

10th International Conference on Materials Structure and Micromechanics of Fracture

On a computational stress-based non-local damage model for quasi-brittle composites

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Abstract

Numerical models for the stress and strain analysis of quasi-brittle composites, such as cement-based ones, supplied by various stiffening particles, under mechanical, thermal, etc. loads should handle i) the creation of certain micro-damaged zones, antecedent to ii) the initiation and propagation of a system of macroscopic cracks. Whereas ii) can be analysed using some extended, generalized or similar finite element technique (XFEM, GFEM, etc. algorithms), i) must rely on a smeared crack formulation, whose regularization properties are derived from a non-local stress evaluation. This paper studies mathematical and computational properties of such model, based on the Eringen's approach, useful in many applications in civil engineering.

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Peer-review under the responsibility of MSMF10 organizers.

Keywords: Quasi-brittle composites; smeared damage; computational modelling.

PACS: 46.50.+a

2020 MSC: 74R10; 65M06; 65M60

1. Introduction

Materials used in building and engineering structures, namely cement-based composites with various reinforcements, suffer from initiation and propagations of micro- and macro-fractured zones, resulting in decrease of their load bearing ability and durability, therefore precise physical and mathematical modelling of related processes is needed, covering both formal verification and practical validation of such models, including reliable identification of material characteristics. The relevance of such modelling increases in the design of advanced materials, structures and technologies where no long-time practical experience is available. Using the classification of (Sumi, 2014), Parts 4 and 5, unlike pure fatigue or brittle crack propagation, the computational analysis of the so-called quasi-brittle fracture in reinforced cement-based and similar composites is more difficult because of i) the initiation of zones of micro-defects, followed by ii) the formation and propagation of a large number of macroscopic cracks, whose complete deterministic description is rarely available, thus some non-local heuristic computational models are needed, although the solvability of related problems, connected with the convergence of sequences of approximated solutions. is far from being transparent.

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Most approaches try to overcome the incomplete information on material micro-structure, using some non-local description of material defects, as introduced by (Hashiguchi, 2014), Section 14.2. However, the dislocation-based considerations like (Mousavi, 2016) cannot be applied in a reliable way to complicated composite, in general non-periodic material structures. However, the well-posedness of the frequently used non-local Eringen's computational model by (Eringen, 1984) and (Eringen, 2002) may not be guaranteed, as evident from both theoretical studies and computational observations; the oft-quoted existence results, relying on (Altan, 1989), were revised by (Evgrafov and Bellido, 2019) substantially. The effort to derive a relatively simple material model, taking the different behaviour of concrete-like materials under compression and tension into account, can be traced by (Fichant et al., 1999), (Pijaudier-Cabot and Mazars, 2001) and (Giry et al., 2011). Its result is referenced as Mazars' model in later studies; for more general reviews of bridging micro- and macroscopic formulations with numerous useful references cf. (Li et al., 2018) and (Sun et al., 2021). In this short paper we shall rely namely on the stress-based approach to smeared damage by (Grassl et al., 2014) and (Havlásek et al., 2016), working with one irreversible damage factor, regardless of its non-negligible limitations, discussed by (Vala, 2021).

Nevertheless, the development of more general physical and mathematical formulations and computational tools are required by engineering applications. The framework of thermodynamics by (Houlsby and Puzrin, 2000) is adopted to ensure the thermodynamic consistency of the model of isotropic damage in (Vu et al., 2017); novel Helmholtz potential and dissipation functions are suggested by (Kamińska and Szwed, 2021). For anisotropic damage in quasi-brittle materials (Vilppo et al., 2021), inspired by (Otossen, 1977), presents a thermodynamic model based on the precise evaluation of the specific Gibbs free energy and the complementary form of the dissipation potential. Unfortunately, related formulations lead in general to complicated, still not completely closed problems, beyond the scope of this paper; also the design of their effective and robust (probably distributed and/or parallel) software solvers can be seen as the significant research challenge for the next time period, due to the progress in both hardware and software abilities.

