JOURNAL OF OPTIMIZATION, DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS (JODEA)
Volume **32**, Issue **1**, June **2024**, pp. 84–96, DOI 10.15421/142405

ISSN (print) 2617–0108
ISSN (on-line) 2663–6824

OPTIMAL STABILIZATION IN SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

Denys Ya. Khusainov*, Josef Diblik†, Andriy V. Shatyrko‡, Zhenya R. Hahurin§

Abstract. This article considers the optimal stabilization problems for complex dynamical systems, which can be described in terms of linear differential equations. At the beginning of the article, general provisions on optimal stabilization and the application of the apparatus of optimal Lyapunov functions for the purpose of solving the formulated problem are given. To ensure consistency and easier understanding of the obtained results, the systems with scalar control are considered first. The main results were obtained for systems with n-dimensional control and the presence of a diagonal matrix in the quality criteria. Finally, the conditions are extended to the case when a matrix of the general form is used in the quality criterion.

Key words: Lyapunov functions method, control, stability, stabilization.

Mathematics Subject Classification MSC2020: 49J15, 93D05, 93D15.

Communicated by Prof. V. V. Semenov

1. Introduction

As is known, when we try to solve problems of optimization of dynamic systems, two approaches are used. The first of them consists in finding a fixed control (program control), at which the system, described by differential equations, reaches a given value and minimizes the integral quality criterion at a finite time period. This method was proposed by L.S. Pontryagin and practically was the transfer of general optimization methods on dynamic systems [1, 2]. The second method was, in essence, not optimal control, but optimal stabilization. It consisted of finding a control function in the form of feedback, at which the zero solution was asymptotically stable (traditionary stabilization), and in addition, some specific integral quality criterion reached a minimum value in an infinite period of time (optimal stabilization). The last approach was based on the second Lyapunov method and was proposed by N.N. Krasovskii [3, 4]. Further development of this direction was carried out, for example in the work [5] and others, which were referred in it. It should be noted that the standard approach to

*Taras Shevchenko National University of Kyiv, 64, Volodymyrska str., Kyiv, 01033, Ukraine, d.y.khusainov@gmail.com
†Brno University of Technology, Technicka 3058/10, Brno, 61600, Czech Republic, diblik@vut.cz
‡Taras Shevchenko National University of Kyiv, 64, Volodymyrska str., Kyiv, 01033, Ukraine, shatyrko.a@knu.ua
§Taras Shevchenko National University of Kyiv, 64, Volodymyrska str., Kyiv, 01033, Ukraine, zhenyahahurin@gmail.com

studying the optimal stabilization problem involves the use of the dynamic programming method [6]. But at the same time, constructing the Bellman function causes significant difficulties for general classes of nonlinear systems. Therefore, the development of other approaches, in particular Lyapunov methods, to solving the problem of optimal stabilization is a current direction in mathematical control theory, which attracts the attention of leading specialists. Initially, and this constituted a certain tradition, stabilization problems were problems of mechanics about stabilizing the movement of certain mechanical systems [4, 7]. Naturally, the area of construction of stabilizing control, and especially its optimization, was not limited to these issues. A number of similar problems were posed and solved in various fields of technology, engineering, etc. [8, 9]. Interesting results specifically in the field of optimal stabilization, both for technical systems and for generalized various mathematical abstractions, have been recently obtained in works [10, 14]. Using the technique of the direct Lyapunov method, research was carried out in the field of stabilization of systems with uncertain coefficients, both the system and the observer [15]. The universality of the Lyapunov method in solving stabilization problems can be evidenced by works in the field of research of difference systems, stochastic systems, systems of functional differential equations [16, 19].

In connection with the rapid development of Computer Internet Technologies, the use of similar approaches, in particular, in the field of Artificial Intelligence (artificial neural networks) is of interest [15, 20].

The present article further examines the main provisions of the method of optimal Lyapunov functions, formulated in [5,12,21] at relation to dynamic systems, which are described by ordinary differential equations.

