Review of Algorithms for Conics in Geometric Algebra, PhD Thesis by Ing. Pavel Loucka

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1. Up to datedness of dissertation topic

The dissertation uses the recently developed hypercomplex algebraic formalism called Geometric Algebra for Conics (GAC), that goes back to the work of Christian Perwass (C. Perwass, Geometric Algebra with Applications in Engineering, Springer, 2009, Section 4.5), and further developed by Hradina, Navrat and Vasik in 2018 (https://doi.org/10.1007/s00006-018-0879-2) and 2019 (see references of the thesis) and independently as conic CGA by Hitzer and Sangwine in 2019 (DOI: 10.1007/s00006-019-1016-6). It combines the traditional matrix based study of conics with the new algebraic techniques, and is in my judgement fully up to date and presenting new findings.

2. Whether the dissertation has met the stated objective

The objectives of presenting new algorithms for conic fitting and conic construction based on GAC, as, e.g., described in the abstract appear to be fully achieved. The dissertation describes the theory, the algorithms, discusses their strength and limitations, provides examples, tests noisy data and provides the computer algebra code.

3. The procedure applied to problem solving and the results of the dissertation stating the specific contribution of the PhD student

The thesis has three chapters. The *first chapter* describes the basics of geometric algebra and of GAC in particular. It then shows (and this is an original contribution) how to include projective geometry in GAC, and ends with the classification of conics.

The *second chapter* reviews the previously existing conic fitting algorithm (Hrdina et al, 2019, see references of the thesis). It then extends the algorithm (this is also an original contribution) to cases with additional geometric constraints of axis alignment, centering at the origin, and the combination of these two. One variant of the algorithms is based on direct algebraic computation from matrices and vectors given by the data. When centering at the origin is required, a symmetrization technique provides another variant (this is also an original contribution).

The previous algorithm (Hrdina et al, 2019) was not translation invariant. This may be due to how the distance measure is defined in these algebras. For example, in conformal geometric algebra, the inner product of a point and a circle gives the length of the tangential distance, not the shortest orthogonal distance from the circle, which leads to distance dependent distortions in the results. The results of iterative fitting (this is also an original contribution) appear to successfully overcome this problem.

The next major group of new algorithms (this is also an original contribution) is designed to fit conics through point clouds with additionally prescribed proper and improper (at infinity, up to two of them) waypoints. Here a Moore-Penrose inverse or a null-space orthonormal basis matrix are used. These algorithms are again compared in detail in a range of representative examples and the influence of noise is tested as well.

Chapter 3 is devoted to the construction of conics using the wedge product of geometric algebra. A key property of C. Perwass design, that has been preserved in GAC and in conic CGA, is the possibility to wedge 5 points and directly obtain the multivector representation of the conic through these 5 points. Furthermore, the meet (intersection) product in GAC of two conics produces the wedge product of the four points of intersection (this appears not to be possible in the alternative formalism of double conformal GA (DCGA)), here aptly named four-point. Wedging a four-point with a fifth point produces a new conic that passes through all 4 points of the four-point and the additional fifth point. Since the choice of the fifth point is free, the totality of conics created by all possible choices of the fifth point creates a pencil of conics. Special attention is paid to the cases of line-pairs and generalized parabolas in pencils of conics. As far as possible geometric algebra techniques are employed and combined with known matrix techniques. Every important case is well illustrated with representative examples. A number of important theorems are proven. It is the most comprehensive study using hypercomplex algebra techniques that I have seen so far, constituting a genuine contribution to this field.

As the authors point out at the end of p. 54, they consider several settings including improper points which are usually only considered geometrically not by analytical (here algebraic) construction. Furthermore (see p. 94), their work appears to be the first general investigation of the construction of parabolas that appear in the pencil of conics generated by two conics.