2. A model problem

As a first model problem, let us introduce a building structure occupying a domain Ω in the 3-dimensional Euclidean space R^3 , supplied by a Cartesian coordinate system $x = (x_1, x_2, x_3)$; such domain represent a deformable body, with its boundary $\partial\Omega$ in R^3 , consisting of 2 disjoint parts: Θ for Dirichlet boundary conditions, Γ for Neumann ones. To avoid technical difficulties, let us consider $\partial\Omega$ as a Lipschitz boundary with the non-zero measure of Θ on $\partial\Omega$ (to avoid insufficient support) for the related Lebesgue, Sobolev, Bochner - Sobolev, etc. (abstract) function spaces we are then allowed to apply the standard notations compatible with (Roubíček, 2005), Parts 1 and 7, working with unit normal vectors $n(x) = (n_1(x), n_2(x), n_3(x))$ on Γ , taken as outward from Ω everywhere. For simplicity, we shall consider the evolution of all quantities just in time t from a closed time interval $I = [0, \tau]$ with certain prescribed finite time τ . We shall also need the simplified notations $H = L^2(\Omega)^3$, $V = \{w \in W^{1,2}(\Omega)^3 : w = o \text{ on } \Theta\}$, $X = L(\Gamma)^3$, $Z = L^2(\Gamma)^3$ and $G = L^\infty(\Gamma)^3$, o being the zero vector in R^3 , together with the following integral products:

$$(v, w) = \int_{\Omega} v(x) \cdot w(x) \, dx, \quad ((\mathfrak{B}, \mathfrak{B})) = \int_{\Omega} \mathfrak{B}(x) : \mathfrak{B}(x) \, dx, \quad \langle v, w \rangle = \int_{\Gamma} v(x) \cdot w(x) \, ds(x), \quad (1)$$

with appropriate triples of functions v and w , whose products are (Lebesgue or Hausdorff) integrable on related sets, or with 3×3 elements of matrices of such functions \mathfrak{B} and \mathfrak{B} ; in the 1st case the symbol \cdot refers to the scalar product in R^3 , in the 2nd case the symbol $:$ similarly to the scalar product in $H \times H$. For any appropriate variable a we shall also use \dot{a} instead of $\partial a / \partial t$ and $a_{,i}$ instead of $\partial a / \partial x_i$ with $i \in \{1, 2, 3\}$, which can be used also for more indices: especially we shall work with the Hamilton operator $\nabla\varphi$, generating the square matrices of order 3, consisting of the elements $\varphi_{i,j}$ for any $\varphi \in V$ and $i, j \in \{1, 2, 3\}$, and with the linear strain operator $\varepsilon(\varphi)$, evaluating $\varepsilon_{ij}(\varphi) = (\varphi_{i,j} + \varphi_{j,i})/2$ similarly. Especially in (1) (v, w) can be seen as the scalar product of $v, w \in H$, $\langle v, w \rangle$ as the scalar product of $v, w \in Z$, $((\mathfrak{B}, \mathfrak{B}))$ as the scalar product of $\mathfrak{B}, \mathfrak{B} \in H \times H$, $(v, w) + ((\nabla v, \nabla w))$ as certain scalar product of $v, w \in V$, alternative to $(v, w) + ((\nabla v, \nabla w))$, justified by the Korn inequality. Let B^* denote the dual space to a selected Banach space B ; especially, due to the Sobolev embedding theorem, we have the Gelfand tripple $V \Subset H \cong H^* \subset V^*$ where \Subset means the compact embedding and \cong refers to the identification of H with H^* by the Riesz representation theorem; moreover $V \Subset Z$ by the trace theorem. Consequently, thanks to the Cauchy - Schwarz inequality, the estimate $(v, w) \leq \|v\|_H \|w\|_H$ holds for all $v, w \in H$, whereas $\langle v, w \rangle \leq \|v\|_Z \|w\|_Z \leq \zeta \|v\|_V \|w\|_Z$ is satisfied for all $v \in V$ and $w \in Z$ with some positive constant ζ , $\|\cdot\|_X$ denoting the norms in the corresponding Hilbert spaces X .