2. Optimal Lyapunov functions of systems of nonlinear differential equations

Let‘s consider general statements about optimal stabilization in differential systems. The second Lyapunov method is chosen as the research apparatus [22, 23]. The problem of optimal stabilization of the zero state of equilibrium $x(t) \equiv 0$ of a system described by ordinary differential equations

$$\dot{x} = f(t, x(t), u(t)), \quad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m, \quad t \geq t_0, \tag{2.1}$$

that is, the task of constructing a control that provides the best quality of the transient process can be written in the form of minimizing the integral criterion of functional quality

$$I\left[x(t), u(x(t))\right] = \int_{t_0}^{\infty} \omega(t, x(t), u(x(t)))\, dt. \tag{2.2}$$

along the solutions of the system (2.1). Here $\omega(t, x, u)$ is a non-negative function defined in the area

$$x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m,\ t \geq t_0, |x| = \left\{ \sum_{i=1}^{n} x_i^2 \right\}^{\frac{1}{2}}, \tag{2.3}$$

which contains the origin. In particular, for systems of linear differential equations with constant coefficients

$$\dot{x} = Ax(t) + Bu(t), \ \ x(t) \in \mathbb{R}^n, \quad u(t) \in \mathbb{R}^m, \quad t \geq t_0,$$

the function of "quality of system dynamics" may have a quadratic form

$$\omega(x, u) = x^T C x + u^T D u$$

with positive-definite matrices C and D.

Consider the following problem. Let the process $x(t)$quality criterion be chosen in the form of an integral (2.2). It is necessary to find a control $u_0(t)$ that ensures asymptotic stability of the undisturbed motion $x(t) \equiv 0$ of the system

$$\dot{x} = f(t, x, u)$$

and at the same time for any other controls $u^*(t)$ the inequality is satisfied

$$\int_{t_0}^{\infty} \omega(t, x_0(t), u_0(x_0(t)))\, dt \leq \int_{t_0}^{\infty} \omega(t, x(t), u^*(x(t)))\, dt.$$

Thus, the problem was called the problem of optimal stabilization. The function $u_o(t)$ was called optimal control. The following expression was introduced

$$L\left[V, t, x, u\right] = \frac{\partial V(x,t)}{\partial t} + \operatorname{grad}_x^T V(x,t) f(t, x, u) + \omega(t, x, u).$$

The conditions for optimal stabilization were formulated and proved in the form of the following theorem.

Theorem 2.1 (on optimal stabilization)**.** *Let for the differential equation of unperturbed motion (2.1) we can find an additively defined function $V_0(x,t)$ and a vector function $u_0(x,t)$ admitting an infinitesimal higher bound, such that in the domain (2.3) the conditions are satisfied:*

1. *the function $\omega(x,t) = \omega(x, u_o(x,t), t)$ is positive-definite;*
2. *the equality holds $L\left[V_0(x,t), t, x, u_0(x,t), \right] = 0$;*
3. *whatever other control functions $u(x,t)$ are, the inequality is strict*

$$B\left[V_0(x,t), t, x, u(x,t)\right] \geq 0.$$

Then the function $u_0(x,t)$ solves the optimal stabilisation problem. And, moreover,

$$\int_{t_0}^{\infty} \omega(t, x(t), u_0(x(t), t))\, dt = \min_u \Big\{ \int_{t_0}^{\infty} \omega(t, x(t), u(x(t), t))\, dt \Big\} = V_0(x(t_0), t_0). \quad (2.4)$$

Proof. The proof is based on the following propositions. Let the control function $u_0(x,t)$ satisfies the conditions of the theorem, and $V_0(x,t)$ some additionally defined function. As follows from the second condition of the optimal stabilization theorem, its full derivative by virtue of system (2.1) is

$$\frac{d}{\mathrm{d}t}V_0(x,t) = -\omega(t,x,u_0(x,t))$$

and is a negatively definite function. Then, as follows from Lyapunov's second theorem, the zero solution of the system will be asymptotically stable. Thus, the control function $u_0(x,t)$ solves the stabilization problem. It is shown that it also solves the problem of optimal stabilization, i.e., with this control, the integral quality criterion reaches a minimum value. Due to the asymptotic stability of the zero state of equilibrium, for any solution $x(t)$ of system (2.1), the following will be true

