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ONLINE COURSE IN MATHEMATICS (PDF OUTPUT)

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1. SETS AND MATHEMATICAL LOGIC

Language of mathematics



Knowledge of elementary mathematics is beneficial for our lives. The entire history of human society is therefore linked to the development of mathematics. In the past, the development of mathematical knowledge was related to the development of natural sciences, which is why mathematics is often ranked among natural sciences. The essence of mathematics is much broader than the study of relations in animate or inanimate nature. Mathematics has created ideal objects based on abstraction, through which it can examine e.g. economic processes, linguistic regularities, or elements of sociological researches. In fact, mathematics is used in all fields of science as a universal language, which helps them to word their laws. Moreover, the level of a given scientific discipline is judged by the extent to which mathematical language, methods, and models are used.

Therefore, it is important to learn mathematical language, which differs from everyday language not only in its symbols, but also in the way its sentences, called propositions, are formed. The reason for using a subject-specific, mathematical language is not only that some mathematical knowledge would be difficult to express in everyday language, but also that everyday language allows for the creation of various paradoxes. For example, suppose we have the following propositions:

1. $2 + 2 = 4$.
2. $2 + 2 = 5$.
3. Exactly one of these three, i.e. 1, 2 and 3, propositions is true.

What about statement 3? If proposition 3 were true, then propositions 1 and 2 must be false, but proposition 1 is obviously true. If, on the other hand, proposition 3 were false, then the other two propositions must also be false, or conversely, the other two propositions must be true. Neither of these options is possible, which means we obtained a paradox. It exists because proposition 3 refers to the truthfulness of proposition 1-3, i.e. among others to the truthfulness of itself.

For a correct understanding of mathematical texts and mathematics in general, it is important to understand mathematical language and its laws. If we imagine mathematics as a castle, its foundations will be mathematical logic and set theory. That is why we will start our course with these two parts of mathematics.

Propositions

By a proposition we mean any declarative sentence for which only one of two possibilities can occur: the sentence can be either true or false. Each proposition therefore has a unique truth value, which is 1 if the proposition is true, or 0 if it is false.

Examples of propositions:

1. 2 is an even number.
2. Prague is the capital of the Czech Republic.
3. $\sqrt{2}$ is a rational number.

Propositions 1 and 2 are true while the last proposition is false.

Imperative and interrogative sentences or incomplete or unclear sentences such as headings or exclamations are not considered propositions. In other words, we regard only comprehensible statements truthfulness of which can be determined.

Propositional formulas

Reasoning on the truthfulness of propositions is done in propositional logic. In addition to relatively simple propositions (called atomic propositions such as the ones in the previous case), propositional calculus deals with the composition of propositions using so-called logical connectives, which we will now introduce one by one.

If A is a proposition, then $\neg A$ means its negation, i.e. a proposition whose truth value is: if A is true, then $\neg A$ is false and vice versa, i.e. if A is false, then $\neg A$ is true. If the proposition A is written in everyday language, we form its negation by using the phrase “it is not true that” replacing “is” with “is not,” or by using one of the negative prefixes. For example, the negation of proposition: “Number 2 is even” is the proposition: “Number 2 is not even” (or: “It is not true that number 2 is even,” or “Number 2 is odd”).

When A and B are propositions written in everyday language, then “ A and B ” is also a proposition. We call it the conjunction of propositions A and B . When propositions A and B are written using mathematical language, we replace the conjunction “and” with the symbol $\wedge$. The conjunction of propositions A and B is true only if both propositions A and B are true.

We can connect proposition A and B also with the conjunction “or,” which we formally write as $A \vee B$. We call such a propositional formula a disjunction of propositions A and B . The disjunction of propositions A and B is true if at least one of the propositions A and B is true.

In mathematics, we most often encounter the phrase “if A , then B .” We formally write such a proposition as $A \Rightarrow B$ and call it an implication with the assumption A and the statement B . We can read it also as “from A there follows B ” or “ A implies B ”. The implication $A \Rightarrow B$ is false only when A is true and B is false – in all other cases it is true.

The last connective used to connect propositions A and B is “ A if and only if B .” We formally write such a compound proposition as $A \Leftrightarrow B$ and call it the equivalence of statements A and B . The equivalence $A \Leftrightarrow B$ is true in cases when propositions A and B have the same truth value.

In mathematical (or propositional) logic, the dependence of the truth of compound propositions on the truth of their atomic propositions is given by the following table (“tv” stands for truth value):

$tv(p)$	$tv(q)$	$tv(\neg p)$	$tv(p \wedge q)$	$tv(p \vee q)$	$tv(p \Rightarrow q)$	$tv(p \Leftrightarrow q)$
1	1	0	1	1	1	1
1	0	0	0	1	0	0
0	1	1	0	1	1	0
0	0	1	0	0	1	1

We do not condition the construction of compound propositions on either the content or formal connection of their components. Therefore, from the point of view of mathematical logic, the statement “If $2+2 = 5$, then there exists a triangle whose angles are all right angles” makes sense. Moreover, this statement is true.

Propositional formulas

A variable which can be true or false only is called a propositional variable. We use logical connectives $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$ to form propositional formulas from propositional variables. Propositions such as $A \wedge B, A \vee B, A \Rightarrow B, A \Leftrightarrow B$ are called compound propositions. Compound propositions can be connected by logical connectives into new compound propositions, or propositional formulas. We will show the procedure for determining the truth value of propositional formulas for the input values of propositional variables using an example.

Example

Exercise 1. Suppose that A, B, C are propositional variables. Find out whether

$$(A \Leftrightarrow B) \Rightarrow ((A \wedge C) \Leftrightarrow (B \wedge C))$$

is a propositional formula. If so, determine its truth values based on the input truth values of propositional variables A, B, C .

Solution. Given the fact that A, B, C are propositional variables and $\Leftrightarrow, \wedge$ are logical connectives, $A \Leftrightarrow B, A \wedge C, B \wedge C$ are propositional formulas. Then also $((A \wedge C) \Leftrightarrow (B \wedge C))$ is a propositional formula. As a result, $(A \Leftrightarrow B) \Rightarrow ((A \wedge C) \Leftrightarrow (B \wedge C))$ is a propositional formula. We have constructed it from the following formulas: $A, B, C, A \Leftrightarrow B, A \wedge C, B \wedge C, ((A \wedge C) \Leftrightarrow (B \wedge C)), (A \Leftrightarrow B) \Rightarrow ((A \wedge C) \Leftrightarrow (B \wedge C))$. This sequence of formulas is called the generating sequence of the formula $(A \Leftrightarrow B) \Rightarrow ((A \wedge C) \Leftrightarrow (B \wedge C))$.

For each propositional formula, we can construct a truth table according to the following instructions:

1. In the header, we write all the terms that are used in the sequence of the formula.
2. Under the variables, we write all possible n -tuples of the truth values 0 and 1 in a row. If the formula has n variables, the number of n -tuples is 2^n . Note the procedure of enumerating these n -tuples. Under the variable C , the values 0 and 1 alternate in each row, under the variable B in every second row, and under the variable A in every fourth row (i.e. alternating in every $2^{n^{th}}$ row). This procedure ensures that no possibility is omitted or repeated.
3. In accordance with the table for the truth values of logical connectives, we fill in the columns under the remaining elements generating the sequence.

For a formula $F = (A \Leftrightarrow B) \Rightarrow ((A \wedge C) \Leftrightarrow (B \wedge C))$ we construct the following table:

A	B	C	$A \Leftrightarrow B$	$A \wedge C$	$B \wedge C$	$(A \wedge C) \Leftrightarrow (B \wedge C)$	F
1	1	1	1	1	1	1	1
1	1	0	1	0	0	1	1
1	0	1	0	1	0	0	1
1	0	0	0	0	0	1	1
0	1	1	0	0	1	0	1
0	1	0	0	0	0	1	1
0	0	1	1	0	0	1	1
0	0	0	1	0	0	1	1

As we can see, the formula $(A \Leftrightarrow B) \Rightarrow ((A \wedge C) \Leftrightarrow (B \wedge C))$ is true for all possible combinations of input variables A, B, C .

Tautology

In mathematics, formulas, truth values of which are always 1 in the last column of their truth table, are particularly important. Such propositional formulas are true for any truth values of the propositional variables. We call them tautologies. Formulas, truth values of which are always 0, are called contradictions.

Using truth tables, you can prove that the following propositional formulas are tautologies:

1. $A \Leftrightarrow \neg(\neg A)$,
2. $(A \wedge B) \Leftrightarrow (B \wedge A)$, $(A \vee B) \Leftrightarrow (B \vee A)$,
3. $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$, $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$,
4. $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$, $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$,
5. $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$, $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$,
6. $(A \Leftrightarrow B) \Leftrightarrow ((A \Rightarrow B) \wedge (B \Rightarrow A))$,
7. $\neg(A \Rightarrow B) \Leftrightarrow A \wedge \neg B$,
8. $A \Rightarrow B \Leftrightarrow \neg B \Rightarrow \neg A$.

Quantifiers and closed formulas

So far we have discussed the process of construction of compound propositions and the dependence of their truth values based on the truth values of their respective components. However, we have not discussed the structure of the propositions.

First, let us explain the meaning of two symbols.