We shall demonstrate that some reasonable results can be derived even from the standard equations of motion, used in structural mechanics, i. e. from the conservation of linear momentum in terms of classical thermodynamics, applied to Boltzmann (nonpolar) continuum, by (Bermúdez de Castro, 2005), Part 1.3, supplied by usual material characteristics, namely by the stiffness tensor $C \in L^\infty(\Omega)_{\text{sym}}^{(3 \times 3) \times (3 \times 3)}$ (i. e. with 21 independent elements, reducible to 2 well-known Lamé factors for the isotropic case) such that $C_{ijkl}b_{ij}b_{kl} \geq cb_{ij}b_{ij}$ for certain constant c and any $b \in R_{\text{sym}}^{3 \times 3}$ (almost) everywhere on Ω , by Einstein summation indices $i, j, k, l \in \{1, 2, 3\}$ and by the always positive material density $\rho \in L^\infty(\Omega)$, Only one non-standard positive multiplicative factor $1 - \mathcal{D}$ for a damage characteristic $\mathcal{D} \in [0, 1]$, whose easy evaluation is not available, here taken as scalar one for simplicity, must be added: C is then allowed to be replaced by $(1 - \mathcal{D})C$ with the aim of incorporation of energy dissipation into the computational model. No viscous, plastic, etc. effects are considered here: in numerous computational models such terms are motivated more by the numerical stabilization than by the need of incorporation of further physical processes. However, the derivation of \mathcal{D} from stress invariants is the most delicate step in this approach, compatible with (Havlásek et al., 2016): its details and limitations are discussed in (Vala, 2021) in scope of the static formulation with certain final damage stage. Unlike this, (Vala and Kozák, 2020) implements the serial viscoelastic Kelvin model to come to the quasi-static formulation, whereas (Vala and Kozák, 2021) shows the possibility of generalization to its fully dynamic version with usual mass and structural damping classes, well-known from common engineering formulations.

Let $f \in L^2(I, H) \cong L^2(\Omega \times I)$ represent the prescribed volume forces and $g \in W^{1,2}(I, Z)$ the prescribed surface forces. Then we are able to seek for some displacement $u \in W^{2,2,2,2}(I, V, H, V^*)$ (for the definition of such space see (Roubíček, 2005), Part 7.1) such that $u = o$ on Θ in any time $t \in I$, which represents the homogeneous Dirichlet boundary conditions, guaranteed by the careful choice of $u \in L^2(I, V)$ above, and that the pair of Cauchy initial conditions for $t = 0$ is satisfied, i. e. $u(., 0) = o$ and $\dot{u}(., 0) = \hat{u}$ on Ω with some prescribed $\hat{u} \in V$: for simplicity we consider all displacements related to the initial geometric configuration here, without any prescribed support motions (useful e. g. in seismic analysis). Then, for a stress tensor $\sigma \in L^2(\Omega)_{\text{sym}}^{3 \times 3}$ then our model problem reads

$$\rho \ddot{u}_i + \sigma_{ij,j}(u) = f_i \text{ on } \Omega, \quad \sigma_{ij}(u) n_j = g_i \text{ on } \Gamma, \quad \sigma_{ij}(u) = (1 - \mathcal{D}(u)) \sigma_{ij}^*(u), \quad \sigma_{ij}^*(u) = C_{ijkl} \varepsilon_{ij}(u) \text{ on } \Omega \quad (2)$$