$$\lim_{t\to+\infty} V_0(x(t),t) = 0.$$

By integrating the full derivative of the Lyapunov function along the solution $x(t)$ and using its limit value, we obtain

$$\int_{t_0}^{\infty} \omega(t,x(t),u_0(x(t),t))\,dt = -\lim_{t\to+\infty} V_0(x(t),t) + V_0(x(t_0),t_0) = V_0(x(t_0),t_0).$$

which is what we needed to show. Condition (2.4) ensures the optimality of the solution to the problem. □

3. Optimal stabilization of linear systems

The problem of stabilization of linear control systems, i.e., the construction of control laws that guarantee asymptotic stability of linear systems

$$\dot{x} = Ax + Bu, A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times n}$$

has been considered for a long time and in detail. The main requirement for linear stationary control systems is to fulfill the controllability condition of the system, i.e., to fulfill the condition

$$\det S_n \neq 0, S_n = \{B, AB, A^2B, ..., A^{n-1}B\}.$$

Let us consider the application of the method of Lyapunov functions to the problem of optimal stabilization of linear stationary systems.

3.1. Linear systems with scalar control

Consider linear systems with scalar control of the form

$$\dot{x} = Ax(t) + bu(x),\ A \in \mathbb{R}^{n\times n},\ \ b \in \mathbb{R}^n,\ x \in \mathbb{R}^n,\ \ u(x) \in \mathbb{R}^1,\ t \geq 0. \tag{3.1}$$

It is necessary to find a control $u_0(x)$ under which the system (3.1) is asymptotically stable and the quality criterion

$$I\left[x(t), u(x(t))\right] = \int_0^\infty \{x^T(t)Cx(t) + du^2(x(t)))\,dt, \tag{3.2}$$

will reach a minimum value. Here $C \in \mathbb{R}^{n\times n}$ is a symmetric, positive definite matrix, $d \in \mathbb{R}^1 > 0$. Let the matrix A be asymptotically stable and symmetric, a H positive definite matrix that is a solution to the Lyapunov matrix equation

$$A^T H + HA = -C. \tag{3.3}$$

The following result occurs.

Theorem 3.1. *Let the matrix A be asymptotically stable. Then the stabilizing control $u_0(x)$ that optimizes the integral quality criterion (3.2) has the form*

$$u_0(x) = -\frac{2}{d} b^T Hx. \tag{3.4}$$

Proof. The solution to the optimal stabilization problem is sought by the method of Lyapunov functions. The function is taken in the form of a quadratic form $V(x) = x^T Hx, H \in \mathbb{R}^{n\times n}$, where the symmetric, additionally defined matrix is the solution of equation (3.3). Checking that the conditions of the optimal stabilization theorem are met.

1. A function $\omega(x, u) = x^T Cx + du^2$ with a positive definite matrix $C \in \mathbb{R}^{n\times n}$ and $d > 0$ is a positive definite function;
2. The function $L\left[V(x), x, u(x)\right]$ has the form

   $$L\left[V(x), x, u(x)\right] = \frac{d}{dt}V(x) + x^T Cx + du^2.$$

   Let's calculate the total derivative of the Lyapunov function $V(x) = x^T Hx$ by virtue of system (3.1). It has the form

   $$\begin{aligned}\frac{d}{dt}V(x(t)) &= \dot{x}^T(t)Hx(t) + x^T(t)H\dot{x}(t)\\ &= \left[Ax(t) + bu(x(t))\right]^T Hx(t) + x^T(t)H\left[Ax(t) + bu(x(t))\right]\\ &= x^T(t)\left[A^T H + HA\right]x(t) + u(x(t))\left[b^T Hx(t) + x^T(t)Hb\right].\end{aligned}$$

   Thus,

   $$L\left[V(x), x, u(x)\right] = x^T\left[A^T H + HA\right]x + u(x)\left[b^T Hx + x^T Hb\right] + x^T Cx + du^2.$$

