alpha * |\hat{u}|^{\alpha+3} \right) |\hat{u}|^{\alpha+3} \mathrm{d}x$, where $h(t)$ is defined by (5.2). Let $S := B(\hat{u}, \delta)$ and $\varepsilon := \min\{\Theta/24, 1, \varrho\delta/8\}$. Then [35, Lemma 2.3] yields a deformation $\eta \in \mathcal{C}([0, 1] \times E, E)$ such that

(i) $\eta(1, u) = u$ if $\Phi_\mu(u) < \hat{m}_\mu - 2\varepsilon$ or $\Phi_\mu(u) > \hat{m}_\mu + 2\varepsilon$;
(ii) $\eta\left(1, \Phi^{\hat{m}_\mu+\varepsilon} \cap B(\hat{u}, \delta)\right) \subset \Phi^{\hat{m}_\mu-\varepsilon}$;
(iii) $\Phi_\mu(\eta(1, u)) \le \Phi_\mu(u), \ \forall\, u \in E$;
(iv) $\eta(1, u)$ is a homeomorphism of E.

Noting that $\Phi_\mu\left(t^2 \hat{u}_t\right) \le \Phi_\mu(\hat{u}) = \hat{m}_\mu$ for $t > 0$, it follows from Corollary 5.3, (5.39) and the above ii) that

$$\Phi_\mu\left(\eta(1, t^2 \hat{u}_t)\right) \le \hat{m}_\mu - \varepsilon, \quad \forall\, t > 0, \ |t - 1| < \delta_1. \tag{5.42}$$

On the other hand, by iii) and (5.40), we have

$$\begin{aligned} \Phi_\mu\left(\eta(1, t^2 \hat{u}_t)\right) &\le \Phi_\mu\left(t^2 \hat{u}_t\right) \\ &\le \hat{m}_\mu - \frac{\alpha + 2 - (\alpha + 3)t^3 + t^{3(\alpha+3)}}{2(\alpha + 3)} \int_{\mathbb{R}^3} \left(I_\alpha * |\hat{u}|^{\alpha+3} \right) |\hat{u}|^{\alpha+3} \mathrm{d}x \\ &\le \hat{m}_\mu - \delta_2, \quad \forall\, t > 0, \ |t - 1| \ge \delta_1, \end{aligned} \tag{5.43}$$

where $\delta_2 := \min\{h(1 - \delta_1), h(1 + \delta_1)\} \int_{\mathbb{R}^3} \left(I_\alpha * |\hat{u}|^{\alpha+3} \right) |\hat{u}|^{\alpha+3} \mathrm{d}x > 0$. The combination of (5.42) and (5.43) yields that

$$\max_{t \in [T_1, T_2]} \Phi_\mu\left(\eta(1, t^2 \hat{u}_t)\right) < \hat{m}_\mu. \tag{5.44}$$

Set $\Psi_0(t) := J\left(\eta\left(1, t^2 \hat{u}_t\right)\right)$ for $t > 0$. It follows from (5.43) and (i) that $\eta(1, \hat{u}_t) = \hat{u}_t$ for $t = T_1$ and $t = T_2$, which, together with (5.41), implies

$$\Psi_0(T_1) = J\left(T_1^2 \hat{u}_{T_1}\right) > 0, \quad \Psi_0(T_2) = J\left(T_2^2 \hat{u}_{T_2}\right) < 0.$$

Since $\Psi_0(t)$ is continuous on $[T_1, T_2]$, then we have that $\eta\left(1, t^2 \hat{u}_t\right) \cap \mathcal{M}_\mu \neq \emptyset$ for some $t_0 \in [T_1, T_2]$, contradicting to the definition of $\hat{m}_\mu$. □

Theorem 1.4 is a direct consequence of Lemmas 5.6, 5.10 and 5.11.

6 Case $p = 6$

In the last section, we establish the non-existence result to (1.1) with $p = 6$, and complete the proof of Theorem 1.5.

Proof of Theorem 1.5 Assume that $\hat{u} \in E$ is a solution of Problem (1.1). Multiplying (1.1) by $\hat{u}$, and then integrating, we have

$$\|\nabla \hat{u}\|_2^2 + \mathcal{N}[\hat{u}] - \mu \|\hat{u}\|_6^6 - \int_{\mathbb{R}^3} \left(I_\alpha * |\hat{u}|^{\alpha+3} \right) |\hat{u}|^{\alpha+3} \mathrm{d}x = 0. \tag{6.1}$$

Recalling the Pohozaev identity as Lemma 2.13, we also have

$$\frac{1}{2} \|\nabla \hat{u}\|_2^2 + \frac{5}{4} \mathcal{N}[\hat{u}] - \frac{\mu}{2} \|\hat{u}\|_6^6 - \frac{1}{2} \int_{\mathbb{R}^3} \left(I_\alpha * |\hat{u}|^{\alpha+3} \right) |\hat{u}|^{\alpha+3} \mathrm{d}x = 0. \tag{6.2}$$

Combining (6.1) with (6.2), we obtain

$$\mathcal{N}[\hat{u}] = 0. \tag{6.3}$$

This shows that $\hat{u} = 0$. □

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Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Declarations

Conflict of interest The authors declare that there is no Conflict of interest. We also declare that this manuscript has no associated data.

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