SOLUTION OF A WEAKLY DELAYED DIFFERENCE SYSTEM

Jan Šafařík
Doctoral Degree Programme (II), FEEC BUT
E-mail: xsafar19@stud.feec.vutbr.cz

Supervised by: Josef Diblík
E-mail: diblik@feec.vutbr.cz

Abstract: The paper solves a weakly delayed difference system

$$x(k+1) = Ax(k) + Bx(k-1)$$

where $k = 0, 1, \ldots$, $A = (a_{ij})_{i,j=1}^{3}$, $B = (b_{ij})_{i,j=1}^{3}$ are constant matrices. An explicit solution is given with a discussion on the number of independent initial data.

Keywords: Discrete system, weak delay, initial problem.

1 INTRODUCTION

We investigate a system of difference equations

$$x(k+1) = Ax(k) + Bx(k-1), \; k = 0, 1, \ldots \tag{1}$$

where A and B are 3 by 3 constant matrices with elements a_{ij} and b_{ij}, $i, j = 1, 2, 3$ respectively and the matrix A has real and mutually different eingenvalues λ_1, λ_2, λ_3 and is given in the Jordan form

$$A = \Lambda = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}. \tag{2}$$

We assume that (1) is a weakly delayed system in the sense of the following definition.

Definition 1 *System* (1) *is called weakly delayed if the characteristic equations for* (1) *and for the system without delay*

$$x(k+1) = Ax(k)$$

have identical roots, that is, if, for every $\lambda \in \mathbb{C} \setminus \{0\}$,

$$\det\left(A + \lambda^{-1}B - \lambda I\right) = \det\left(A - \lambda I\right).$$

Applying Definition 1 to system (1) we get conditions under which the system is weakly delayed:

Theorem 1 *System* (1) *is a weakly delayed system if and only if*

$$b_{11} = b_{22} = b_{33} = 0, \tag{3}$$

$$b_{12}b_{23}b_{31} + b_{13}b_{21}b_{32} = 0, \tag{4}$$

$$b_{12}b_{21} + b_{13}b_{31} + b_{23}b_{32} = 0, \tag{5}$$

$$\lambda_3 b_{12}b_{21} + \lambda_2 b_{13}b_{31} + \lambda_1 b_{23}b_{32} = 0. \tag{6}$$

We omit the proof and referring to [1], Theorem 3.

The purpose of the paper is to give an explicit solution of the initial problem

$$x_i(0) = x_{i,0},\ x_i(-1) = x_{i,-1},\ i = 1,2,3 \tag{7}$$

to system (1). If the system considered is weakly delayed, some of the initial data in (7) do not influence the solution's behaviour. We discuss this problem as well.

2 RESULTS

System (1) can be transformed into a system without delay. Below we describe this transformation. Taking into account the above mentioned properties of matrix B, we have

$$B = \begin{pmatrix} 0 & b_{12} & b_{13} \\ b_{21} & 0 & b_{23} \\ b_{31} & b_{32} & 0 \end{pmatrix} \tag{8}$$

and its elements satisfy (3)–(6). From these formulas obviously follows that B is a nilpotent matrix (all its eigenvalues are equal to zero). Below we assume that the geometrical multiplicity of the zero eigenvalue equals 2.

I.e., the system (1) is

$$\begin{aligned}
x_1(k+1) &= \lambda_1 x_1(k) + b_{12}x_2(k-1) + b_{13}x_3(k-1),\\
x_2(k+1) &= \lambda_2 x_2(k) + b_{21}x_1(k-1) + b_{23}x_3(k-1),\\
x_3(k+1) &= \lambda_3 x_3(k) + b_{31}x_1(k-1) + b_{32}x_2(k-1)
\end{aligned}$$

with initial data

$$\begin{aligned}
x_1(0) &= x_{1,0}, & x_1(-1) &= x_{1,-1},\\
x_2(0) &= x_{2,0}, & x_2(-1) &= x_{2,-1},\\
x_3(0) &= x_{3,0}, & x_3(-1) &= x_{3,-1}.
\end{aligned}$$