   Equating the resulting value to zero, and moving the last two terms to the right, we get

   $$x^T\left[A^T H + HA\right]x + u(x)\left[b^T Hx + x^T Hb\right] = -x^T Cx - du^2(x).$$

By equating the corresponding terms, we obtain a system of two equations

$$\begin{cases} x^T \left[A^T H + HA\right] x = -x^T Cx, \\ u(x) \left[b^T Hx + x^T Hb\right] = -du^2(x). \end{cases}$$

From the first system, we obtain that H is the solution of the matrix Lyapunov equation (3.3). It is known that if a matrix A is asymptotically stable, then for any positive definite matrix C, the Lyapunov matrix equation (3.3) has a unique solution, a positive definite matrix H. Let's consider the second system. Since the control $u(x)$ is a scalar quantity, we can reduce it by

$$b^T Hx + x^T Hb = -du(x).$$

Thus

$$2b^T Hx = -du(x).$$

And, if the control $u(x)$ is taken as (3.4), we will get

$$L\left[V(x), x, u_0(x)\right] \equiv 0.$$

3. We will show that the third condition is also met. Namely, when $u(x) \neq u_0(x)$

$$L\left[V(x), x, u_0(x)\right] \neq 0.$$

Let $u(x) = u_0(x) + \Delta u(x)$. Then

$$\begin{aligned} L\left[V(x), x, u_0(x) + \Delta u(x)\right] &= x^T \left[A^T H + HA\right] x \\ &+ (u_0(x) + \Delta u(x)) \left[b^T Hx + x^T Hb\right] + x^T Cx \\ &+ d(u_0(x) + \Delta u(x))^2 = x^T \left[A^T H + HA\right] x \\ &+ u_0(x) \left[b^T Hx + x^T Hb\right] \\ &+ \Delta u(x) \Big[b^T Hx + x^T Hb\Big] + x^T Cx \\ &+ d(u_0^2(x) + \Delta u^2(x) + 2u_0(x)\Delta u(x)). \end{aligned}$$

As follows from the choice of the function $u_0(x)$, the equality

$$x^T \left[A^T H + HA\right] x + 2u_0(x) b^T Hx + x^T Cx + du_0^2(x) = 0.$$

Therefore, it remains

$$L[V(x), x, u_0(x) + \Delta u(x)] = 2\Delta u(x) b^T Hx + 2du_0(x)\Delta u(x) + d(\Delta u(x))^2.$$

Or

$$L[V(x), x, u_0(x) + \Delta u(x)] = \Delta u(x)[2b^T Hx + 2du_0(x) + d\Delta u(x)].$$

The resulting expression can be zero

(a) or at $\Delta u(x) = 0$;

(b) or at $\Delta u(x) = -\frac{2}{d}\Big(b^T Hx + du_0(x)\Big)$.

The first case cannot be assumed. Let's consider the second case. Substituting for $u_0(x)$ its value, we get

$$\Delta u(x) = -\frac{2}{d}\Big(b^T Hx - 2b^T Hx\Big) = \frac{2}{d}b^T Hx = -\Delta u_0(x) \neq 0.$$

by assumption. Thus, under the control of (3.4), the system (3.1) will be asymptotically stable and the quality criterion (3.2) will reach a minimum value equal to $I\left[x_0(t), u_0(x(t))\right] = x_0^T H x_0$.

□

Example 3.1. Let the system have the form

$$\frac{d}{dt}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} u(x(t), y(t)),$$

with a quality criterion

$$I\left[x(t), y(t), u(x(t), y(t))\right] = \int_0^{\infty} (3x^2(t) + 3y^2(t) + u^2(x(t), y(t))\, dt,$$

Here we have $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $b = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $d = 1$. In this case we get a matrix equation to determine the matrix H

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} = -\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}.$$

Solving it, we get $h_{11} = 1, h_{22} = 1, h_{21} = h_{12} = \frac{1}{2}$. Thus, the optimal Lyapunov function is

$$V_0(x, y) = (x, y)\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = x^2 + xy + y^2,$$

and the control function gets the following form

$$u_0(x, y) = -2(1, 2)\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = -4x + 5y.$$