4. Importance of practice or development of the respective scientific discipline

The use of geometric algebra in applied mathematics, engineering, computer graphics, robotics, neural networks, physics, geographic information science, electric engineering, image and signal processing, and many other fields is rapidly growing (see e.g. https://doi.org/10.1002/mma.9575). The techniques developed in this thesis are a most welcome expansion in so far as now not only points, point pairs, lines and circles can be used, but the much more flexible class of conic curves is added to the set of elementary geometric objects with elegant multivector representations, rotor transformations, and algebraic multivector operations. The work thus has rich potential applications in many areas, beyond the realm of pure geometric algebra.

5. Formal arrangement of the dissertation and its linguistic level

The thesis is well structured and the language clear, rational and easy to comprehend. That said, I felt in a number of cases definite articles were missing, but that maybe something particularly difficult for a non-native writer.

6. Recommending the award of the academic degree

Yes, I do strongly recommend to award the academic degree of PhD to Ing. Pavel Loucka.

7. Additional comments

Because I have read the thesis in full detail, and found a number of technical points to comment on, I decide to add these here in the following list for the consideration of the author.

1. I think the original conformal conic space theory of C. Perwass (C. Perwass, Geometric Algebra with Applications in Engineering, Springer, 2009, Section 4.5) in Cl(5,3), has an important difference in conformal point embedding: Comparing the point embeddings (1.1) in the thesis with that in the work C. Perwass (4.118) shows that the quadratic terms are *considerably* simpler in C. Perwass original approach. It also means for the conic represented as vector in conformal conic space theory, that compared to (1.14) in the thesis, the coefficients of the quadratic equation for a conic (q_11 to q_33) are in Perwass' approach identical to the coefficients of the vector in conic CGA up to signs and factors of 2, see (4.119) in C. Perwass book. The same applies to the matrix equation (1.17). The previous published work on GAC and the current thesis appear not to explain why they

prefer the more complicated embedding (1.1) to the simpler original formulation of C. Perwass. We note that a continuation of the conformal conic space theory of Perwass using Perwass' original embedding (4.118) has been undertaken in E. Hitzer, S.J. Sangwine, *Foundations of Conic Conformal Geometric Algebra and Compact Versors for Rotation, Translation and Scaling*, Adv. of App. Cliff. Algs., (2019) **29**(5):96, 16 pages, DOI: 10.1007/s00006-019-1016-6, which we recommend to at least refer to in the introduction of the thesis.

2. In (1.14), line 3, the upper index of vbar on the left side should be plus not minus.
3. Regarding determinant computations in (1.21). How determinants of multivectors can be computed in GA has been well explained by D. Shirokov: https://arxiv.org/abs/1108.5447 . It is therefore to be expected that in the future equations like (1.21) can also be fully expressed in GA. Concerning linear maps in GA, their determinants have always been known to be computable as det(f) = f(I)I^{-1}, where I is the pseudoscalar of the GA, see the book of Hestenes and Sobczy, Clifford Algebra to Geometric Calculus, Springer, 1984.
4. P. 13, line 11: depend → depends (there are more instances where the plural is needed but missing in this thesis).
5. The expression on top of page 14 "w.r.t. inner product in GAC" is unnecessary. It is a fundamental property of all geometric algebras that the square of a vector equals the inner product.
6. 2nd sentence on p. 15: The matrix D is at this point in the text not well defined.
7. Line 4: ... matrix P, such that this block is not further used in the subsequent computations.
8. P. 16, under item 3, line 2: ... shown that all conics of types ...
9. Section 2.5 on influence of noise: Since the embedding of points in Cl(5,3) and the expressions for conics are simpler in the original work of C. Perwass (as explained above), it is to be expected that the original formulation may have better performance in situations with noise.
10. Is there a particular reason to only use ellipses in Sections 2.5.1 and 2.5.2?
11. In Section 2.5.3 tests with more than 2 waypoints seem to be missing.
12. In Fig. 2.13, in the first four cases with green hyperbolas look very close to the reference parabola. So perhaps another measure than (2.25) could be meaningfully used to evaluate the similarity.
13. Top of page 29: ... i.e. center position, etc. I recommend to be more explicit here and replace etc. by the concrete geometric characteristics in question.
14. Above Fig. 2.14: ... hyperbolas and ellipses are characterized by the same sets of characterists. This is not clear to me, because I think hyperbolas and ellipses are quite different.
15. The two lines and the zero outer product equation under (3.1) simply mean that the 5 points are not linearly independent. I think this is more meaningful than saying "cannot be determined".
16. Page 53, last paragraph: "parallels". The magnitude of the pseudoscalar constructed in Def. 1.2 on page 4 C(p) wedge A_0 wedge nbar wedge nbar_x is the determinant: (C(p) wedge A_0 wedge nbar wedge nbar_x) I^{-1}, using the I of page 5. So C(p) wedge A_0 wedge nbar wedge nbar_x = 0 is the same as (3.4) or (3.5). It is not just a parallel.
17. Section 3.2, line 3, after ref. [5], reference should also be made to [7].
18. In GA there are general identities that relate inner and outer products: (IA) wedge B = I (A dot B) and A wedge (BI) = (A dot B)I. Therefore I expect that the double wedge product in (3.7) can be rewritten as ((Q1 wedge Q2)* wedge P)* = p/m (Q1 wedge Q2) dot P. Here p/m means a plus or minus sign. The authors should check this.
19. Def. 3.2: t_1 and t_2 need to be defined clearly.
20. The notation in the third paragraph of page. 71 does not agree with the labeling in Fig. 3.22.