The notation $\exists x$ reads “there exists an x ”. The symbol $\exists$ is called the existential quantifier. We use it whenever we want to say that there is at least one element in the set we are working with for which the formula after the quantifier holds. The notation $\forall x$ reads “for every x ”(or “for all x ”). The symbol $\forall$ is called the universal quantifier. We use it whenever we want to say that for all elements of the set we are working with, the formula after the quantifier holds.

For example, $\exists x \in \mathbb{R}: (x^2 - 1 = 0)$ means that the equation $x^2 - 1 = 0$ has at least one real solution, i.e. that there exists at least one real number such that $x^2 - 1 = 0$. On contrary, $\forall x \in \mathbb{R}: (x^2 + 1 \neq 0)$ means that the equation $x^2 + 1 = 0$ does not have any real solution, i.e. that there is $x^2 + 1 \neq 0$ for all real numbers.

In the following chapter we will discuss the notion of a set. Before doing so we must explain one more concept – a closed formula. This is a formula containing variables which, after substituting a permissible constant for the variables, becomes a proposition. Equations and inequations are typical examples of closed formulas. Suppose that the closed formula A has a variable x . When we substitute some element $a \in A$ for x , we may or may not obtain a proposition. For example, if the domain of the following closed formula

$$4 - x = \frac{3}{x}$$

are all real numbers and we substitute number 1 for x , we get a true proposition. However, if we substitute 2, we get a false proposition. Finally, if we substitute 0 for x , we get an expression $\frac{3}{0}$ which does not represent any real number. The set of all elements, substitution of which results in true propositions, is called the truth domain of the closed formula.

When defining sets, we usually do so by enumerating their elements. However, we can also give the truth domain of closed formulas, e.g. as

$$M = \{x \in A: V(x)\}.$$

This means that the set M contains all elements of the set A , for which the formula V with a variable x is a true proposition. For example, the interval $[1, 3)$ can be written as

$$M = \{x \in \mathbb{R}: 1 \leq x < 3\}.$$

Sets

The concept of a set is one of the most fundamental concepts in mathematics. One can say that set theory is a suitable foundation for all mathematical disciplines. By a set we mean a collection (group, set) of mutually different objects that – using a suitable criterion – form a single whole. A set is well defined when it is possible to decide whether each object is or is not its element. If an element x is an element of set A , we write $x \in A$. In this is not the case, we write $x \notin A$. For example, if the set A consists of numbers 1, 2, 3 and the set A has no other elements, we write $A = \{1, 2, 3\}$. Writing $A = \{2, 1, 3\}$ is also correct because the order of the elements does not matter. In this case, we say that the set is defined by the enumeration of its elements. An empty set, i.e., a set which has no elements, is denoted by $\emptyset$ or $\{\}$. Notice that the set $\{\emptyset\}$ is, on the contrary, a one-element set, the element of which is the empty set.

The notation used for the usual numerical sets – number domains – is as follows:

- $\mathbb{N}$ – the set of all natural numbers,
- $\mathbb{Z}$ – the set of all integers,
- $\mathbb{Q}$ – the set of all rational numbers,
- $\mathbb{R}$ – the set of all real numbers,
- $\mathbb{C}$ – the set of all complex numbers.

Definition 2. We say that the set A is a subset of the set of the set B if each element of the set A is the element of the set B . In such a case we write $A \subseteq B$. If we want to stress that $A \subseteq B$ and $A \neq B$, we write $A \subset B$ and say that A is a proper subset of B .

Remark 3. There is $\emptyset \subseteq A$, $A \subseteq A$ for each set A .

Definition 4. We say that sets A, B are equal if there is $A \subseteq B$ and simultaneously $B \subseteq A$.

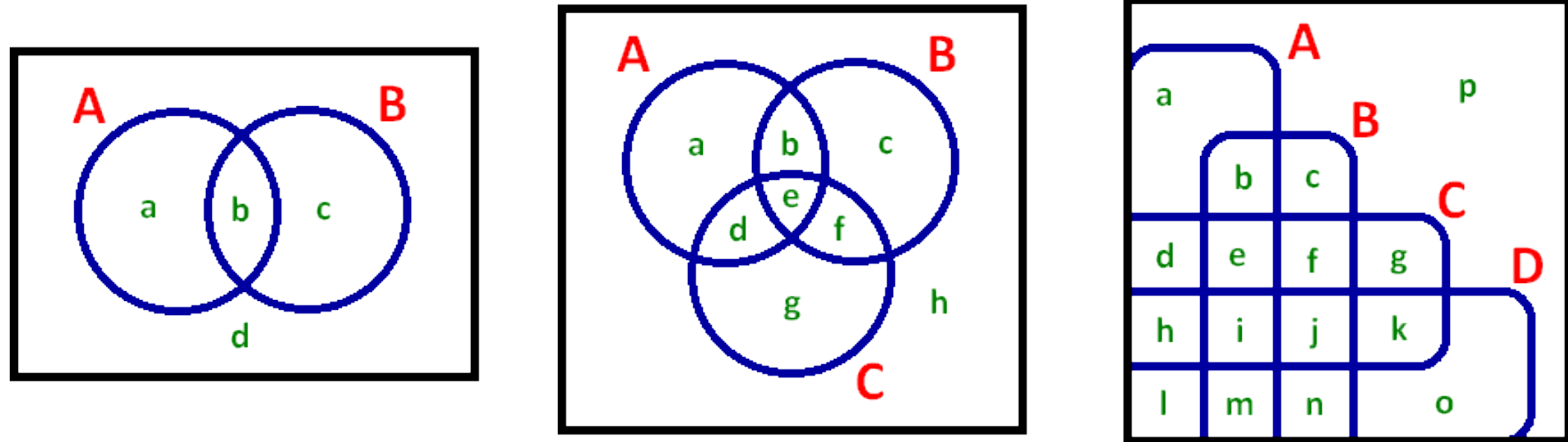
As we have mentioned above, sets can be defined by giving the truth domain of some closed formula. This way of defining sets is useful especially when we need to define infinite sets. We recall it when defining intervals of real numbers.

- A closed is defined as $[a, b] = \{x; a \leq x \leq b\}$,
- An open interval is defined as $(a, b) = \{x; a < x < b\}$,
- A left open and right closed interval is defined as $(a, b] = \{x; a < x \leq b\}$,
- A left closed and right open interval is defined as $[a, b) = \{x; a \leq x < b\}$.

Remark 5. In the list above there is always $a < b$.

Venn diagram style

For informal understanding, it is useful to display set operations and relationships using a Venn diagram style, which is capable of showing all possible relationships between several sets. For two, three, and four sets, we use the following diagrams:



We will explain the idea of Venn diagrams for two sets. In the figure, we have sets: *A* and *B* and four areas: *a*, *b*, *c*, and *d*. Area *a* contains elements which belong to set *A* but do not belong to set *B*. Area *c* contains elements which belong to set *B* but not to set *A*. Area *b* contains elements which are common to both sets *A* and *B*, and area *d* contains elements that do not belong to either set *A* or *B*. We use a similar procedure to show the relationship between multiple sets.

Set operations: set union

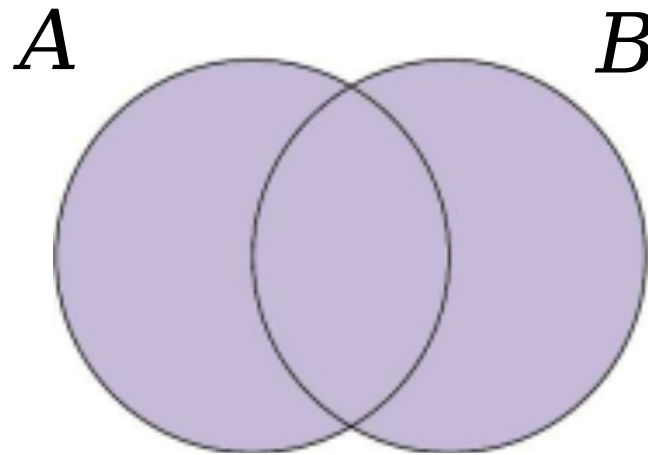
For each pair of sets A, B a new set

$$A \cup B = \{x; x \in A \vee x \in B\}$$

can be constructed. Such a set is called the union of sets A and B . There is

$$x \in A \cup B \Leftrightarrow x \in A \vee x \in B.$$

The set union can be graphically depicted as follows:



For the set union we have:

- $A \cup \emptyset = A$,
- $A \cup A = A$,
- $A \cup B = B \cup A$,
- $A \cup (B \cup C) = (A \cup B) \cup C$.

Set operations: set intersection

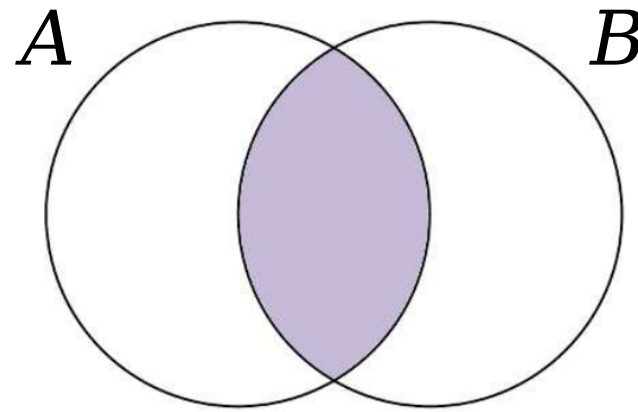
For each pair of sets A, B a new set

$$A \cap B = \{x; x \in A \wedge x \in B\},$$

can be constructed. Such a set is called the intersection of sets A and B . There is

$$x \in A \cap B \Leftrightarrow x \in A \wedge x \in B.$$

The set intersection can be graphically depicted as follows:



For the set intersection we have:

- $A \cap \emptyset = \emptyset$,
- $A \cap A = A$,
- $A \cap B = B \cap A$,
- $A \cap (B \cap C) = (A \cap B) \cap C$.

