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# Monotonicity character of convex combinations of two sequences generated by the square of the cube root function

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## Abstract

There is a natural connection between the finite sums of the reciprocals of the squares of the cube roots of positive natural numbers and two definite integrals. By using the differences between the sums and the integrals are formed in a natural way two sequences of real numbers converging to the same finite limit. We completely determine the monotonicity character of the sequences which are obtained by convex combinations of the two sequences, in an elegant way. We also present an interesting result about monotonicity character of a real function on the interval  $[1, +\infty)$ , which is used in the proof of the main result.

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## 1 Introduction and preliminaries

Let  $\mathbb{N} = \{1, 2, 3, \dots\}$ ,  $\mathbb{Z}$  be the set of integers,  $\mathbb{N}_s = \{j \in \mathbb{Z} : j \geq s\}$ ,  $s \in \mathbb{Z}$ ,  $\mathbb{R}$  be the set of real numbers, and

$$C_k^\alpha = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!},$$

where  $\alpha \in \mathbb{R}$  and  $k \in \mathbb{N}_0$ , where we regard that  $0! = 1$ .

Recall that a sequence

$$(a_n)_{n \in \mathbb{N}_s} \subset \mathbb{R} \tag{1}$$

is monotone, if it is monotone on the whole domain of definition of indices, that is, on the set  $\mathbb{N}_s$ , and that it is eventually monotone if it is monotone on a proper subset  $\mathbb{N}_t$  of the domain. The following papers and books: [1–4, 8, 9, 12–18, 21, 23, 25–31, 37–39], among other things, deal with such sequences, including some eventually constant sequences. Some of the sequences in the references are given by some recursive relations, that is, by difference equations, whereas some of them are given by some explicit formulas or are

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described in some other ways. The investigation of their monotonicity strongly depends on the ways how they are given or described.

Determining eventual monotonicity of the sequence in (1) is a simpler task than the problem of determining its monotonicity character on the whole domain  $\mathbb{N}_s$ . We will demonstrate this fact also in an example below.

Our deep analysis of the problem proposed in [37] showed that it is sometimes possible to give a complete picture about the monotonicity character of the convex combinations of some sequences, more precisely, of the known sequences ([12, 18, 19, 25]):

$$x_k = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{pk}$$

and

$$y_k = \frac{1}{k} + \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{pk},$$

for  $k \in \mathbb{N}$ , with  $p = 2$  and  $p = 3$  on their whole domains.

This fact was a bit unexpected for us, and was proved in [33]. Therein can be also found many remarks, as well as applications of the Hermite-Hadamard inequalities ([10, 11, 17]). A solution to the problem in [37] can be also found in [38].

A natural question is whether or not this fact is something which is an isolated case, or there are other pairs of sequences of real numbers for which it is possible to give a complete picture about the monotonicity character of their convex combinations. It should be also noticed that it is not so rare that sequences are studied simultaneously in pairs (see, for instance, the pairs of sequences in [2, 5–7, 9, 14, 15, 18, 20–22, 24, 25, 28, 29, 32, 33, 35–38]).

The sequences

$$a_n = 1 + \frac{1}{\sqrt[3]{2^2}} + \dots + \frac{1}{\sqrt[3]{n^2}} - 3\sqrt[3]{n}, \tag{2}$$

$$b_n = 1 + \frac{1}{\sqrt[3]{2^2}} + \dots + \frac{1}{\sqrt[3]{n^2}} - 3\sqrt[3]{n+1}, \tag{3}$$

for  $n \in \mathbb{N}$ , looked as some good prototypes for dealing with the above-mentioned problem. They are concrete and some basic results about them are known [12]. They can be obtained, for instance, from the following sequences, which have been considered considerably for a long time ([15, 18, 25, 39]):

$$a_n(h) = \sum_{j=1}^n h(j) - \int_0^n h(t)dt, \quad b_n(h) = \sum_{j=1}^n h(j) - \int_0^{n+1} h(t)dt, \quad n \in \mathbb{N},$$