Define new dependent functions z_1, z_2 and z_3 by formulas

$$\begin{aligned}
z_1(k) &= x_1(k-1), & &\Rightarrow & z_1(k+1) &= x_1(k),\\
z_2(k) &= x_2(k-1), & &\Rightarrow & z_2(k+1) &= x_2(k),\\
z_3(k) &= x_3(k-1), & &\Rightarrow & z_3(k+1) &= x_3(k)
\end{aligned}$$

and, instead of (1), consider a new system without delay

$$\begin{aligned}
x_1(k+1) &= \lambda_1 x_1(k) + b_{12}z_2(k) + b_{13}z_3(k),\\
x_2(k+1) &= \lambda_2 x_2(k) + b_{21}z_1(k) + b_{23}z_3(k),\\
x_3(k+1) &= \lambda_3 x_3(k) + b_{31}z_1(k) + b_{32}z_2(k),\\
z_1(k+1) &= x_1(k),\\
z_2(k+1) &= x_2(k),\\
z_2(k+1) &= x_3(k).
\end{aligned}$$

Unifying the notation of the dependent variables as

$$y_i(k) = x_i(k),\ i = 1,2,3,\ y_{j+3}(k) = z_j(k),\ j = 1,2,3$$

we have

$$\begin{array}{rclccccc}
y_1(k+1) & = & \lambda_1 y_1(k) & & & & +b_{12}y_5(k) & +b_{13}y_6(k), \\
y_2(k+1) & = & & \lambda_2 y_2(k) & & +b_{21}y_4(k) & & +b_{23}y_6(k), \\
y_3(k+1) & = & & & \lambda_3 y_3(k) & +b_{31}y_4(k) & +b_{32}y_5(k), & \\
y_4(k+1) & = & y_1(k), & & & & & \\
y_5(k+1) & = & & y_2(k), & & & & \\
y_6(k+1) & = & & & y_3(k). & & &
\end{array}$$

The matrix form of the last system is

$$y(k+1) = \mathcal{A}y(k) \tag{9}$$

where

$$\mathcal{A} = \left(\begin{array}{ccc|ccc}
\lambda_1 & 0 & 0 & 0 & b_{12} & b_{13} \\
0 & \lambda_2 & 0 & b_{21} & 0 & b_{23} \\
0 & 0 & \lambda_3 & b_{31} & b_{32} & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right) = \left(\begin{array}{c|c} \Lambda & B \\ \hline E & \Theta \end{array}\right) \tag{10}$$

or

$$y(k+1) = \begin{pmatrix} \Lambda & B \\ E & \Theta \end{pmatrix} \cdot y(k).$$

The initial data for system (9) are

$$\begin{aligned}
y_1(0) &= x_1(0), \\
y_2(0) &= x_2(0), \\
y_3(0) &= x_3(0), \\
y_4(0) &= x_1(-1), \\
y_5(0) &= x_2(-1), \\
y_6(0) &= x_3(-1).
\end{aligned}$$

We will transform system (9) using $y(k) = Sw(k)$ where S is a regular transient matrix and $w(k)$ is a new dependent vector into a system with a matrix of the Jordan form. We get

$$Sw(k+1) = \mathcal{A}Sw(k)$$

or

$$w(k+1) = \gamma w(k) \tag{11}$$

where

$$\gamma = S^{-1}\mathcal{A}S.$$

with the initial data for (11) being

$$w(0) = S^{-1}y(0).$$

It is easy to prove that the eigenvalues of the matrix $\mathcal{A}$ are λ_1, λ_2, λ_3, $\lambda_4 = \lambda_5 = \lambda_6 = 0$. Then the solution of the (11) is

$$w(k) = \gamma^k w(0),\ k = 1,2,3,\ldots$$

where the powers of γ "transient" forms for $k=1$ and $k=2$, i.e.,

$$\gamma = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3 \end{array}\right) = \left(\begin{array}{c|c} \Lambda_1 & \Theta \\ \hline \Theta & \Lambda \end{array}\right)$$

and

$$\gamma^2 = \left(\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^2 \end{array}\right) = \left(\begin{array}{c|c} \Lambda_1^2 & \Theta \\ \hline \Theta & \Lambda^2 \end{array}\right).$$