for any $i \in \{1, 2, 3\}$ in an arbitrary time $t \in I$. The 1st equation (2) represents the Cauchy equilibrium conditions, the 2nd one the Neumann boundary conditions, whereas the 3rd and 4th ones form the empirical constitutive equation, composed as a modification of the classical Hooke law, with some expected properties of a continuous function $\mathcal{D}(w)$ with an argument $w \in V$: for a sequence $\{w^m\}_{m=1}^\infty \subset V$ weakly convergent to certain $w \in V$ we shall need $\{\mathcal{D}(w^m)\}_{m=1}^\infty$ tending to w strongly, which requires some regularization, here relying on the Eringen’s approach to non-local elasticity and stress invariants, derived from eigenvalues of σ^* . Moreover, after such evaluation at a fixed $t \in I$, we must take $\mathcal{D}(u)$ as the minimum of the calculated value and all corresponding values in all times between 0 and t , to force the damage irreversibility in time; we shall consider initial $\mathcal{D}(.) = 0$ (no damage at $t = 0$) for simplicity.

Multiplying (2) by an admissible displacement v on Ω , integrating the result over Ω , at least in the distributive sense, and applying the integration by parts by the Green - Ostrogradkiĭ theorem, we receive the integral form of (2)

$$(v, \rho \ddot{u}) + ((\varepsilon(v), (1 - \mathcal{D}(u)) C \varepsilon(u))) = (v, f) + \langle v, g \rangle \quad (3)$$

for any $t \in I$ (which is not highlighted explicitly for brevity) and $v \in V$. The initial linearity of the hyperbolic system of partial equations of evolution (3) is disturbed by the rather inconvenient evaluation of $\mathcal{D}(u)$ in (3), thus the availability of its exact solution in some (semi-)analytical form (except very special cases, useful for software testing) cannot be expected. Consequently the design of some sequence of approximate solution of (3), accompanied by the analysis of its convergence, is required.

3. Exact and approximate solutions

Let us start with the following linearization of (3): taking, instead of $t \in I$, step-by-step, $t_s = sh$ with any positive integer step s and certain positive h , $f_s \in H$ and $g_s \in Z$ using the Clément quasi-interpolation of f and g related to $I_s = (t_{s-1}, t_s) \subset I$, following (Roubíček, 2005), Part 8.2, and u_s instead of unknown $u(., sh)$; moreover $u_0 = o$ and $u_{-1} = u_0 - h\hat{u} = -h\hat{u}$ formally. Then it is natural to set $\mathcal{D}_s = \mathcal{D}(u_{s-1})$ (its iterative improvement, if needed, could be performed) and to replace the 2nd derivative in the 1st left-hand-side additive term of (3) by the 2nd relative difference. Thus the semi-discretized form of (3) for the evaluation of $u_s \in V$ is

$$\frac{1}{h^2} (v, \rho(u_s - u_{s-1}) - \rho(u_{s-1} - u_{s-2})) + ((\varepsilon(v), (1 - \mathcal{D}_s) C \varepsilon(u_s))) = (v, f_s) + \langle v, g_s \rangle. \quad (4)$$

Introducing the Rothe sequences $\{u^m\}_{m=1}^\infty, \{\bar{u}^m\}_{m=1}^\infty, \{u_-^m\}_{m=1}^\infty, \{\bar{u}_-^m\}_{m=1}^\infty$ and $\{\check{u}^m\}_{m=1}^\infty$ with $m = \tau/h$, defined as $u^m(t) = u_{s-1} + (t - (s-1)h)u_s/h$, extensible to $s = 0$ and $I_0 = (-h, 0]$, too (standard linear Lagrange splines), $u_-^m(t) = u^m(t-h)$ (retarded linear Lagrange splines), $\bar{u}^m(t) = u_s$ (standard simple functions), $\bar{u}_-^m(t) = u_{s-1}$ (retarded simple functions) and $\check{u}^m(t) = u^m(t) - u_-^m(t)$ (2nd relative differences) for any $t \in I_s$, we can rewrite (4) on I in the form

$$(v, \rho \check{u}^m) + ((\varepsilon(v), (1 - \mathcal{D}(\bar{u}_-^m)) C \varepsilon(\bar{u}^m))) = (v, f^m) + \langle v, g^m \rangle. \tag{5}$$