with

$$h(t) = \frac{1}{t^{2/3}}. \tag{4}$$

From (2) and (3), we have

$$\begin{aligned}
 a_n - b_n &= 3(\sqrt[3]{n+1} - \sqrt[3]{n}) \\
 &= \frac{3}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}} > 0,
 \end{aligned}
 \tag{5}$$

for  $n \in \mathbb{N}$ ,

$$\begin{aligned}
 a_{n+1} - a_n &= \frac{1}{\sqrt[3]{(n+1)^2}} - 3(\sqrt[3]{n+1} - \sqrt[3]{n}) \\
 &= \frac{1}{\sqrt[3]{(n+1)^2}} - \frac{3}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}} \\
 &= \frac{\sqrt[3]{(n+1)n} - \sqrt[3]{(n+1)^2} + \sqrt[3]{n^2} - \sqrt[3]{(n+1)^2}}{\sqrt[3]{(n+1)^2}(\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2})} < 0,
 \end{aligned}$$

for  $n \in \mathbb{N}$ , and

$$\begin{aligned}
 b_{n+1} - b_n &= \frac{1}{\sqrt[3]{(n+1)^2}} - 3(\sqrt[3]{n+2} - \sqrt[3]{n+1}) \\
 &= \frac{1}{\sqrt[3]{(n+1)^2}} - \frac{3}{\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} + \sqrt[3]{(n+1)^2}} \\
 &= \frac{\sqrt[3]{(n+2)^2} - \sqrt[3]{(n+1)^2} + \sqrt[3]{(n+2)(n+1)} - \sqrt[3]{(n+1)^2}}{\sqrt[3]{(n+1)^2}(\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} + \sqrt[3]{(n+1)^2})} > 0,
 \end{aligned}$$

for  $n \in \mathbb{N}$ .

Hence,  $(a_n)_{n \in \mathbb{N}}$  is strictly decreasing, whereas  $(b_n)_{n \in \mathbb{N}}$  is strictly increasing, and both sequences are bounded, so convergent.

Letting  $n \rightarrow \infty$  in (5) we get

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a^*,
 \tag{6}$$

for some  $a^* > 0$ .

We study here the sequences

$$c_n^{(\lambda)} := \lambda a_n + (1 - \lambda)b_n, \quad n \in \mathbb{N},
 \tag{7}$$

for  $\lambda \in [0, 1]$ . It should be noticed that due to (6),  $c_n^{(\lambda)}$  converge to  $a^*$ , but this fact does not tell us practically anything about their monotonicity.

As we have mentioned determining the eventual monotonicity is a simpler problem than the problem of describing the monotonicity on the whole domain. One of the reasons for this is the fact that we can apply some known asymptotic formulas in dealing with the eventual monotonicity (see, for instance, some of the problems on real sequences in [15, 18, 25]).

Using the well known formula

$$(1 + t)^\alpha = 1 + C_1^\alpha t + C_2^\alpha t^2 + C_3^\alpha t^3 + O(t^4)$$

as  $t \rightarrow 0$  ([15, 18, 39]), it follows that

$$\begin{aligned}
 c_{n+1}^{(\lambda)} - c_n^{(\lambda)} &= \lambda(a_{n+1} - a_n) + (1 - \lambda)(b_{n+1} - b_n) \\
 &= \frac{1}{\sqrt[3]{(n+1)^2}} - 3(\sqrt[3]{n+2} - \sqrt[3]{n+1}) + 3\lambda(\sqrt[3]{n+2} - 2\sqrt[3]{n+1} + \sqrt[3]{n}) \\
 &= \frac{1}{\sqrt[3]{n^2}} \left(1 + \frac{1}{n}\right)^{-2/3} - 3\sqrt[3]{n} \left( (2\lambda - 1) \left(1 + \frac{1}{n}\right)^{1/3} + (1 - \lambda) \left(1 + \frac{2}{n}\right)^{1/3} - \lambda \right) \\
 &= \frac{1}{\sqrt[3]{n^2}} \left(1 - \frac{2}{3n} + \frac{5}{9n^2} - \frac{40}{81n^3} + O\left(\frac{1}{n^4}\right)\right) \\
 &\quad - 3\sqrt[3]{n} \left( (2\lambda - 1) \left(1 + \frac{1}{3n} - \frac{1}{9n^2} + \frac{5}{81n^3} + O\left(\frac{1}{n^4}\right)\right) \right. \\
 &\quad \left. + (1 - \lambda) \left(1 + \frac{2}{3n} - \frac{4}{9n^2} + \frac{40}{81n^3} + O\left(\frac{1}{n^4}\right)\right) - \lambda \right) \\
 &= \frac{1 - 2\lambda}{3n^{5/3}} + \frac{30\lambda - 20}{27n^{8/3}} + O\left(\frac{1}{n^{11/3}}\right). \tag{8}
 \end{aligned}$$