3.2. Systems with a diagonal matrix in quality integral control

This section describes the full management system

$$\dot{x}(t) = Ax(t) + Bu(x(t)),\ A \in \mathbb{R}^{n\times n},\ B \in \mathbb{R}^{m\times n},\ x(t) \in \mathbb{R}^n.$$

We should find the control function $u_0(x)$ at which the system is asymptotically stable and integral quality criterion

$$I\left[x(t), u(x(t))\right] = \int_0^\infty \left(x^t(t)Cx(t) + u^t(x(t))\Lambda_d u(x(t))\right) dt$$

reaches a minimum value. Here $C \in \mathbb{R}^{n\times n}$ is symmetrical, positive-definite matrix, $\Lambda_d = diag(d_{jj}), d_{jj} > 0, j = \overline{1,\mathrm{m}}$ is a diagonal matrix. The following statement is true.

Theorem 3.2. *Suppose the matrix A is asymptotically stable and H the solution of the matrix equation (3.3) with a positively definite matrix C is included in the integral quality criterion (3.2). Then the stabilizing control $u_0(x)$, at which the integral quality criterion reaches a minimum value, is as follows*

$$u_0(x) = -2\Lambda_{\frac{1}{d}} B^T Hx, \quad \Lambda_{\frac{1}{d}} = \begin{bmatrix} \frac{1}{d_{11}} & 0 & ... & 0 \\ 0 & \frac{1}{d_{22}} & ... & 0 \\ . & ... & . & . \\ 0 & 0 & ... & \frac{1}{d_{mm}} \end{bmatrix}.$$

Proof. In this case, the Lyapunov function is also taken as a quadratic form $V(x) = x^T Hx$. its total derivative is

$$\frac{d}{dt}V(x(t)) = \left[Ax(t) + Bu(x(t))\right]^T Hx(t) + x^T(t)H\left[Ax(t) + Bu(x(t))\right].$$

By equating it to the sub-integral expression of quality criterion (3.2), we get

$$\begin{aligned}\left[Ax(t) + Bu(x(t))\right]^T Hx(t) + x^T(t)H\left[Ax(t) + Bu(x(t))\right] \\ = -x^t(t)Cx(t) - u^t(x(t))\Lambda_d u(x(t)).\end{aligned}$$

By equating the corresponding terms, we obtain a system of two equations

$$\begin{cases} x^T(t)\left[A^T H + HA\right]x(t) &= -x^T(t)Cx(t), \\ u^T(x(t))B^T Hx(t) + x^T(t)HBu(x(t)) &= -u^T(x(t))\Lambda_d u(x(t)). \end{cases}$$

The matrix H , as in the previous case, is a solution of the Lyapunov matrix equation (3.3). Let‘s take a look at the second equation and we consider the following $b_1 = \begin{pmatrix} b_{11} \\ b_{21} \\ ... \\ b_{n1} \end{pmatrix}, ..., b_m = \begin{pmatrix} b_{1m} \\ b_{2m} \\ ... \\ b_{nm} \end{pmatrix}, h_1 = \begin{pmatrix} h_{11} \\ h_{21} \\ ... \\ h_{n1} \end{pmatrix}, ..., h_n = \begin{pmatrix} h_{1n} \\ h_{2n} \\ ... \\ h_{nn} \end{pmatrix}$ Then the second equation can be rewritten as

$$(u_1, u_2, ..., u_m) \begin{bmatrix} b_1^T h_1 & b_1^T h_2 & ... & b_1^T h_n \\ b_2^T h_1 & b_2^T h_2 & ... & b_2^T h_n \\ . & . & ... & . \\ b_m^T h_1 & b_m^T h_2 & ... & b_m^T h_n \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ ... \\ x_n \end{pmatrix}$$

$$+ (x_1, x_2, ..., x_n) \begin{bmatrix} h_1^T b_1 & h_1^T b_2 & ... & h_1^T b_m \\ h_2^T b_1 & h_2^T b_2 & ... & h_2^T b_m \\ . & . & ... & . \\ h_n^T b_1 & h_n^T b_2 & ... & h_n^T b_m \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_m \end{pmatrix}$$

$$= -(u_1, u_2, ..., u_m) \begin{bmatrix} d_{11} & 0 & ... & 0 \\ 0 & d_{22} & ... & 0 \\ . & . & ... & . \\ 0 & 0 & ... & d_{mm} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_m \end{pmatrix}.$$