When there is $A \cap B = \emptyset$ for some sets A, B , we say that the sets A and B are disjoint.

Set operations: set difference

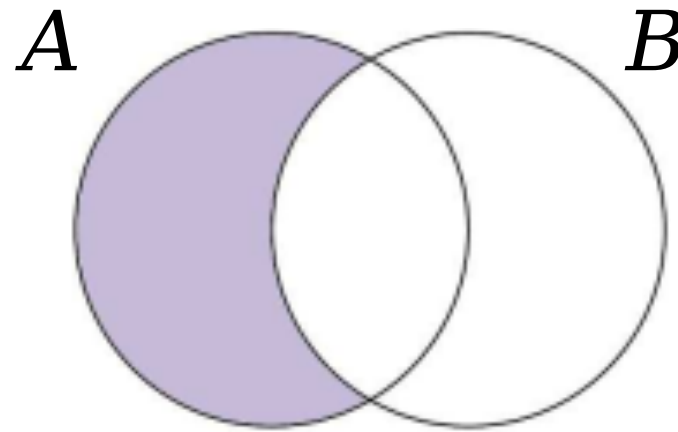
For each pair of sets A, B a new set

$$A \setminus B = \{x; x \in A \wedge x \notin B\},$$

can be constructed. Such a set is called the set difference of sets A and B . There is

$$x \in A \setminus B \Leftrightarrow x \in A \wedge x \notin B.$$

The set difference can be graphically depicted as follows:



Properties of the set difference include:

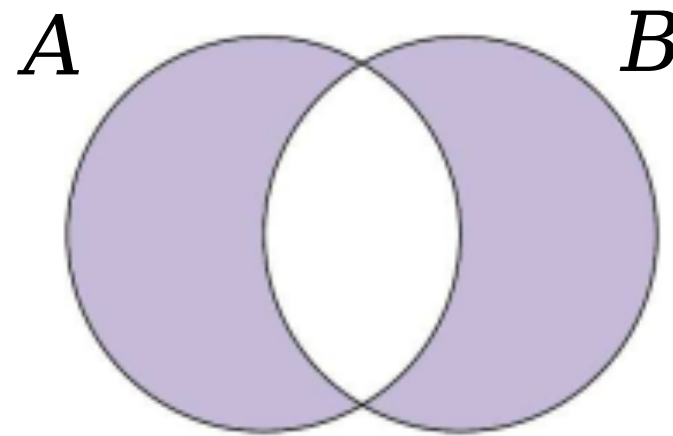
- $A \setminus \emptyset = A$,
- $A \setminus A = \emptyset$,
- $A \setminus B \neq B \setminus A$,
- $A \setminus (B \setminus C) \neq (A \setminus B) \setminus C$.

Set operations: symmetric difference

For each pair of sets A, B a new set

$$A \Delta B = (A \setminus B) \cup (B \setminus A),$$

can be constructed. Such a set is called the symmetric difference of sets A and B . The symmetric difference can be graphically depicted as follows:



Properties of the set difference include:

- $A \Delta B = B \Delta A$,
- $A \Delta (B \Delta C) = (A \Delta B) \Delta C$,
- $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

Set operations: complement

Suppose that M is some “basic set” and that there is $A \subseteq M$. Then

$$A'_M = M \setminus A$$

is called the complement of set A with respect to set M . There is

$$x \in A'_M \Leftrightarrow x \in M \wedge x \notin A.$$

This set operation is special in the sense that we always need to know, which basic set the complement is related to. If e.g. $M = \{1, 2, 3\}$ and $A = \{1, 2\}$, there is $A'_M = \{3\}$ while for $P = \{1, 2, 3, 4\}$ there is $A'_P = \{3, 4\}$. For the complement of A with respect to M we have:

- $(A'_M)'_M = A$,
- $(A \cup B)'_M = A'_M \cap B'_M$,
- $(A \cap B)'_M = A'_M \cup B'_M$,
- $(A \setminus B)'_M = A'_M \cup B$.

Cartesian product

In mathematics we often make use of ordered pairs. Two ordered pairs are equal if and only if there is:

$$[a, b] = [c, d] \Leftrightarrow a = c \wedge b = d.$$

Remark 6. *Students often ignore the difference between $\{a, b\}$ and $[a, b]$. Notice that $\{a, b\}$ is a two-element set and as we already know, it does not matter whether we write $\{a, b\}$ or $\{b, a\}$. However, $[a, b]$ is an ordered pair for which there is $[a, b] \neq [b, a]$.*

Cartesian product is a notion closely related to ordered pairs.

Definition 7. *By the Cartesian product $A \times B$ of sets A and B we mean the set of all ordered pairs $[a, b]$ such that their first component is the element of set A and their second component is the element of set B , i.e.*

$$A \times B = \{[x, y]; x \in A \wedge y \in B\}.$$

There is

$$[x, y] \in A \times B \Leftrightarrow x \in A \wedge y \in B.$$

Exercise

Exercise 8. Enumerate all elements of the set M which is defined as

$$M = \{n \in \mathbb{N} : 5 \leq n < 10\}.$$

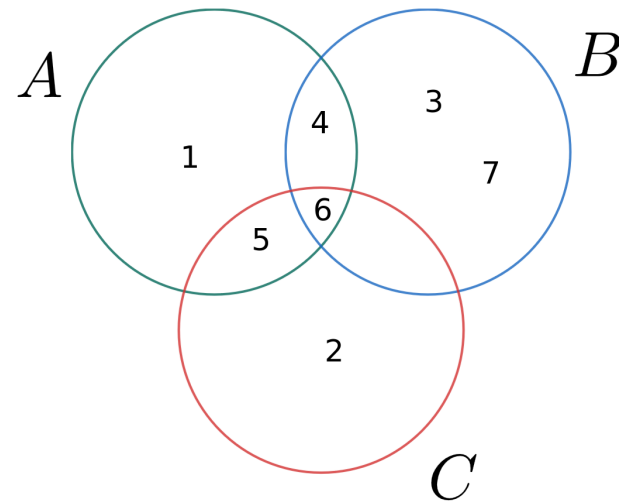
Solution. For each element n of the set M there must be $5 \leq n < 10$. Moreover, we must keep in mind that we work with natural numbers only. Therefore:

$$M = \{5, 6, 7, 8, 9\}.$$

Exercise

Exercise 9. Regard the following sets: $A = \{1, 4, 5, 6\}$, $B = \{3, 4, 6, 7\}$, $C = \{2, 5, 6\}$. Determine $A \cup B$, $A \cap C$, $C \setminus B$, $A \setminus (C \setminus B)$, $A \cap (B \setminus C)$.

Solution. First of all, we place the elements of all three sets into a Venn diagram scheme. After that we can write the answers immediately.



- $A \cup B = \{1, 3, 4, 5, 6, 7\}$,
- $A \cap C = \{5, 6\}$,
- $C \setminus B = \{2, 5\}$,
- $A \setminus (C \setminus B) = \{1, 4, 6\}$,
- $A \cap (B \setminus C) = \{4\}$.

Exercise

Exercise 10. Regard the following sets: $A = \{1, 2, 3, 13\}$, $B = \{2, 4, 6, 13\}$, $C = \{1, 3, 6, 13\}$. Determine $A\Delta B$, $A\Delta(B\Delta C)$.

Solution. Since the symmetric difference of sets A and B consists of elements which are not included in their intersection, there is

$$A\Delta B = \{1, 3, 4, 6\}.$$

The second question is somewhat more difficult. We make use of the fact that the symmetric difference is associative, i.e. that there is $\forall A, B, C: A\Delta(B\Delta C) = (A\Delta B)\Delta C$ and we make use of the previous result. Then there is

$$A\Delta(B\Delta C) = (A\Delta B)\Delta C = \{1, 3, 4, 6\}\Delta\{1, 3, 6, 13\}$$