For $k \geq 3$ the powers γ^k are given by

$$\gamma^k = \left(\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1^k & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda_2^k & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda_3^k \end{array}\right) = \left(\begin{array}{c|c} \Theta & \Theta \\ \hline \Theta & \Lambda^2 \end{array}\right).$$

The solution of (9) is

$$y(k) = Sw(k) = S\gamma^k w(0),\ k = 1,2,3,\dots.$$

Using an auxiliary matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = (E|\Theta)$$

we can write the solution of the initial system (1) in the form

$$x(k) = QS\gamma^k w(0),\ k = 1,2,3,\dots \tag{12}$$

where

$$w(0) = S^{-1}x(0). \tag{13}$$

Therefore, the following theorem holds.

Theorem 2 *Let matrix A have the form* (2) *where* λ_1, λ_2, λ_3 *are different real numbers, let matrix B have the form* (8)*, with its elements satisfying* (4)–(6)*, and let the geometrical multiplicity of its zero eigenvalue equal 2. Then the solution of system* (1) *is given by formula* (12) *with* $w(0)$ *given by* (13).

The initial-value problem to system (1) is specified by 6 initial values (7). In the case considered the solution determined by (7) depends, for $k \to \infty$, not on 6 parameters, but only 5 parameters are necessary. This can be seen from system (9) with the matrix $\mathcal{A}$ given by (10). Since in the matrix B only two of its rows are linearly independent, one of the three initial values $y_4(0)$, $y_5(0)$, $y_6(0)$ is unnecessary.

ACKNOWLEDGEMENT

The author was supported by the Grant FEKT-S-14-2200 of Faculty of Electrical Engineering and Communication, BUT.

REFERENCES

[1] Šafařík, J., Diblík, J., Halfarová, H.: *Weakly Delayed Systems of Linear Discrete Equations in* $\mathbb{R}^3$. In *MITAV 2015 (Matematika, informační technologie a aplikované vědy), Post-conference proceedings of extended versions of selected papers*. Brno: Univerzita obrany v Brně, 2015. s. 105-121. ISBN: 978-80-7231-436-2.

[2] Elaydi, S. N.: *An Introduction to Difference Equations*, Third Edition, Springer, 2005.

[3] Khusainov D. Ya., Benditkis D. B., Diblík J.: *Weak delay in systems with an aftereffect.* Functional Differential Equations, 9 (2002), 385–404.

[4] Diblík J., Khusainov D. Ya., Šmarda Z.: *Construction of the general solution of planar linear discrete systems with constant coefficients and weak delay*. Adv. Difference Equ. 2009, Art. ID 784935, 18 pp.

[5] Diblík J., Halfarová H.: *Linear system of discrete equations with a weak delay.* In *Dynamical System Modelling and Stability Investigation*. Kyjev, Ukrajina: University of Kyiv, UA, 2011. s. 26-27. ISBN: 9667652009.

[6] Diblík J., Halfarová H.: *Discrete systems of linear equations with weak delay.* In *XXIX International Colloquium on the Management of Educational Process*. Brno: 2011. s. 1-4. ISBN: 978-80-7231-779- 0.

[7] Diblík J., Halfarová H.: *Explicit general solution of planar linear discrete systems with constant coefficients and weak delays.* Adv. Difference Equ. 2013, Art. number: 50, doi:10.1186/1687-1847-2013-50, 1–29.

[8] Diblík J., Halfarová H.: *General explicit solution of planar weakly delayed linear discrete systems and pasting its solutions.* Abstr. Appl. Anal. 2014, doi:10.1155/2014/627295, 1–37.