And we can rewrite it and get

$$(u_1, u_2, ..., u_m) \begin{pmatrix} b_1^T (h_1 x_1 + h_2 x_2 + ... + h_n x_n) \\ b_2^T (h_1 x_1 + h_2 x_2 + ... + h_n x_n) \\ ... \\ b_m^T (h_1 x_1 + h_2 x_2 + ... + h_n x_n) \end{pmatrix}$$

$$+ ((h_1^T x_1 + h_2^T x_2 + ... + h_n^T x_n) b_1, (h_1^T x_1 + h_2^T x_2 + ... + h_n^T x_n) b_2, \dots,$$

$$(h_1^T x_1 + h_2^T x_2 + ... + h_n^T x_n) b_m) \begin{pmatrix} u_1 \\ u_2 \\ ... \\ u_m \end{pmatrix}$$

$$= -d_{11} u_1^2 - d_{22} u_2^2 - ... - d_{mm} u_m^2$$

Equating corresponding terms we obtain a system of equations

$$\begin{cases} 2(h_1^T x_1 + h_2^T x_2 + ... + h_n^T x_n) b_1 u_1 = -d_{11} u_1^2, \\ 2(h_1^T x_1 + h_2^T x_2 + ... + h_n^T x_n) b_2 u_2 = -d_{22} u_2^2, \\ ... \\ 2(h_1^T x_1 + h_2^T x_2 + ... + h_n^T x_n) b_m u_m = -d_{mm} u_m^2. \end{cases}$$

Considering and transforming each of the equations separately, we obtain that the optimal control is $u_0(x) = (u_1^0(x), u_2^0(x), ..., u_m^0(x))$, where

$$\begin{cases} u_1^0(x) = -\dfrac{2}{d_{11}} (b_1^T h_1 x_1 + b_1^T h_2 x_2 + ... + b_1^T h_n x_n), \\ u_2^0(x) = -\dfrac{2}{d_{22}} (b_2^T h_1 x_1 + b_2^T h_2 x_2 + ... + b_2^T h_n x_n), \\ ... \\ u_m^0(x) = -\dfrac{2}{d_{mm}} (b_m^T h_1 x_1 + b_m^T h_2 x_2 + ... + b_m^T h_n x_n). \end{cases}$$

Or, in a more compact form

$$u_0(x) = -2\Lambda_{\frac{1}{d}} B^T H x, \Lambda_{\frac{1}{d}} = \begin{bmatrix} \frac{1}{d_{11}} & 0 & ... & 0 \\ 0 & \frac{1}{d_{22}} & ... & 0 \\ . & ... & . & . \\ 0 & 0 & ... & \frac{1}{d_{mm}} \end{bmatrix}.$$

□

Example 3.2. Consider system (3.1) with quality criterion (3.2). Let the matrices have the following form

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; C = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \; D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Matrix equation

$$A^T H + HA = -C,$$

as in the previous example, has the solution $H = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$. And the optimal control is

$$\begin{cases} u_1^0(x,y) = -2\left[(1,0)x + (0,1)y\right] \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -2x - 2y, \\ u_2^0(x,y) = -2\left[(1,0)x + (0,1)y\right] \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -2y. \end{cases}$$

3.3. Linear systems with matrices of general form in the quality integral

Consider linear systems with a quality criterion of the general form

$$I\left[x(t), u(x(t))\right] = \int_0^{\infty} (x^T(t)Cx(t) + u^T(x(t))Du(x(t)))\, dt,$$

Here the matrix D is symmetric, but not diagonal.