Inserting $v = u_s - u_{s-1}$ into (4), performing the Einstein summation over $s \in \{1, \dots, r\}$ for an arbitrary $r \in \{1, \dots, m\}$, utilizing the Cauchy - Schwarz and the Young inequalities, taking the above introduced properties of V, H and Z into account, after rather extensive formal computations we come to the estimate

$$\frac{1}{h^2} \sum_{s=1}^r \|u_s - 2u_{s-1} + u_{s-2}\|_H^2 + \frac{1}{h^2} \|u_r - u_{r-1}\|_H^2 + \|u_r\|_V^2 \leq \kappa \left(1 + h \left(\sum_{s=1}^r \|u_{s-1}\|_V^2 + \sum_{s=1}^r \frac{1}{h^2} \|u_s - u_{s-1}\|_H^2 \right) \right) \tag{6}$$

where κ is a positive constant. The discrete Gronwall lemma, in terms of (5), yields by (6) the uniform boundedness of $\{u^m\}_{m=1}^\infty, \{\bar{u}^m\}_{m=1}^\infty, \{u_-^m\}_{m=1}^\infty$ and $\{\bar{u}_-^m\}_{m=1}^\infty$ in $L^\infty(I, V)$, of $\{\check{u}^m\}_{m=1}^\infty$ in $L^\infty(I, H)$ and of $\{\check{u}^m\}_{m=1}^\infty$ in $L^2(I, V^*) \cong L^2(I, V)^*$.

Thanks to the Eberlein - Shmul'yan theorem, as formulated by (Drábek and Milota, 2013), Theorem 2.1.25, up to subsequences, we are able to supply $\{\bar{u}^m\}_{m=1}^\infty$, and $\{\bar{u}_-^m\}_{m=1}^\infty$ by their weak limits \bar{u} and \bar{u}_- in $L^\infty(I, V)$, $\{\check{u}^m\}_{m=1}^\infty$ by its weak limit \check{u} in $L^\infty(I, H)$ and $\{\check{u}^m\}_{m=1}^\infty$ by its weak limit \check{u} in $L^\infty(I, V^*)$. The more detailed identification of these limits needs the exploitation of the Aubin - Lions lemma (see (Roubíček, 2005), Part 7.3): since $W^{1,2,2}(I, V, H) \Subset L^2(I, H)$, $\{u^m\}_{m=1}^\infty$ converges strongly to u in $L^2(I, H)$, taking $u(\cdot, t)$ as the integral of $u'(\cdot, \tilde{t})$ over $\tilde{t} \in [0, t]$ for any $t \in I$. We have $(v, u - \bar{u}) = \lim_{m \rightarrow \infty} (v, u^m - \bar{u}^m) = \lim_{m \rightarrow \infty} h(v, \check{u}^m) = 0$ for each $v \in V$, thus $u = \bar{u}$ and $\dot{u} = u'$. Similarly we can derive $\dot{\bar{u}} = \dot{\bar{u}}_-' = \check{u}$, as well as the identity of weak limits of $\{u^m\}_{m=1}^\infty, \{\bar{u}^m\}_{m=1}^\infty, \{u_-^m\}_{m=1}^\infty$ and $\{\bar{u}_-^m\}_{m=1}^\infty$. Consequently the limit passage $m \rightarrow \infty$ from (5) to (3), exploiting the strong convergence of $\{\mathcal{D}(\bar{u}_-^m)\}_{m=1}^\infty$ to $\mathcal{D}(u)$ in $L^\infty(\Omega)$ for any $t \in I$, is available.