and

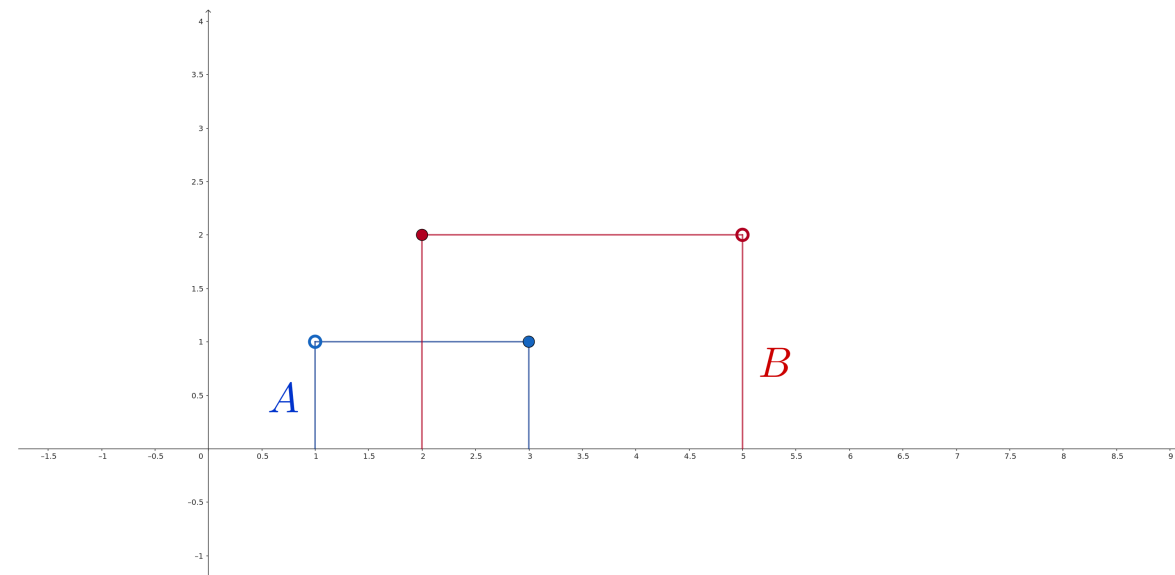
$$A\Delta(B\Delta C) = \{4, 13\}.$$

Remark 11. Notice that the set $A\Delta(B\Delta C)$ consists only of elements which belong to all three sets (such as 13) or of elements which are contained in exactly one of the three sets (such as 4). This is not a coincidence. This is true for an arbitrary triplet of sets.

Exercise

Exercise 12. For $A = (1, 3]$, $B = [2, 5)$ determine $A \cup B$, $A \cap B$, $B \setminus A$, $A \setminus (A \setminus B)$, $B \setminus (B \setminus A)$, $A \setminus (B \setminus A)$, $B \setminus (A \setminus B)$, $(A \setminus B) \cap (B \setminus A)$.

Solution. Since we work with intervals of real numbers, we first position the relevant numbers on the real axis.



- $A \cup B = (1, 5)$,
- $A \cap B = \langle 2, 3 \rangle$,
- $B \setminus A = (3, 5)$,
- $A \setminus (A \setminus B) = \langle 2, 3 \rangle$,
- $B \setminus (B \setminus A) = \langle 2, 3 \rangle$,
- $A \setminus (B \setminus A) = (1, 3)$,
- $B \setminus (A \setminus B) = \langle 2, 5 \rangle$,
- $(A \setminus B) \cap (B \setminus A) = \emptyset$.

Exercise

Exercise 13. For $A = \{1, 2, 3\}$, $B = \{\circ, \star\}$ determine $A \times B$, $B \times A$.

Solution. Given the definition of the Cartesian product, we get that:

$$A \times B = \{[1, \circ], [1, \star], [2, \circ], [2, \star], [3, \circ], [3, \star]\},$$

$$B \times A = \{[\circ, 1], [\star, 1], [\circ, 2], [\star, 2], [\circ, 3], [\star, 3]\}.$$

Maintaining the correct order of the sets and systematic listing of pairs is essential when solving tasks like this one. When writing $A \times B$, we chose the first element of the set A and made all pairs with this element as the first component while the second position was occupied by the other elements of B – one by one. We continue in this way until we list all elements of the set A .

Exercise

Exercise 14. Find possible sets X and Y so that there was: $X \cap Y = \emptyset$, $X \cup Y = \{1, 2, 3, 4, 5, 6, 7\}$ and to each $a \in X$ there exists $b \in Y$ such that $b = a + 4$.

Solution. The condition $\forall a \in X \exists b \in Y: b = a + 4$ results in the fact that $X \subseteq \{1, 2, 3\}$. If there was e.g. $4 \in X$, then the condition $8 \in Y$ would result into $X \cup Y = \{1, 2, 3, 4, 5, 6, 7\}$. Therefore, we can conclude with the following options:

1. $X_1 = \emptyset, Y_1 = \{1, 2, 3, 4, 5, 6, 7\}$,
2. $X_2 = \{1\}, Y_2 = \{2, 3, 4, 5, 6, 7\}$,
3. $X_3 = \{2\}, Y_3 = \{1, 3, 4, 5, 6, 7\}$,
4. $X_4 = \{3\}, Y_4 = \{1, 2, 4, 5, 6, 7\}$,
5. $X_5 = \{1, 2\}, Y_5 = \{3, 4, 5, 6, 7\}$,
6. $X_6 = \{1, 3\}, Y_6 = \{2, 4, 5, 6, 7\}$,
7. $X_7 = \{2, 3\}, Y_7 = \{1, 4, 5, 6, 7\}$,
8. $X_8 = \{1, 2, 3\}, Y_8 = \{4, 5, 6, 7\}$.

Make sure that you understand why the required conditions are valid for $X_1 = \emptyset$ as well. The condition must hold for all elements of the set X_1 . Since X_1 does not contain any elements, it is valid by default. Finally, notice that when we list the sets systematically, we minimize the risk of omitting some options.

Exercise

Exercise 15. Give an example of sets A and B such that there is $A \in B$ and at the same time $A \subseteq B$.

Solution. Let us start with set A . Each element of this set must be the element of the set B yet in such a way that the whole set A must be the element of the set B . Suppose that $A = \{2\}$. Then there must be $2 \in B$ because $2 \in A$. However, there must be also $\{2\} \in B$ because there must be $A \in B$. Therefore, there is $B = \{2, \{2\}\}$.

1 TEST

1. If $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 6, 7, 8\}$, there is $A \setminus B =$
 - (a) $\{\}$
 - (b) $\{1, 3\}$
 - (c) $\{6, 7, 8\}$
 - (d) $1, 3$
2. If $A = [1, 5]$, $B = (-1, 3)$, then there is $A \cap B =$
 - (a) $[1, 3)$
 - (b) $[1, 3]$
 - (c) $(3, 1]$
 - (d) $[3, 1]$
3. If $A = [1, 5]$, $B = (-1, 3)$, then there is $A \setminus B =$
 - (a) $[3, 5]$
 - (b) $[5, 3]$
 - (c) $(-1, 5)$
 - (d) $(-1, 1)$
4. Which pairs of sets are not disjoint?
 - (a) $A = (1, 2)$, $B = \left[-1, \frac{3}{2}\right]$
 - (b) $A = (1, 2)$, $B = [-1, 1]$
 - (c) $A = (-1, 2)$, $B = \left[2, \frac{5}{2}\right]$
 - (d) $A = (1, 2]$, $B = [2, 7]$
5. If there is $M = \{n \in \mathbb{R} : 5 \leq n < 10\}$, then
 - (a) $M = (10, 5)$
 - (b) $M = [5, 10)$
 - (c) $M = [5, 10]$
 - (d) $M = \{5, 6, 7, 8, 9\}$
6. If there is $A = \{1, 2, 3\}$, $B = \{4, 5\}$, then $A \times B =$
 - (a) $\{[1, 4], [1, 5], [2, 4], [2, 5], [3, 4], [3, 5]\}$
 - (b) $\{\{1, 4\}, \{1, 5\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\}\}$
 - (c) $\{[4, 1], [5, 1], [4, 2], [5, 2], [4, 3], [5, 3]\}$
 - (d) $\{4, 5, 8, 10, 12, 15\}$
7. If there is $x \in [1, 5] \wedge x \in (-1, 3)$, then $x \in$

- (a) $\langle 1, 3 \rangle$
- (b) $(1, 3)$
- (c) $(-1, 5)$
- (d) $[1, 3)$

8. If there is $x \in [1, 5] \vee x \in (-1, 3)$, then $x \in$

- (a) $[-1, 5]$
- (b) $(-1, 5)$
- (c) $(-1, 5]$
- (d) $\langle 1, 3 \rangle$

9. Suppose that $A = (-\infty, x + 2]$, $B = \left[\frac{x}{2}, 4\right]$. For $x = 4$ there is

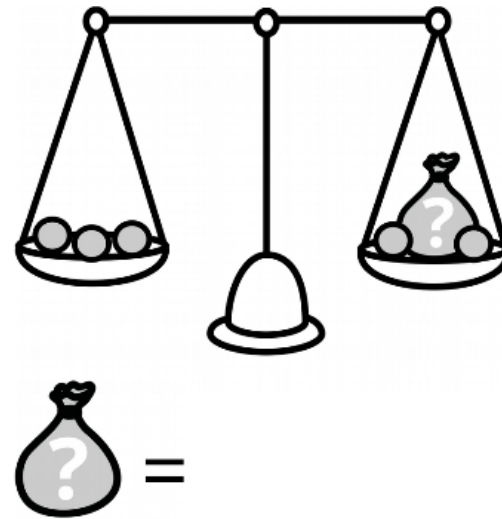
- (a) $A \cap B = \emptyset$
- (b) $A \subseteq B$
- (c) $B \subseteq A$
- (d) $A = B$

10. Suppose that $A = (-\infty, x)$, $B = \left[\frac{x}{2}, 6\right]$. For $x = 4$ there is

- (a) $A \cap B = \emptyset$
- (b) $A \subseteq B$
- (c) $B \subseteq A$
- (d) $A \cap B \neq \emptyset$

2. EQUATIONS AND INEQUALATIONS

Equations and inequations



Linear equations

By a linear equation we mean an equation with one variable, which is present at most in its first power. The normal form of linear equations is

$$ax + b = 0.$$

If $a \neq 0$, the equation has exactly one solution $x = -\frac{b}{a}$. However, the equation can be given also e.g. in the following form:

$$2 \cdot (3 - x) - 3 \cdot (1 - 3x) = 2 \cdot (3 - 4x).$$

Usually, we obtain the normal form of a linear equation by applying some equivalence transformations. We will discuss them one by one.