21. Under (3.22), line 3, I suggest to replace deconstruct with analyze.
22. Line 2 under (3.23): replace not zero matrix by a non-zero matrix.
23. P. 78, line 3: could be → can be.
24. Lemma 3.2, item 3: does “complex solutions” mean the same as “imaginary parabola”? This should be clarified, the terms complex and imaginary cannot be used interchangeably in mathematics.
25. I do not quite understand the difference between simply having a dash for the case B1 in Table 3.3 and the statement no parabola under B2. Is there an essential difference?
26. 5 lines above Th. 3.6: conics has → conics have
27. P. 83, 3rd line from bottom: Replace abar by pbar.
28. P. 85, line 7 from bottom (towards end of item 3) mentions “a pari of imaginary ones” (parabolas). I do not understand why here and elsewhere imaginary parabolas are not shown in the figures. Their expressions are imaginary, but they still mean locations in a plane.
29. Last line of item 3: $\det(N) = 0$ must be $\det(N) > 0$.
30. Line 6, p. 86: ... for these particular matrices, the generalized eigenproblem (3.28) has no ...
31. P. 86, 2nd par.] will take unusual ... → will have an unusual
32. Just below page center: range of lambda → range of λ
33. P. 86, last line: be it at → in this case at
34. P. 87, 3rd equation at the end: $7x$ → $7x^2$.
35. Center of p. 87: acquirement of P^2 → but the acquirement of P^2
36. P. 87, line 13 from bottom: there is no real
37. Line 4 from bottom: but may still be ...
38. Under (3.30): where elements → where the components ...
39. Th. 3.8, proof, 3rd line: equivalent with → equivalent to
40. Page 90, case (b) should have gamma_12 =0 according to the end of p. 89 and not gamma_12 not equal 0.
41. In item (2) on p. 92 the use of the wedge product should be made explicit, as mentioned in the figure caption of Fig. 3.30.
42. Above Remark 3.21: ... of the respective kind of pencil.
43. 2 lines from bottom of p. 92: using which → with which
44. In the middle of page 95, the authors exclude complex forms of matrices, and I do not understand why they need to be excluded. GA has plenty of elements that square to minus 1 (https://link.springer.com/chapter/10.1007/978-3-0348-0603-9_7).
45. Fig. 3.31 appears not to be referred to in the text.
46. P. 98, 4th paragraph, line 7: wedge can be → wedging can be