Relation (8) shows that  $c_n^{(\lambda)}$  is eventually increasing for  $\lambda \in [0, \frac{1}{2})$ , and eventually decreasing for  $\lambda \in (\frac{1}{2}, 1]$ .

For  $\lambda = \frac{1}{2}$ , from (8) we get

$$c_{n+1}^{(1/2)} - c_n^{(1/2)} = -\frac{5}{27n^{8/3}} + O\left(\frac{1}{n^{11/3}}\right),$$

so,  $c_n^{(1/2)}$  is eventually decreasing.

Thus,  $c_n^{(\lambda)}$  is eventually decreasing when  $\lambda \in [1/2, 1]$ , and it is eventually increasing when  $\lambda \in [0, 1/2)$ . However, from these facts we cannot conclude almost anything about the monotonicity character of  $c_n^{(\lambda)}$  on the whole domain.

In the section that follows we give a complete picture about the monotonicity character of the sequences  $(c_n^{(\lambda)})_{n \in \mathbb{N}}$  for each value of the parameter  $\lambda$  belonging to the interval  $[0, 1]$ .

### 2 Main results

Here we first present a nontrivial lemma, which is employed in the proof of our main theorem. The lemma is one of the most important parts of the proof.

**Lemma 1** *Consider the function*

$$\begin{aligned}
 g(t) &:= \frac{\sqrt[3]{(t+2)^2} + \sqrt[3]{(t+2)(t+1)} - 2\sqrt[3]{(t+1)^2}}{\sqrt[3]{(t+2)^2} + \sqrt[3]{(t+2)(t+1)} - \sqrt[3]{(t+1)t} - \sqrt[3]{t^2}} \\
 &\quad \times \frac{\sqrt[3]{(t+1)^2} + \sqrt[3]{(t+1)t} + \sqrt[3]{t^2}}{\sqrt[3]{(t+1)^2}}, \tag{9}
 \end{aligned}$$

for  $t \in [1, +\infty)$ . Then,  $g$  is strictly increasing on  $[1, +\infty)$  and also the following relation holds

$$\lim_{t \rightarrow +\infty} g(t) = \frac{3}{2}. \tag{10}$$

*Proof* Let

$$\begin{aligned} a(t) &:= \sqrt[3]{t}, \\ b(t) &:= \sqrt[3]{t+1}, \\ c(t) &:= \sqrt[3]{t+2}, \end{aligned} \tag{11}$$

then we have

$$c^3(t) - b^3(t) = b^3(t) - a^3(t) = 1, \tag{12}$$

for  $t \geq 0$ .