4. Smeared damage implementation

The evaluation of $\mathcal{D}(\cdot)$ is the crucial step of the above sketched approach; its non-local character cannot be avoided: for the non-local damage phenomena near boundaries see (Grassl et al., 2014), for the design of kernels for compact linear operators see (Fasshauer and Ye, 2011). without the loss of the needed strong convergence property. Its potential implementation, compatible with (2) and (3), based on the evaluation of principal stresses, can be

$$\sigma^* = C \varepsilon(u), \quad \det(\sigma^* - \sigma_I^*) = \det(\sigma^* - \sigma_{II}^*) = \det(\sigma^* - \sigma_{III}^*) = 0, \quad \varepsilon^* = \mathcal{F}(\sigma_I^*, \sigma_{II}^*, \sigma_{III}^*), \tag{7}$$

$$\bar{\varepsilon}(x) = \int_{\Omega} \mathcal{K}(x, \xi) \varepsilon^*(\xi) d\xi \text{ for any } x \in \Omega, \tag{8}$$

$$\mathcal{D}^*(u) = \omega(\bar{\varepsilon}), \quad \sigma(u) = (1 - \mathcal{D}(u)) \sigma^* \tag{9}$$

for any $t \in I$; $\mathcal{D}(u)$ in the last equation of (9) has to be evaluated as $\mathcal{D}(u(\cdot, t)) = \max(\mathcal{D}^*(u(\cdot, t)), \sup_{\tilde{t} \in [0, t]} \mathcal{D}(u(\cdot, \tilde{t}))$.

Each block of relations (7), (8) and (9) introduces one new material characteristic. In (7) certain comparable local strain ε^* is evaluated using a characteristic \mathcal{F} , presented in its simple form by (Grassl et al., 2014) as a continuous function of 3 real principal stresses $\sigma_I^*, \sigma_{II}^*, \sigma_{III}^*$ (to preserve the objectivity); the natural generalization, covering different behaviour of cement-based composites under compression and tension, suggested by (Pijaudier-Cabot and Mazars, 2001), and further improvements are available. In (8) the non-local strain $\bar{\varepsilon}$ is obtained using an appropriate kernel $\mathcal{K} \in L^2(\Omega \times \Omega)$ in the sense of (Drábek and Milota, 2013), Example 2.2.5; for its suitable numerical approximations cf. (Skala, 2016). Finally in (9) an characteristic ω as a real continuous function is necessary to generate the final values of $\mathcal{D}(u)$; for much more details shown on an instructive example cf. both graphs of (Vala, 2021).

5. Extended finite element technique

A model problem must be modified and generalized in several directions to cover realistic engineering computations. Firstly some discretization of Ω, Θ and Γ and an approximation of V by some its finite-dimensional approximation V^δ , characterized by an appropriate positive parameter δ , with the aim of convergence of V^δ to V assuming $\delta \rightarrow 0$ in a reasonable sense; in particular V^δ may be a subspace of V . Typically in the Galerkin approach to the finite element method (FEM) the basis of V^δ consists of some functions of 3 real variables with small compact supports on Ω, Θ and Γ (not very complicated, as linear Lagrange splines again) to obtain all approximations of (4) in every step $s \in \{1, \dots, m\}$ as sparse systems of linear algebraic equations. However, such approach is not able to handle the initiation and development of macroscopic cracks directly.

An intuitive remedy could be to rewrite our model problem for Ω consisting of a finite number of domains, separated by mutual interfaces, whose union Λ must be supplied by some additional interface conditions. Especially for the cohesive interfaces by (Pike and Oskay, 2015) we can consider, in addition to the a priori known g on $\Gamma \times I$, also certain g^\times on $\Lambda \times I$, whose evaluation needs some specific cohesive function ϕ as a new material characteristics, working (at least) with the differences of normal components of u from both sides of Λ . Although this brings an additional source of nonlinearity to (3), a linearized scheme like (4) can be (under reasonable assumptions on ϕ) formulated and the existence and convergence results remain valid.