Transformations of equations done separately on each side

All the following transformations can be done at will because all they do is change the visual appearance of the expressions at respective sides of the equation and the expressions before and after the transformation are in fact the same.

Addition and subtraction:

We work with numbers and variables in the usual way such as when e.g. in the equation

$$2x-3+5x-3=3x-7x+2+6$$

we make the sum of numbers -3 and -3 at the left-hand side and get

$$2x-6+5x=3x-7x+2+6.$$

Then we, at the left-hand side of the equation, add the expressions containing x :

$$7x-6=3x-7x+2+6.$$

Then we repeat this procedure for the right-hand side of the equation:

$$7x-6=-4x+8.$$

Transformations of equations done separately on each side

Expanding brackets and taking out of the brackets:

- Mind the signs when expanding brackets! See e.g.:

$$\begin{aligned}2(x - 5) - 3(1 - x) &= 0 \\2 \cdot x - 2 \cdot 5 + (-3) \cdot 1 + (-3) \cdot (-x) &= 0 \\2x - 10 - 3 + 3x &= 0\end{aligned}$$

- The following example shows how to take out of the brackets:

$$3(x - 1) - 5(x - 1) = 0$$

Both expressions contain the difference $x - 1$, which we take out. We have to be careful when determining the sign in front of the second bracket:

$$(x - 1)(3 - 5) = 0 \iff (-2)(x - 1) = 0$$

Transformations of equations done separately on each side

Putting over a common denominator:

In the following equation there are fractions with different denominators. Before we apply transformations of the equation, we need to combine the fractions into a single fraction such that its denominator is the least common multiple of the respective denominators. In other words, we are looking for such a number that is divisible by all denominators and at the same time is the smallest of such numbers. The transformation in question is expanding fractions with suitable numbers. For example, if we want that the fraction $\frac{3}{4}$ had denominator 12 and its value did not change, we need to expand it by three (i.e. multiply it by three over three), i.e. write

$$\frac{3}{4} = \frac{3 \cdot 3}{4 \cdot 3} = \frac{9}{12}.$$

In the following equation we want to combine fractions into one:

$$\frac{x}{2} + \frac{x-2}{3} = 2x - \frac{x}{6}.$$

On both sides, their common denominator is 6 (of course, this is just a coincidence, the common denominators can be different on each side), which means that we expand the fractions in a suitable way:

$$\frac{x \cdot 3}{2 \cdot 3} + \frac{(x-2) \cdot 2}{3 \cdot 2} = \frac{2x \cdot 6}{1 \cdot 6} - \frac{x}{6}$$

and get

$$\frac{3x}{6} + \frac{2x-4}{6} = \frac{12x}{6} - \frac{x}{6}.$$

Now we can add the fractions and get

$$\frac{3x+2x-4}{6} = \frac{12x-x}{6}.$$

Transformation of equations

Adding or subtracting the same expression to both sides of the equation.

The solution of the equation does not change if we add the same number, such as -3 in the following equation, to both sides of the equation:

$$\begin{array}{l} x + 3 = 5 - 2x \\ x + 3 + (-3) = 5 - 2x + (-3) \quad \left. \vphantom{x + 3 + (-3)} \right\} \text{add } -3 \\ x = 2 - 2x \end{array}$$

The same holds for subtraction, such as e.g. 3 in this case:

$$\begin{array}{l} x + 3 = 5 - 2x \\ x + 3 - 3 = 5 - 2x - 3 \quad \left. \vphantom{x + 3 - 3} \right\} \text{subtract } 3 \\ x = 2 - 2x \end{array}$$

The same holds for algebraic expressions such as e.g. $2x$ in this case:

$$\begin{array}{l} x = 2 - 2x \\ x + 2x = 2 - 2x + 2x \quad \left. \vphantom{x + 2x} \right\} \text{add } 2x \\ 3x = 2 \end{array}$$

or $-2x$ here

$$\begin{array}{l} x = 2 - 2x \\ x - (-2x) = 2 - 2x - (-2x) \quad \left. \vphantom{x - (-2x)} \right\} \text{subtract } -2x \\ 3x = 2 \end{array}$$

Transformation of equations

Multiplying the equation by a nonzero number.

The solution of the equation does not change if we multiply it by a non-zero number such as e.g. 2 in the following case:

$$\begin{array}{l} \frac{x}{2} + 3 = 5 - 2x \\ \left(\frac{x}{2} + 3 \right) \cdot 2 = (5 - 2x) \cdot 2 \quad \downarrow \text{multiply by } 2 \\ \frac{x}{2} \cdot 2 + 3 \cdot 2 = 5 \cdot 2 - 2x \cdot 2 \\ x + 6 = 10 - 4x \end{array}$$

Exercise

Exercise 1. Solve the following equation in the real domain:

$$2 \cdot (3 - x) - 3 \cdot (1 - 3x) = 2 \cdot (3 - 4x).$$

Solution.

$$\begin{array}{lcl}
 2 \cdot (3 - x) - 3 \cdot (1 - 3x) = 2 \cdot (3 - 4x) & & \left. \begin{array}{l} \\ \end{array} \right\} \text{expand the brackets} \\
 6 - 2x - 3 + 9x = 6 - 8x & & \left. \begin{array}{l} \\ \end{array} \right\} \text{adjust the left- and right-hand sides} \\
 3 + 7x = 6 - 8x & & \left. \begin{array}{l} \\ \end{array} \right\} \text{add } 8x \text{ to both sides, subtract } 3 \\
 7x + 8x = 6 - 3 & & \left. \begin{array}{l} \\ \end{array} \right\} \text{transform the left hand side, transform the right-hand side} \\
 15x = 3 & & \\
 x = \frac{3}{15} & & \\
 x = \frac{1}{5} & &
 \end{array}$$

The solution is $x = \frac{1}{5}$.

We verify the correctness of the solution. This is done in such a way that we substitute the solution, in our case $\frac{1}{5}$, into both sides of the equation.

$$L = 2 \cdot \left(3 - \frac{1}{5}\right) - 3 \cdot \left(1 - 3 \cdot \frac{1}{5}\right) = 2 \cdot \frac{14}{5} - 3 \cdot \frac{2}{5} = \frac{22}{5},$$

$$R = 2 \cdot \left(3 - 4 \cdot \frac{1}{5}\right) = 2 \cdot \frac{11}{5} = \frac{22}{5}.$$

Since both results are the same, we know that our solution is correct.

Linear inequations

The normal form of linear inequations is:

$$ax + b > 0.$$

Alternatively, other inequality signs, such as $<$, $\leq$, $\geq$ are used instead of $>$. Solutions of linear inequations can have the form of the emptyset (i.e. no real number fits the inequation), finite set or an interval of real numbers. Similarly as in the case of linear equations we obtain the normal form of the linear inequation after application of several transformations. Transformations of equations done separately on each side are the same as in the case of linear equations.

Transformations of linear inequations

Adding or subtracting the same expression to both sides of the inequation

The solution of the inequation does not change if we add the same number, such as -3 in the following inequation, to both sides of the inequation:

$$\begin{array}{l} x + 3 < 5 - 2x \\ x + 3 + (-3) < 5 - 2x + (-3) \quad \leftarrow \text{add } -3 \\ x < 2 - 2x \end{array}$$

The same holds for subtraction, such as e.g. 3 in this case:

$$\begin{array}{l} x + 3 < 5 - 2x \\ x + 3 - 3 < 5 - 2x - 3 \quad \leftarrow \text{subtract } 3 \\ x < 2 - 2x \end{array}$$

The same holds for algebraic expressions such as e.g. $2x$ in this case:

$$\begin{array}{l} x < 2 - 2x \\ x + 2x < 2 - 2x + 2x \quad \leftarrow \text{add } 2x \\ 3x < 2 \end{array}$$

or $-2x$ here:

$$\begin{array}{l} x < 2 - 2x \\ x - (-2x) < 2 - 2x - (-2x) \quad \leftarrow \text{subtract } -2x \\ 3x < 2 \end{array}$$

Transformations of linear inequations

Multiplying the inequation by a positive number

The solution of the inequation does not change if we multiply it by a positive number such as by 2 in the following case:

$$\begin{array}{l} \frac{x}{2} + 3 > 5 - 2x \\ \left(\frac{x}{2} + 3\right) \cdot 2 > (5 - 2x) \cdot 2 \quad \left. \begin{array}{l} \nearrow \\ \searrow \end{array} \right\} \text{multiply by } 2 \\ \frac{x}{2} \cdot 2 + 3 \cdot 2 > 5 \cdot 2 - 2x \cdot 2 \\ x + 6 > 10 - 4x \end{array}$$

Notice that this is true when we multiply by a positive number only!