By using some algebraic calculations, as well as (11) and (12), it follows that

$$\begin{aligned} g(t) &= \frac{(c^2(t) + c(t)b(t) - 2b^2(t))(b^2(t) + a(t)b(t) + a^2(t))}{b^2(t)(c^2(t) + c(t)b(t) - a(t)b(t) - a^2(t))} \\ &= \frac{(c^2(t) + c(t)b(t) - 2b^2(t))}{c^2(t) + c(t)b(t) - a(t)b(t) - a^2(t)} \left( 1 + \frac{a(t)}{b(t)} + \left( \frac{a(t)}{b(t)} \right)^2 \right) \\ &= \frac{(c(t) - b(t))(c(t) + 2b(t))}{(c(t) - a(t))(c(t) + a(t) + b(t))} \left( 1 + \frac{a(t)}{b(t)} + \left( \frac{a(t)}{b(t)} \right)^2 \right) \\ &= \frac{(c(t) + 2b(t))(c^2(t) + c(t)a(t) + a^2(t))}{2(a(t) + b(t) + c(t))(c^2(t) + c(t)b(t) + b^2(t))} \left( 1 + \frac{a(t)}{b(t)} + \left( \frac{a(t)}{b(t)} \right)^2 \right) \\ &= \frac{c^3(t) + 2b(t)c^2(t) + a(t)c^2(t) + 2a(t)b(t)c(t) + a^2(t)c(t) + 2a^2(t)b(t)}{c^3(t) + 2b(t)c^2(t) + 2b^2(t)c(t) + b^3(t) + a(t)c^2(t) + a(t)b(t)c(t) + a(t)b^2(t)} \\ &\quad \times \frac{1}{2} \left( 1 + \frac{a(t)}{b(t)} + \left( \frac{a(t)}{b(t)} \right)^2 \right). \end{aligned} \tag{13}$$

Since the function

$$\frac{a(t)}{b(t)} = \sqrt[3]{\frac{t}{t+1}} = \sqrt[3]{1 - \frac{1}{t+1}}$$

is positive and strictly increasing for  $t > 0$ , so is also the function

$$1 + \frac{a(t)}{b(t)} + \left( \frac{a(t)}{b(t)} \right)^2,$$

from which along with (13) it is enough to prove that the function

$$\frac{c^3(t) + 2b(t)c^2(t) + a(t)c^2(t) + 2a(t)b(t)c(t) + a^2(t)c(t) + 2a^2(t)b(t)}{c^3(t) + 2b(t)c^2(t) + 2b^2(t)c(t) + a(t)c^2(t) + a(t)b(t)c(t) + a(t)b^2(t) + b^3(t)} \tag{14}$$

is positive and strictly increasing for  $t > 0$ .

The positivity of the function in (14) follows from the positivity of the functions in (11).

Let  $A(t)$  and  $B(t)$  be the numerator and the denominator of the function in (14), respectively. Then, by using some calculations, (12) and the relation

$$B(t) = (a(t) + b(t) + c(t))(c^2(t) + c(t)b(t) + b^2(t)),$$

we have

$$\begin{aligned}
 \frac{A(t)}{B(t)} &= 1 + \frac{A(t) - B(t)}{B(t)} \\
 &= 1 + \frac{a(t)b(t)c(t) + a^2(t)c(t) + 2a^2(t)b(t) - 2b^2(t)c(t) - a(t)b^2(t) - b^3(t)}{B(t)} \\
 &= 1 + \frac{(a(t) - b(t))(b(t)c(t) + c(t)(a(t) + b(t)) + a(t)b(t) + b(t)(a(t) + b(t)))}{B(t)} \\
 &= 1 - \frac{(b(t) - a(t))(2b(t)c(t) + a(t)c(t) + 2a(t)b(t) + b^2(t))}{B(t)} \\
 &= 1 - \frac{b^2(t) + 2b(t)c(t) + a(t)c(t) + 2a(t)b(t)}{B(t)(b^2(t) + b(t)a(t) + a^2(t))} \\
 &= 1 - \frac{1}{B(t)\left(1 + \frac{a(t)}{b(t)} + \left(\frac{a(t)}{b(t)}\right)^2\right)} \\
 &\quad - \frac{2}{b(t)c(t)(a(t) + b(t) + c(t))\left(1 + \frac{a(t)}{b(t)} + \left(\frac{a(t)}{b(t)}\right)^2\right)\left(1 + \frac{b(t)}{c(t)} + \left(\frac{b(t)}{c(t)}\right)^2\right)} \\
 &\quad - \frac{a(t)\left(\frac{1}{b(t)} + \frac{2}{c(t)}\right)}{b(t)c(t)(a(t) + b(t) + c(t))\left(1 + \frac{a(t)}{b(t)} + \left(\frac{a(t)}{b(t)}\right)^2\right)\left(1 + \frac{b(t)}{c(t)} + \left(\frac{b(t)}{c(t)}\right)^2\right)}. \tag{15}
 \end{aligned}$$