This approach is available even without consideration of any damage factor; however, its disadvantage is in the strictly prescribed crack surfaces, whose heuristic reset by the classical FEM adaptive meshes may be not physically realistic and leads to expensive computations typically. This justifies the significance of the development of various types of local enrichments of above mentioned basis functions, presented as the generalized finite element method (GFEM), extended finite element method (XFEM), etc.; for relevant references see (Kozák and Vala, submitted), together with the demonstration of an exemplary problem of such crack growth modelling in a cement-based composite, up to its software implementation and comparative numerical results.

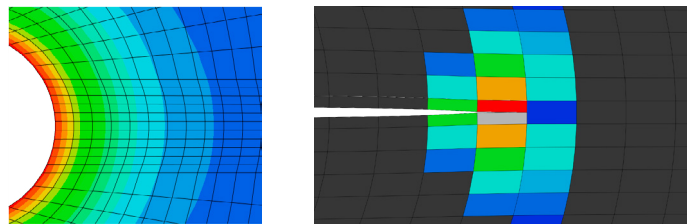


Fig. 1. Initial loading conditions and the detail of final maximum principal stress distribution for single elements.

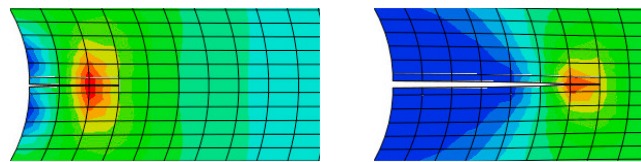


Fig. 2. Mazars' smeared damage exponential model during crack growth.

Fig. 1 and Fig. 2 here present illustrative results for quasi-stationary two-dimensional modelling (thus only σ_I^* , σ_{II}^* are utilized). The loads are applied on a body with a prior circular crack; initial stress concentrated on the surface of the hole is 60 MPa. The cement paste has the Young modulus 3.2 GPa, the Poisson constant is 0.3, The fibres are 2.5 mm long and have the Young modulus 190 GPa. Red colour refer to 60 MPa, blue to 35 MPa and green to 15 MPa. Degradation and softening behaviour of cement paste, including threshold parameters for different material behaviour in tension and compression, is compatible with (Pijaudier-Cabot and Mazars, 2001).

6. Conclusions

The aim of this short conference paper was to show the possibility of development of a proper physical, mathematical and computational deterministic model covering both micro- and macroscopic fracture in quasi-brittle, namely cement-based composites, closely connected to the software development and more extensive numerical examples of (Kozák and Vala, submitted). Some simplifications were accepted only to avoid long proofs; nevertheless, the number of still open questions increases with the removal of any artificial linearization in general.

In the more general context the considerations on incomplete and complete damage and fracture, including their connection to (elasto-)plasticity, presented by (Mielke and Roubíček, 2015), are inspirational. Some results for the analysis of cracked media beyond domains with Lipschitz boundaries have been derived by (Cianchi and Maz'ya, 2016), namely the Korn inequality on irregular domains by (Jiang and Kauranen, 2015). For the fully discrete combination of the Rothe and Galerkin approaches the discrete formulation of the Aubin - Lions lemma by (Dreher and Jüngel, 2012) is available; for useful generalizations of this lemma cf. (Chen et al., 2013) and (Moussa, 2016). For large strains and strongly non-linear constitutive equations its much more complicated version is needed, as evident from the theory for moving domains, derived by (Muha and Čanić, 2019). This motivates also the search for an effective and robust computational scheme for time integration in such case, as discussed by (Bybordiani and Dias-da-Costa, 2021). Incorporation of these particular results into a generalized model can be seen as the research challenge for the near future.

Acknowledgement

This study has been supported by the project of specific university research at Brno University of Technology No. FAST-S-22-7867.

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