Theorem 3.3. *Let H be the solution of matrix equation (3.3) with positive-definite matrix C, included in the integral criterion. Then the stabilising control $u_0(x)$ that minimizes the integral criterion (3.2) has the following form*

$$u_0(x) = -S\Lambda_{\frac{1}{d}} B^T H x.$$

S is orthogonal matrix, leading to matrix D to a diagonal form.

Proof. Let‘s do the following transformation. Vector control function $u_0(x) \in \mathbb{R}^m$ is found as

$$u_0(x) = Sz_0(x),$$

where S is some non-specific matrix. The quality criterion takes the form

$$I\left[x(t), Sz(x)\right] = \int_0^{\infty} \left(x^T(t)Cx(t) + z^T(x(t))S^T DSz(x(t))\right) dt.$$

If the matrix D is real, symmetric, then it is orthogonally similar to some diagonal matrix Λ_d, i.e. there exists real orthogonal matrix S , that

$$S^{-1}DS = \Lambda_d, \Lambda_d = \mathrm{diag}\left(d_{11}, d_{22}, ..., d_{mm}\right).$$

Here $0 \le d_{11} \le d_{22} \le ... \le d_{mm}$ are eigenvalues of matrix D. Since for an orthogonal matrix $S^{-1} = S^T$, then the quadratic form $u^T Du$ under orthogonal transformation $u = Sz(SS^T = 1)$ changes to the form

$$z^T \Lambda_d z = \sum_{j=1}^{n} d_{jj} z_j^2.$$

Thus, after transforming (3.2) the quality criterion for the control system will has the following form

$$I\left[x(t), z(x)\right] = \int_0^{\infty} \left(x^T(t)Cx(t) + z^T(x(t))\Lambda_d z(x(t))\right) dt,$$

with diagonal matrix Λ_d. And back to the previous case. The optimal control would be

$$z_0(x) = -2b^T \Lambda_{\frac{1}{d}} Hx.$$

And the output control

$$u_0(x) = -2Sb^T \Lambda_{\frac{1}{d}} Hx, \Lambda_{\frac{1}{d}} = \begin{bmatrix} \frac{1}{d_{11}} & 0 & ... & 0 \\ 0 & \frac{1}{d_{22}} & ... & 0 \\ . & ... & . & . \\ 0 & 0 & ... & \frac{1}{d_{mm}} \end{bmatrix}.$$

□

4. Conclusion

Based on the application of Lyapunov's direct method and using the apparatus of Lyapunov's optimal functions, the authors proved sufficient conditions for stabilization in systems of linear ordinary differential equations to the state of asymptotic stability with the provision of optimality of additional quality criteria.

In the future, applying the proposed methodology and the ideas of the this article authors, which was presented in papers [14, 24], it is possible to conduct similar studies with nonlinear control systems and complex systems, the functioning of which is described in terms of functional-differential or difference equations.

5. Acknowledgements

This work is conducted under the Agreement on scientific cooperation between Faculty of Computer Science and Cybernetics Taras Shevchenko National University of Kyiv, Ukraine and the Faculty of Electrical Engineering and Communication, Brno University of Technology, Brno, Czech Republic. J. Diblik has been supported by the project of specific university research at Brno University of Technology FEKT-S-23-8179 (Faculty of Electrical Engineering and Communication).