Further, we have

$$\begin{aligned}
 &\frac{a(t)}{b(t)c(t)(a(t) + b(t) + c(t))} \\
 &= \frac{\sqrt[3]{t}}{\sqrt[3]{(t+1)(t+2)}(\sqrt[3]{t} + \sqrt[3]{t+1} + \sqrt[3]{t+2})} \\
 &= \frac{1}{\sqrt[3]{(t+1)(t+2)} + \sqrt[3]{\frac{(t+1)^2(t+2)}{t}} + \sqrt[3]{\frac{(t+1)(t+2)^2}{t}}} \\
 &= \frac{1}{\sqrt[3]{(t+1)(t+2)} + \sqrt[3]{t^2 + 4t + 5 + \frac{2}{t}} + \sqrt[3]{t^2 + 5t + 8 + \frac{4}{t}}}. \tag{16}
 \end{aligned}$$

Let

$$h_1(t) := t^2 + 4t + 5 + \frac{2}{t}$$

and

$$h_2(t) := t^2 + 5t + 8 + \frac{4}{t}.$$

Since

$$h'_1(t) = 2t + 4 - \frac{2}{t^2} \geq 4, \tag{17}$$

for  $t \geq 1$ , and

$$h'_2(t) = 2t + 5 - \frac{4}{t^2} \geq 3, \tag{18}$$

for  $t \geq 1$ , we see that the functions  $h_1$  and  $h_2$  are strictly increasing for  $t \geq 1$ , from which it follows that the function in (16) is strictly decreasing for  $t \geq 1$ .

Using this fact, the fact that the function

$$\frac{b(t)}{c(t)} = \sqrt[3]{\frac{t+1}{t+2}} = \sqrt[3]{1 - \frac{1}{t+2}}$$

as well as the functions  $a(t)$ ,  $b(t)$  and  $c(t)$  are strictly increasing for  $t \geq 0$ , from relation (15) it follows that the function in (14) is strictly increasing for  $t \geq 1$ .

Finally, we have

$$\begin{aligned} \lim_{t \rightarrow +\infty} g(t) &= \lim_{t \rightarrow +\infty} \frac{1 + 2\frac{b(t)}{c(t)} + \frac{a(t)}{c(t)} + 2\frac{a(t)}{c(t)}\frac{b(t)}{c(t)} + \frac{a^2(t)}{c^2(t)} + 2\frac{a^2(t)}{c^2(t)}\frac{b(t)}{c(t)}}{1 + 2\frac{b(t)}{c(t)} + 2\frac{b^2(t)}{c^2(t)} + \frac{b^3(t)}{c^3(t)} + \frac{a(t)}{c(t)} + \frac{a(t)}{c(t)}\frac{b(t)}{c(t)} + \frac{a(t)}{c(t)}\frac{b^2(t)}{c^2(t)}} \\ &\quad \times \frac{1}{2} \left( 1 + \frac{a(t)}{b(t)} + \left( \frac{a(t)}{b(t)} \right)^2 \right) = \frac{3}{2}, \end{aligned} \tag{19}$$

since

$$\lim_{t \rightarrow +\infty} \frac{a(t)}{c(t)} = \lim_{t \rightarrow +\infty} \frac{b(t)}{c(t)} = \lim_{t \rightarrow +\infty} \frac{a(t)}{b(t)} = 1,$$

which is very easily verified by some well known methods, completing the proof of the lemma. □

*Remark 1* We did not show the monotonicity of the function  $g(t)$  defined in (9) by using the standard method of differentiation, since the calculation of the derivatives of the function looked even more complicated, than the method that we have used in the proof of Lemma 1. It is expected that the first part of Lemma 1 can be also proved by using the differentiation. We leave the problem to the interested reader as an exercise.