Transformations of linear inequations

Multiplication of the inequation by a negative number

The solution of the inequation **changes profoundly** after we multiply it by a negative number such as by -2 in the following case:

$$\begin{aligned} \frac{x}{2} + 3 &> 5 - 2x \\ \left(\frac{x}{2} + 3\right) \cdot (-2) &< (5 - 2x) \cdot (-2) && \left. \begin{array}{l} \text{multiply by } -2 \\ \text{use the opposite inequality sign} \end{array} \right\} \\ \frac{x}{2} \cdot (-2) + 3 \cdot (-2) &< 5 \cdot (-2) - 2x \cdot (-2) \\ -x - 6 &< -10 + 4x \end{aligned}$$

Exercise

Exercise 2. In the real domain solve the following inequation:

$$4x - \frac{x-5}{2} > \frac{5x}{3} + 1.$$

Solution.

$$\begin{array}{l} 4x - \frac{x-5}{2} > \frac{5x}{3} + 1 \\ 24x - 3x + 15 > 10x + 6 \end{array} \left. \begin{array}{l} \\ \end{array} \right\} \begin{array}{l} \text{multiply both sides of the inequation by the smallest common} \\ \text{multiple of numbers 2 and 3 i.e. by 6} \end{array}$$

$$21x + 15 > 10x + 6$$

$$11x > -9$$

$$x > -\frac{9}{11}$$

The solution is formed by all real numbers from the interval $\left(-\frac{9}{11}, \infty\right)$.

Exercise

Exercise 3. *In the real domain solve the following inequation:*

$$2x - \frac{x-5}{3} > \frac{5x}{3} + 1.$$

Solution.

$$\begin{aligned} 2x - \frac{x-5}{3} &> \frac{5x}{3} + 1 \\ 6x - x + 5 &> 5x + 3 \quad \left. \begin{array}{l} \text{multiply both sides of the inequation by 3} \end{array} \right\} \\ 5x + 5 &> 5x + 3 \\ 5 &> 3 \end{aligned}$$

Since we obtained a true proposition ($5 > 3$), the solution of the inequation is formed by the set of all real numbers.

Remark

The need to solve inequations appears also in the set theory. Further on, we show two such tasks.

Exercise

Exercise 4. Find all real numbers x such that intervals $[\frac{x-1}{2}, 3]$ and $(-\infty, x+2]$ have a non-empty intersection.

Solution. Obviously, the starting point of each interval must be smaller than its ending point. As a result we get condition:

$$\frac{x-1}{2} < 3.$$

This is equivalent to

$$x-1 < 6, \text{ therefore } x < 7.$$

Intervals are supposed to have a non-empty intersection, which gets us one more condition:

$$\frac{x-1}{2} \leq x+2,$$

We multiply both sides of the inequation by 2 and get

$$x-1 \leq 2x+4,$$

which means that

$$-5 \leq x.$$

Both conditions must hold at the same time, which means that

$$x < 7 \wedge -5 \leq x,$$

i.e.

$$x \in [-5, 7).$$

Exercise

Exercise 5. Find all real numbers x such that the intersection of intervals $\left(\frac{4x}{3}, \infty\right)$ and $\left(-11, \frac{6x-2}{4}\right)$ is empty.

Solution. As in the previous exercise, we obtain a condition for the second interval:

$$-11 < \frac{6x-2}{4}.$$

This is equivalent to

$$-44 < 6x - 2, \quad \text{therefore} \quad -42 < 6x,$$

i.e. to

$$-7 < x.$$

The intersection of the intervals is supposed to be empty, which gets us one more condition:

$$\frac{6x-2}{4} < \frac{4x}{3}.$$

We multiply both sides of the inequation by 12 and get

$$18x - 6 < 16,$$

which means that

$$2x < 6 \Rightarrow x < 3.$$

Both conditions must hold at the same time, which means that

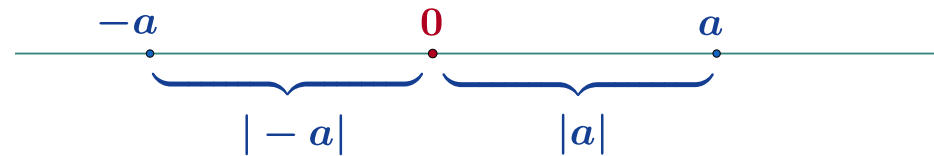
$$-7 < x \wedge x < 3.$$

As a result, the solution of our exercise is

$$x \in (-7, 3).$$

Absolute value

Absolute value of a number is its distance from the origin of the coordinate system, i.e. in the case of real numbers which we place on the real axis, from zero.



Absolute value of number a is denoted by $|a|$. For positive a there is $|a| = a$ while for negative a there is $|a| = -a$. In other words, the absolute value of a number is either the number itself or its opposite. We use this fact when determining absolute values of expressions such as $|x - 1|$. Below we show how to evaluate absolute values:

- $|1| = 1$,
- $|-2| = 2$,
- $4 - |5 - 8| = 4 - |-3| = 4 - 3 = 1$,

What is the geometrical interpretation of the equation $|x - 1| = 2$? There are two options: $x - 1 = 2$ or $x - 1 = -2$. The solution of the former is $x = 3$, the solution of the latter is $x = -1$. The geometrical interpretation of this equation is that we are looking for such numbers x that the distance of $x - 1$ from 0 is 2. In other words, we are looking for such numbers that their distance from 1 is 2. In general, the geometrical interpretation of the equation $|x - a| = b$ is that we are looking for all x , distance of which from number a is b .

When solving equations and inequations with absolute values, it is imperative to remove all absolute values. The way of removing absolute values is demonstrated in the following exercise. How do we evaluate e.g. the expression $|x - 3|$?

- The expression $x - 3$ evaluates to zero for $x = 3$, is negative to the left of zero and positive to the right of zero.
- For $x \in (-\infty, 3)$ there is $x - 3 < 0$, which means that removing the absolute value results in the change of sign, i.e. $|x - 3| = -(x - 3) = 3 - x$.
- For $x \in (3, \infty)$ there is $x - 3 > 0$, which means that removing the absolute value does not affect the sign, i.e. $|x - 3| = x - 3$.

Exercise

Exercise 6. In the real domain solve the equation:

$$|2x + 4| = 6.$$

Solution. First of all, we transform the equation

$$|2x + 4| = 6 \iff |2(x + 2)| = 6 \iff |2| \cdot |x + 2| = 6 \iff |x + 2| = 3.$$

We can solve the exercise using two different strategies:

- *First strategy:* The absolute value is distance, which means that we are looking for a set of real numbers such that their distance from number -2 is 3 . These are numbers -5 and 1 . Therefore, the solution of our equation is the set $\{-5, 1\}$.
- *Second strategy:* The expression $x + 2$ changes its sign at $x = -2$, which means that we will solve the task in intervals $(-\infty, -2)$ and $[-2, \infty)$:

– Suppose that $x \in (-\infty, -2)$. We remove the absolute value:

$$-x - 2 = 3$$

$$x = -5 \quad \wedge \quad -5 \in (-\infty, -2).$$

– Suppose that $x \in [-2, \infty)$. We remove the absolute value:

$$x + 2 = 3$$

$$x = 1 \quad \wedge \quad 1 \in [-2, \infty).$$

The solution is therefore $\{-5, 1\}$.

Exercise

Exercise 7. In the real domain solve the equation:

$$|x - 1| + |3 + x| = 3.$$

Solution. Expressions $(x - 1)$, $(3 + x)$ change signs at $x = 1$ and $x = -3$, respectively. These two points divide the real axis into intervals $(-\infty, -3)$, $[-3, 1)$ and $[1, \infty)$. Therefore we will solve the task in those intervals.

- Suppose that $x \in (-\infty, -3)$. We remove absolute values:

$$1 - x - x - 3 = 3$$

$$-2x - 2 = 3$$

$$-2x = 5$$

$$x = -\frac{5}{2} \quad \wedge \quad -\frac{5}{2} \notin (-\infty, -3).$$

Since we work in the interval $(-\infty, -3)$ and $-\frac{5}{2} \notin (-\infty, -3)$, the equation has no solution in $(-\infty, -3)$.

- Suppose that $x \in [-3, 1)$. We remove absolute values:

$$1 - x + x + 3 = 3$$

$$4 = 3.$$

The last equality obviously does not hold, which means that the equation has no solution in $[-3, 1)$ either.

- Suppose that $x \in [1, \infty)$. We remove absolute values:

$$x - 1 + x + 3 = 3$$

$$2x + 2 = 3$$

$$2x = 1$$

$$x = \frac{1}{2} \quad \wedge \quad \frac{1}{2} \notin [1, \infty).$$

We can see that the equation has no solution in $[1, \infty)$ either.

Since the equation has no solution in none of the intervals, we conclude that it has no solution.

Exercise

Exercise 8. In the real domain solve the inequation:

$$|x - |1 - x|| < 2.$$

Solution. First of all we remove the internal absolute value. The expression $1 - x$ changes its sign at $x = 1$, which means that we will solve the equation in intervals $(-\infty, 1)$ and $[1, \infty)$ respectively. This means that $|1 - x| = 1 - x$ and $|1 - x| = -(1 - x) = x - 1$, respectively.