References

1. Pontryagin L.S., Boltyanskii V.G., Gamkrelidze R.V., Mishchenko E.F., *Mathematical theory of optimal processes*, M.: Nauka, 1983.
2. Alekseev V.M., Tikhomirov V.M., Fomin S.V., *Optimal control*, M.: Nauka, 1979.
3. Krasovskii N.N., *Stability of Motion*, Ed. by J. L. Brenner. Stanford University Press, 1963.
4. Malkin I.G., *Control theory of motion*, M.: Nauka, 1966.
5. V. Lakshmikantham, S. Leela, A. Martynyuk, *Stability Analysis of Nonlinear Systems.*, Birkhauser, 2015.
6. Bellman, R., *Dynamic Programming*, Princeton University Press, Princeton, NJ., 1957.
7. A. Zuyev, A. Ignatiev A. Kovalev, *Stability and Stabilization of Nonlinear Systems*, Naukova Dumka, Kyiv, 2013.
8. Onishchenko S.M., *Optimal Stabilization of the Earth Artificial Satellite with Redundant System of Fly-Wheels* , Journal of Automation and Information Sciences, 2016.
9. Loveikin V.S., Romasevich Yu.A., Khoroshun A.S., *Optimal Control of Stabilization of Inverted Pendulum with Flywheel*, J. International Applied Mechanics, Vol.66, No.4 (2020), 78–86.
10. Onishchenko S.M., *Hard Optimal Stabilization of Nonlinear Dynamic Systems*, Journal of Automation and Information Sciences, Vol.46, No.7 (2014), 12–27.
11. Martynyuk A.A., Chernienko V.O., *On stabilization of the motion of non-autonomous polynomial systems*, J. International Applied Mechanics, Vol.57, No.5 (2021), 35–45.
12. Martynyuk A.A., Chernienko V.O., *On Estimate of Lyapunov Function and Stability of Motion of System with Asymptotic Expansion of Right-Hand Side of Equations of Perturbed Motion*, J. International Applied Mechanics, Vol.57, No.1 (2021), 20–28.
13. Burke J.V., Lewis A.S., Overton M.L., *Optimal stability and eigenvalue multiplicity*, Foundations of Computational Math, Vol.1, No.2 (2001), 205–255.

14. Khusainov D.Ya., Shatyrko A.V., Hahurin Y.R. , *Optimal stabilization for differential equations*, J. of Numerical and Applied Mathematics, Vol.2, (2022), 158–164.
15. Bo Zhao, Derong Liu, Chaomin Luo, *Reinforcement Learning-Based Optimal Stabilization for Unknown Nonlinear Systems Subject to Inputs With Uncertain Constraints* IEEE Transactions on Neural Networks and Learning Systems, Vol.31, No.10 (2020), 4330–4340.
16. Junsoo Lee; Wassim M. Haddad; Manuel Lanchares, *Finite Time Stability and Optimal Finite Time Stabilization for Discrete-Time Stochastic Dynamical Systems*, IEEE Transactions on Automatic Control, (Early Access), August 2022. P. 1–13.
17. 17. Yuan Gong Sun, Long Wang, *Stabilization of planar discrete-time switched systems: Switched Lyapunov functional approach* , Nonlinear Analysis: Hybrid Systems, Vol.2, No.4 (2008), 1062–1068.
18. Hanna Demchenko, Josef Diblik and Denys Khusainov, *Optimal Control of the Heating Process with Delay*, AIP Conference Proceedings 2293, 2020.
19. R. Gabasov, F. Kirillova, N. V. Balashevich, *Optimal synthesis and stabilization of time-delay dynamical systems.* Mathematics IFAC, Proceedings. July 1999, p. 2807–2812.
20. Xiao Liang; Juanjuan Xu; Huanshui Zhang., *Optimal Control and Stabilization for Networked Control Systems with Asymmetric Information* , IEEE Transactions on Control of Network Systems, Vol.7, No.3 (2020), 1355–1365.
21. M.N. Monfared , M.J. Yazdanpanah, *Optimal Dynamic Lyapunov Function and The Largest Estimation of Domain of Attraction*, IFAC-PapersOnLine, Vol.50, No.1 (2017), 2645–2650.
22. Lyapunov A.M. , *The general problem of the stability of motion*, M.-L., State publishing house of technical and theoretical literature, 1950. 2013.
23. Alexander M. Lyapunov. , *The general problem of the stability of motion*, De Gruyter series in Nonlinear Analysis and Applications:4, International Journal of Control, Vol.55, No.3 (1992), 531–534.
24. Demchenko H., Diblik J., Khusainov D.Ya., *Optimal stabilization for differential systems with delays - Malkin's approach*, Journal of the Franklin Institute, Vol.356, No.9 (2019), 4811–4841.

Received 09.02.2024