*Remark 2* It is interesting that the monotonicity of the functions  $h_1$  and  $h_2$  on the interval  $[1, +\infty)$  can be proved without using the derivatives, that is, without using the inequalities (17) and (18), so that there is an elementary way for proving the strict monotonicity of the function in (9). Indeed, we have

$$\begin{aligned} h_1(t) - h_1(s) &= t^2 + 4t + 5 + \frac{2}{t} - \left( s^2 + 4s + 5 + \frac{2}{s} \right) \\ &= (t - s) \left( t + s + 4 - \frac{2}{ts} \right) \geq 4(t - s), \end{aligned} \tag{20}$$

for  $t, s \in [1, +\infty)$ , and

$$\begin{aligned} h_2(t) - h_2(s) &= t^2 + 5t + 8 + \frac{4}{t} - \left( s^2 + 5s + 8 + \frac{4}{s} \right) \\ &= (t - s) \left( t + s + 5 - \frac{4}{ts} \right) \geq 3(t - s), \end{aligned} \tag{21}$$

for every  $t, s \in [1, +\infty)$ .

If  $t > s$ , then from (20) and (21) it follows that

$$h_1(t) > h_1(s) \quad \text{and} \quad h_2(t) > h_2(s),$$

that is, the functions  $h_1$  and  $h_2$  are strictly increasing on the interval  $[1, +\infty)$ , as claimed.

Now we are in a position to formulate and prove the main result in this paper.

**Theorem 1** *Assume that  $\lambda \in [0, 1]$ ,  $(c_n^{(\lambda)})_{n \in \mathbb{N}}$  is given in (7), and*

$$\lambda_k = g(k)/3, \quad k \in \mathbb{N}_0, \tag{22}$$

where the function  $g$  is defined in (9).

Then, the following statements hold.

- (a) Assume  $\lambda \in [1/2, 1]$ , then  $(c_n^{(\lambda)})_{n \in \mathbb{N}}$  is strictly decreasing.
- (b) Assume  $\lambda \in [0, \lambda_1)$ , then  $(c_n^{(\lambda)})_{n \in \mathbb{N}}$  is strictly increasing.
- (c) Assume  $\lambda = \lambda_1$ , then  $(c_n^{(\lambda)})_{n \in \mathbb{N}_2}$  is strictly increasing and  $c_1^{(\lambda)} = c_2^{(\lambda)}$ .
- (d) Assume  $\lambda \in (\lambda_k, \lambda_{k+1})$  for some  $k \in \mathbb{N}$ , then  $(c_n^{(\lambda)})_{n \in \mathbb{N}_{k+1}}$  is strictly increasing and  $c_n^{(\lambda)}$  is strictly decreasing for  $1 \leq n \leq k + 1$ .
- (e) Assume  $\lambda = \lambda_{k+1}$  for some  $k \in \mathbb{N}$ , then  $(c_n^{(\lambda)})_{n \in \mathbb{N}_{k+2}}$  is strictly increasing,  $c_{k+1}^{(\lambda)} = c_{k+2}^{(\lambda)}$ , and  $c_n^{(\lambda)}$  is strictly decreasing for  $1 \leq n \leq k + 1$ .

*Proof* Using some calculation, it follows that

$$\begin{aligned} c_{n+1}^{(\lambda)} - c_n^{(\lambda)} &= \lambda(a_{n+1} - a_n) + (1 - \lambda)(b_{n+1} - b_n) \\ &= \frac{1}{\sqrt[3]{(n+1)^2}} - 3(\sqrt[3]{n+2} - \sqrt[3]{n+1}) \\ &\quad + 3\lambda(\sqrt[3]{n+2} - \sqrt[3]{n+1} - (\sqrt[3]{n+1} - \sqrt[3]{n})) \\ &= \frac{1}{\sqrt[3]{(n+1)^2}} - \frac{3}{\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} + \sqrt[3]{(n+1)^2}} \\ &\quad + 3\lambda \left( \frac{1}{\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} + \sqrt[3]{(n+1)^2}} \right. \\ &\quad \left. - \frac{1}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}} \right) \\ &= \frac{\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} - 2\sqrt[3]{(n+1)^2}}{\sqrt[3]{(n+1)^2}(\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} + \sqrt[3]{(n+1)^2})} \\ &\quad + 3\lambda \left( \frac{\sqrt[3]{(n+1)n} + \sqrt[3]{n^2} - \sqrt[3]{(n+2)^2} - \sqrt[3]{(n+2)(n+1)}}{\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} + \sqrt[3]{(n+1)^2}} \right. \\ &\quad \left. \times \frac{1}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}} \right) \\ &= I_n(g(n) - 3\lambda), \end{aligned} \tag{23}$$