- Suppose that $x \in (-\infty, 1)$. We remove the internal absolute value:

$$|x - (1 - x)| < 2$$

$$|x - 1 + x| < 2$$

$$|2x - 1| < 2$$

$$2\left|x - \frac{1}{2}\right| < 2$$

$$\left|x - \frac{1}{2}\right| < 1$$

The last equation means that we are looking for such real numbers that their distance from $x = \frac{1}{2}$ is less than 1. These are $x \in \left(-\frac{1}{2}, \frac{3}{2}\right)$. Since we work in interval $(-\infty, 1)$, we accept only $x \in \left(-\frac{1}{2}, \frac{3}{2}\right) \cap (-\infty, 1)$, which is equivalent with the fact that $x \in \left(-\frac{1}{2}, 1\right)$.

- Suppose that $x \in [1, \infty)$. We remove the internal absolute value:

$$|x - (x - 1)| < 2$$

$$|x - x + 1| < 2$$

$$|1| < 2$$

$$1 < 2$$

Since $1 < 2$ is a true proposition and all transformations have been equivalent ones, this is true for all $x \in \mathbb{R}$. Since in this part of the solution we worked in $[1, \infty)$, there is $x \in \mathbb{R} \cap [1, \infty)$, which means that $x \in [1, \infty)$.

To sum up, the solution of the inequality is $\left(-\frac{1}{2}, 1\right) \cup [1, \infty)$, i.e. the interval $\left(-\frac{1}{2}, \infty\right)$.

Quadratic equations

Quadratic equations are equations with one variable which is present in at most the second power. Their normal form is

$$ax^2 + bx + c = 0: a, b, c \in \mathbb{R}, a \neq 0.$$

The term ax^2 is called quadratic, the term bx is called linear, and c is the absolute term.

Special types of quadratic equations

- **Equations without absolute terms**

Such equations reduce to:

$$ax^2 + bx = 0,$$

which can be written as:

$$x(ax + b) = 0 \iff x = 0 \vee ax + b = 0 \iff x = 0 \vee x = -\frac{b}{a}.$$

- **Equations without linear terms**

Such equations reduce to:

$$ax^2 + c = 0,$$

which can be written as:

$$ax^2 = -c \iff x^2 = -\frac{c}{a}.$$

For $-\frac{c}{a} < 0$, the equation has no solution. For $-\frac{c}{a} = 0$, the equation has exactly one solution $x = 0$. Finally, for $-\frac{c}{a} > 0$, there is $|x| = \sqrt{-\frac{c}{a}}$, which means that the equation has two solutions.

Solving quadratic equations

- Generally speaking, we solve quadratic equations using the discriminant D , which equals

$$D = b^2 - 4ac.$$

Three cases are possible:

- for $D < 0$ the equation has no solution in the real domain
 - for $D = 0$ the equation has one real solution $x = -\frac{b}{2a}$
 - for $D > 0$ the equation has two mutually different solutions $x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}$.
- Quadratic equations can be solved also using Viete formulas, which make use of the fact that for the solutions of each quadratic equation there is:

$$x_1 + x_2 = -\frac{b}{a} \wedge x_1 \cdot x_2 = \frac{c}{a}.$$

Important relations and transformations

The following is a typical exercise with a quadratic equation:

$$x \cdot (x - 1) - 2x^2 + 2 = x^2 + x - 2.$$

Once again, we obtain the normal form of the quadratic equation using some transformations. These are the same transformations as in the case of linear equations. Moreover, we often make use the following formulas:

$$(a + b)^2 = a^2 + 2ab + b^2,$$

$$(a - b)^2 = a^2 - 2ab + b^2,$$

$$a^2 - b^2 = (a + b) \cdot (a - b).$$

The complete square transformation

Very often we use the transformation which turns the expression

$$ax^2 \pm bx + c$$

into

$$(x \pm m)^2 + n.$$

We show this transformation on the following example. We want to transform the quadratic expression $x^2 - 3x + 7$ into $(x - m)^2 + n$. We make use of the formula $(a - b)^2 = a^2 - 2ab + b^2$. As we can see, there is $2ab = 3x$, where $a = x$. As a result $b = \frac{3}{2}$. Obviously, there is

$$\left(x - \frac{3}{2}\right)^2 = x^2 - 3x + \frac{9}{4} = (x^2 - 3x) + \frac{9}{4},$$

which means that

$$x^2 - 3x = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4}.$$

As a result

$$x^2 - 3x + 7 = (x^2 - 3x) + 7 = \left(x - \frac{3}{2}\right)^2 - \frac{9}{4} + 7 = \left(x - \frac{3}{2}\right)^2 + \frac{19}{4}.$$

This form is extremely useful because we can immediately see that the expression $x^2 - 3x + 7$ attains its smallest value at $x = \frac{3}{2}$, and that its value at $x = \frac{3}{2}$ is $\frac{19}{4}$. This follows from the fact that for other values of x the bracket $\left(x - \frac{3}{2}\right)^2$ attains greater values than 0.

Exercise

Exercise 9. In the real domain solve the equation:

$$x \cdot (x - 1) - 2x^2 + 2 = x^2 + x - 2.$$

Solution.

$$x \cdot (x - 1) - 2x^2 + 2 = x^2 + x - 2$$

$$x \cdot (x - 1) - 2 \cdot (x^2 - 1) = (x - 1) \cdot (x + 2)$$

$$x \cdot (x - 1) - 2 \cdot (x - 1) \cdot (x + 1) = (x - 1) \cdot (x + 2)$$

$$(x - 1) \cdot (x - 2(x + 1)) = (x - 1) \cdot (x + 2)$$

$$(x - 1) \cdot (x - 2x - 2) = (x - 1) \cdot (x + 2)$$

$$(x - 1) \cdot (-x - 2) = (x - 1) \cdot (x + 2)$$

From this point on, two strategies can be followed to successfully solve the exercise. We will show both of them.

- Option 1:

$$\begin{aligned} (x - 1) \cdot (-x - 2) &= (x - 1) \cdot (x + 2) \\ (x - 1) \cdot (-x - 2) - (x - 1) \cdot (x + 2) &= 0 && \left. \begin{array}{l} \text{move everything from the right-hand side} \\ \text{to the left-hand side of the equation} \end{array} \right\} \\ (x - 1) \cdot (-x - 2 - x - 2) &= 0 && \left. \begin{array}{l} \text{take out } (x - 1) \end{array} \right\} \\ (x - 1) \cdot (-2x - 4) &= 0 && \left. \begin{array}{l} \text{simplify the second bracket} \end{array} \right\} \\ (-2) \cdot (x - 1) \cdot (x + 2) = 0 &\iff x = 1 \vee x = -2. \end{aligned}$$

The solution is $\{-2, 1\}$.

- Option 2: On condition that $x \neq 1$ we can divide both sides of the equation by $(x - 1)$

$$\begin{aligned} \cancel{(x - 1)} \cdot (-x - 2) &= \cancel{(x - 1)} \cdot (x + 2) \\ -x - 2 &= x + 2 \\ -4 &= 2x \\ -2 &= x. \end{aligned}$$

For $x = 1$, i.e. the value we have ignored so far, we get:

$$\begin{aligned} (1 - 1) \cdot (-1 - 2) &= (1 - 1) \cdot (1 + 2) \\ 0 \cdot (-3) &= 0 \cdot 3, \text{ which is true }^* \end{aligned}$$

The solution is $\{-2, 1\}$.

* What would be the conclusion if we got e.g. $0 \cdot 2 = 1 \cdot (-2)$? In other words, what would be our conclusion if the equality did not hold? In such a case $x = 1$ would not be a solution of the equation.

Quadratic inequations

The normal form of quadratic inequations is:

$$ax^2 + bx + c > 0: a, b, c \in \mathbb{R}, a \neq 0.$$

Alternatively, other inequality signs, such as $<$, $\leq$, $\geq$ are used instead of $>$. Solutions of quadratic inequations can have the form of the emptyset (i.e. no real number fits the inequation), finite set or an interval of real numbers. Once again, in order to obtain the normal form, one or more transformations have to be applied.

Transformations of quadratic inequations

We give several important rules, which help us during the process of solution of quadratic inequations. At some stage during the process of solution we often obtain an inequation of the following type: $x^2 > d$: $d \geq 0$, (or $x^2 < d$: $d \geq 0$). This can be solved in the following way:

$$x^2 > d: d \geq 0 \iff |x| > \sqrt{d} \iff x \in (-\infty, -\sqrt{d}) \cup (\sqrt{d}, \infty),$$

$$(x^2 < d: d \geq 0 \iff |x| < \sqrt{d} \iff x \in (-\sqrt{d}, \sqrt{d})).$$

Alternatively, we may factorize the expression and tell the solution depending on the signs of respective terms:

$$x^2 > d: d \geq 0 \iff x^2 - d > 0 \iff (x - \sqrt{d}) \cdot (x + \sqrt{d}) > 0 \iff x \in (-\infty, -\sqrt{d}) \cup (\sqrt{d}, \infty),$$

$$(x^2 < d: d \geq 0 \iff x^2 - d < 0 \iff (x - \sqrt{d}) \cdot (x + \sqrt{d}) < 0 \iff x \in (-\sqrt{d}, \sqrt{d})).$$

If we end up with the inequation $x^2 > d$: $d < 0$, all real numbers fit it. If we end up with the inequation $x^2 < d$: $d < 0$, no real numbers fit it.