for  $n \in \mathbb{N}$ , where

$$I_n = \frac{\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} - \sqrt[3]{(n+1)n} - \sqrt[3]{n^2}}{\sqrt[3]{(n+2)^2} + \sqrt[3]{(n+2)(n+1)} + \sqrt[3]{(n+1)^2}} \times \frac{1}{\sqrt[3]{(n+1)^2} + \sqrt[3]{(n+1)n} + \sqrt[3]{n^2}}, \quad n \in \mathbb{N}.$$

Note that

$$I_n > 0, \tag{24}$$

for every  $n \in \mathbb{N}$ .

(a) By using Lemma 1, we have that

$$g(t) < 3/2 \quad \text{for } t \geq 1. \tag{25}$$

From (23), (24) and (25), it follows that

$$c_{n+1}^{(\lambda)} - c_n^{(\lambda)} = I_n(g(n) - 3\lambda) \leq I_n\left(g(n) - \frac{3}{2}\right) < 0,$$

for  $\lambda \in [1/2, 1]$  and  $n \in \mathbb{N}$ , that is,  $(c_n^{(\lambda)})_{n \in \mathbb{N}}$  is strictly decreasing in this case.

(b) Since  $\lambda \in [0, \lambda_1)$ ,

$$\lambda_1 = \frac{g(1)}{3}$$

and the function  $g$  is strictly increasing for  $t \geq 1$ , from (23) and (24), it follows that

$$c_{n+1}^{(\lambda)} - c_n^{(\lambda)} = I_n(g(n) - 3\lambda) \geq I_n(g(1) - 3\lambda) > 0,$$

for  $n \in \mathbb{N}$ , that is,  $(c_n^{(\lambda)})_{n \in \mathbb{N}}$  is strictly increasing in this case.

(c) Since

$$\lambda = \lambda_1 = \frac{g(1)}{3}$$

and the function  $g$  is strictly increasing for  $t \geq 1$ , from (23) and (24), it follows that

$$c_{n+1}^{(\lambda_1)} - c_n^{(\lambda_1)} = I_n(g(n) - 3\lambda_1) \geq I_n(g(2) - 3\lambda_1) > I_n(g(1) - 3\lambda_1) = 0,$$

for  $n \in \mathbb{N}_2$ , that is,  $(c_n^{(\lambda_1)})_{n \in \mathbb{N}_2}$  is strictly increasing, and since

$$c_2^{(\lambda_1)} - c_1^{(\lambda_1)} = I_1(g(1) - 3\lambda_1) = 0,$$

we have

$$c_1^{(\lambda_1)} = c_2^{(\lambda_1)}.$$

(d) Since  $\lambda \in (\lambda_k, \lambda_{k+1})$  for a fixed  $k \in \mathbb{N}$ , and the function  $g$  is strictly increasing for  $t \geq 1$ , from (23) and (24), it follows that

$$c_{n+1}^{(\lambda)} - c_n^{(\lambda)} = I_n(g(n) - 3\lambda) \geq I_n(g(k+1) - 3\lambda) > I_n(g(k+1) - 3\lambda_{k+1}) = 0,$$

for  $n \in \mathbb{N}_{k+1}$ , that is,  $(c_n^{(\lambda)})_{n \in \mathbb{N}_{k+1}}$  is strictly increasing, and since

$$c_{n+1}^{(\lambda)} - c_n^{(\lambda)} = I_n(g(n) - 3\lambda) \leq I_n(g(k) - 3\lambda) < I_n(g(k) - 3\lambda_k) = 0,$$

for  $1 \leq n \leq k$ , we see that the sequence is strictly decreasing for  $1 \leq n \leq k + 1$ .

(e) Since  $\lambda = \lambda_{k+1}$  for a fixed  $k \in \mathbb{N}$ , and the function  $g$  is strictly increasing for  $t \geq 1$ , from (23) and (24), it follows that

$$c_{n+1}^{(\lambda_{k+1})} - c_n^{(\lambda_{k+1})} = I_n(g(n) - 3\lambda_{k+1}) > I_n(g(k+1) - 3\lambda_{k+1}) = 0,$$

for  $n \in \mathbb{N}_{k+2}$ , that is,  $(c_n^{(\lambda)})_{n \in \mathbb{N}_{k+2}}$  is strictly increasing, since

$$c_{n+1}^{(\lambda_{k+1})} - c_n^{(\lambda_{k+1})} = I_n(g(n) - 3\lambda_{k+1}) < I_n(g(k+1) - 3\lambda_{k+1}) = 0,$$

for  $1 \leq n \leq k$ , we see that the sequence is strictly decreasing for  $1 \leq n \leq k + 1$ , and since

$$c_{k+2}^{(\lambda_{k+1})} - c_{k+1}^{(\lambda_{k+1})} = I_{k+1}(g(k+1) - 3\lambda_{k+1}) = 0,$$

we have

$$c_{k+2}^{(\lambda_{k+1})} = c_{k+1}^{(\lambda_{k+1})},$$

finishing the proof of the theorem. □

*Remark 3* Note that the results in Theorem 1 include all the possible cases for the parameter  $\lambda$  belonging to the interval  $[0, 1]$ . Indeed, since

$$\lambda_k = \frac{g(k)}{3}, \quad k \in \mathbb{N},$$

by Lemma 1 we have that the sequence is strictly increasing. Employing (10), we have

$$\lim_{k \rightarrow +\infty} \lambda_k = \frac{1}{2}.$$

From this, it follows that

$$\bigcup_{k=1}^{\infty} [\lambda_k, \lambda_{k+1}) = \left[ \lambda_1, \frac{1}{2} \right). \tag{26}$$

The case (a) deals with  $\lambda \in [1/2, 1]$ , whereas the case (b) deals with  $\lambda \in [0, \lambda_1)$ . Due to the fact in (26), the other cases in Theorem 1 deal with  $\lambda \in [\lambda_1, \frac{1}{2})$ , from which the claim follows.

**Remark 4** One of the reasons why we have chosen the function in (4) in this investigation, beside the fact that it is a good prototype, is based on the fact that we have managed to determine the behaviour at infinity and the monotonicity character of the function  $g$  defined in Lemma 1 on the domain  $[1, +\infty)$ , which naturally occurs in the proof of Theorem 1. In the case of the other functions of the form  $h_\alpha(t) = \frac{1}{t^\alpha}$ , where  $\alpha \in (0, 1) \setminus \{1/2\}$ , some other methods must be used which are more technical than the ones which are employed in the case  $\alpha = 2/3$  (the case  $\alpha = 1/2$  has been considered recently in [34]). This makes the corresponding investigations in the other cases more complex. Even if we take the function  $h_{3/4}(t)$  instead of the function in (4), we have not managed to modify the proof of Lemma 1 to get the corresponding monotonicity result up to this moment. It is expected that this can be done, but it needs a lot of calculations and clever tricks, related to the ones in the proof of Lemma 1, to get the corresponding monotonicity result. Besides, if this is done, the proof will be, in a way, exotic, and also it will be difficult to generalize in the case of general  $\alpha$ . Generally speaking, as far as we can see, to get such a result is a highly nontrivial technical problem which needs some deeper investigations of the included functions.

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#### Author contributions

SS initiated the investigation, proposed some preliminary ideas, and conducted some detailed investigations. BI, WK and ZŠ analyzed the proposed ideas, made some calculations, and gave some ideas and comments. All authors read and approved the final manuscript.

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#### Data availability

No datasets were generated or analysed during the current study.

## Declarations

#### Competing interests

The authors declare no competing interests.

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