Transformations of quadratic inequations

Multiplication by a negative number is again tricky (recall linear inequations). In the case of quadratic inequations the complications are even greater. For example, in the inequation

$$x(x - 1) < x(2x + 3)$$

students often do the following mistake:

$$\cancel{x}(x - 1) < \cancel{x}(2x + 3)$$

and continue looking for a solution of a linear inequation, the result of which is $x \in (-4, \infty)$. This is a mistake because we have no idea about the value of x . It can be zero, it can be negative. Therefore, the correct way of solving the exercise is as follows:

$$\begin{aligned} x(x - 1) &< x(2x + 3) \\ x(x - 1) - x(2x + 3) &< 0 && \left. \begin{array}{l} \text{move everything from the right-hand side to the left-hand side} \\ \text{take out } x \end{array} \right\} \\ x(x - 1 - 2x - 3) &< 0 \\ x(-x - 4) &< 0 \\ -x(x + 4) &< 0 \\ x(x + 4) &> 0 && \left. \begin{array}{l} \text{multiply by } -1 \\ \text{the inequality sign changes} \end{array} \right\} \\ x &\in (-\infty, -4) \cup (0, \infty). \end{aligned}$$

Exercise

Exercise 10. In the real domain solve the inequation:

$$x^2 - 9 > 0.$$

Solution. We can look for the solution using two strategies. We will show both of them.

- Option 1:

$$x^2 - 9 > 0$$

$$(x - 3) \cdot (x + 3) > 0$$

The expression $x - 3$ is equal to zero for $x = 3$ while the expression $x + 3$ is equal to zero for $x = -3$. These numbers divide the real axis into intervals $(-\infty, -3)$, $(-3, 3)$, $(3, \infty)$. Since the left-hand side is zero for $x = 3$ and $x = -3$ and since the inequality is strict, we ignored these values. In order to find the solution we construct the following table:

	$(-\infty, -3)$	$(-3, 3)$	$(3, \infty)$
$x - 3$	−	−	+
$x + 3$	−	+	+
$(x - 3)(x + 3)$	+	−	+

The signs for expressions $x + 3$ and $x - 3$ can be determined easily by substituting an arbitrary value from the given interval. The resulting signs of the expression $(x - 3)(x + 3)$ follow from the properties of the product. Therefore, there is $x \in (-\infty, -3) \cup (3, \infty)$.

- Option 2:

$$x^2 - 9 > 0$$

$$x^2 > 9$$

$$|x| > 3$$

$$x \in (-\infty, -3) \cup (3, \infty).$$

Mind a common mistake:

$$x^2 - 9 > 0$$

$$x^2 > 9$$

$$x > 3$$

) *mistake*

$x \in (3, \infty)$, which is wrong because we ignore the fact that $(-3)^2$ is 9 as well.

Exercise

Exercise 11. In the real domain solve the inequation:

$$(x + 3) \cdot (x - 2) > (x - 2) \cdot (3x - 5).$$

Solution. First of all we show a wrong way of solving the task:

$$(x + 3) \cdot \cancel{(x - 2)} > \cancel{(x - 2)} \cdot (3x - 5) \quad \text{mistake}^*$$

$$x + 3 > 3x - 5$$

$$8 > 2x$$

$$4 > x$$

$$x \in (-\infty, 4).$$

* Since the expression $x - 2$ can be equal to zero or negative, such a division is not an equivalent transformation.

The correct way of solving the task:

$$(x + 3) \cdot (x - 2) > (x - 2) \cdot (3x - 5)$$

$$(x + 3) \cdot (x - 2) - (x - 2) \cdot (3x - 5) > 0$$

$$(x - 2) \cdot (x + 3 - (3x - 5)) > 0$$

$$(x - 2) \cdot (8 - 2x) > 0$$

$$2 \cdot (x - 2) \cdot (4 - x) > 0$$

$$x \in (2, 4).$$

Exercise

Exercise 12. *In the real domain solve the inequation:*

$$x^2 - 5x + 7 > 0.$$

Solution. *We make use of the complete square transformation and rewrite the left-hand side of the inequation:*

$$x^2 - 5x + 7 > 0$$

$$\left(x - \frac{5}{2}\right)^2 - \left(\frac{5}{2}\right)^2 + 7 > 0$$

$$\left(x - \frac{5}{2}\right)^2 - \frac{25}{4} + \frac{28}{4} > 0$$

$$\left(x - \frac{5}{2}\right)^2 + \frac{3}{4} > 0$$

Since the value of the expression on the left-hand side of the inequation is at least $\frac{3}{4}$, any real number fits it. Therefore, $x \in \mathbb{R}$.

TEST

1. The solution of the equation $|x - 1| = 2$ is
 - (a) $\{-1, 3\}$
 - (b) $(-1, 3)$
 - (c) $[-1, 3)$
 - (d) $\emptyset$
2. The solution of the inequation $|x - 2| > 2$ is
 - (a) $\{0, 4\}$
 - (b) $(0, 4)$
 - (c) $(-\infty, 0) \cup (4, \infty)$
 - (d) $\{0, 1, 2, 3, 4\}$
3. The solution of the equation $x^2 - 2 = 2$ is
 - (a) $\{2\}$
 - (b) $\{-2, 2\}$
 - (c) $\{0\}$
 - (d) $\emptyset$
4. The solution of the inequation $x^2 - 2 < 2$ is
 - (a) $(-\infty, 2)$
 - (b) $(-2, 2)$
 - (c) $(-\infty, 2) \cup (2, \infty)$
 - (d) $\emptyset$
5. The solution of the equation $x^2 - 2x = 15$ is
 - (a) $(-3, 5)$
 - (b) $\{-3, 5\}$
 - (c) $\{-5, 3\}$
 - (d) $[-5, 3]$
6. The solution of the equation $|x - 1| + |5 - x| = 10$ is
 - (a) $(-2, 8)$
 - (b) $[-2, 8]$
 - (c) $\{-2, 3, 8\}$
 - (d) $\{-2, 8\}$

7. The solution of the equation $|1 - |x - 3|| = 6$ is

- (a) $\{-4, 10\}$
- (b) $\{-4, -2, 10\}$
- (c) $\{-4, -2, 8, 10\}$
- (d) $\{-4, 8, 10\}$

8. The solution of the inequation $3x - x^2 \leq 2$ is

- (a) $(-\infty, 1] \cup [2, \infty)$
- (b) $[1, 2]$
- (c) $[-2, -1]$
- (d) $(-\infty, -2] \cup [-1, \infty)$

9. The solution of the inequation $|1 - x| - |2 + x| \geq 2$ is

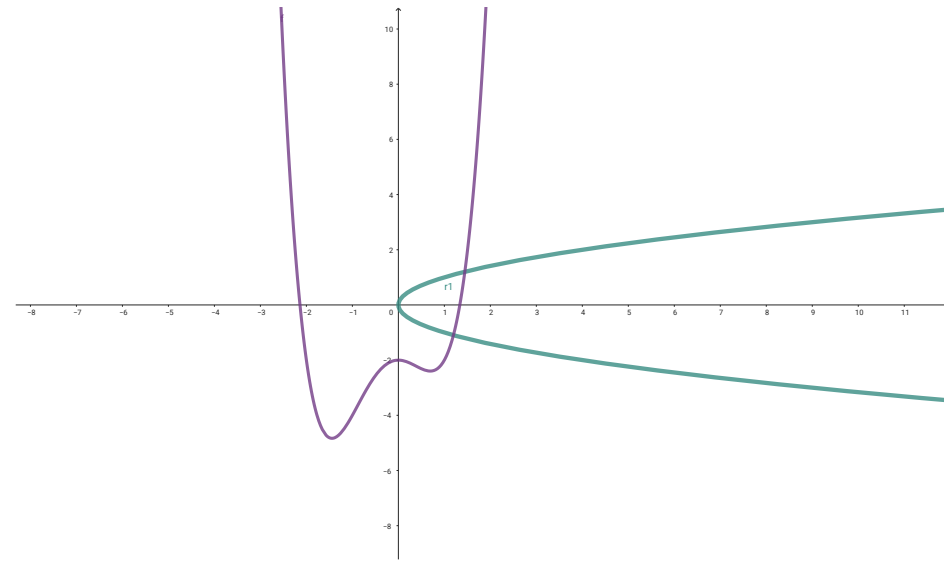
- (a) $(-\infty, -\frac{3}{2})$
- (b) $(-\infty, -2]$
- (c) $(-\infty, -\frac{3}{2}]$
- (d) $\mathbb{R}$

10. The solution of the inequation $|x - 2||x + 1| \geq 3$ is

- (a) $(-\infty, -1] \cup [1, \infty)$
- (b) $\emptyset$
- (c) $(-\infty, -\frac{5}{3}] \cup [1, \infty)$
- (d) $\mathbb{R}$

3. POLYNOMIAL FUNCTION

Funkce



Function

What exactly is a function? We often use this word when talking about various machines or gadgets. For example, when we press **A** on a keyboard, we expect that the letter “a” appears on the screen. Or, when we press **SHIFT** and **A** at the same time, we expect that “A” appears. In other words, by a “function-like” we mean something that generates the same result for each unique input. Mathematically, this idea can be written as follows:

By a function on the set $D \subseteq \mathbb{R}$ we mean a formula that, to each real number, assigns another (yet unique) real number.

We usually denote functions by letter f or letters following f such as $g, h, \dots$.

The argument of the function (or the function input) is usually denoted by x .

The function value (or the function output) is usually denoted by $f(x)$ or y .

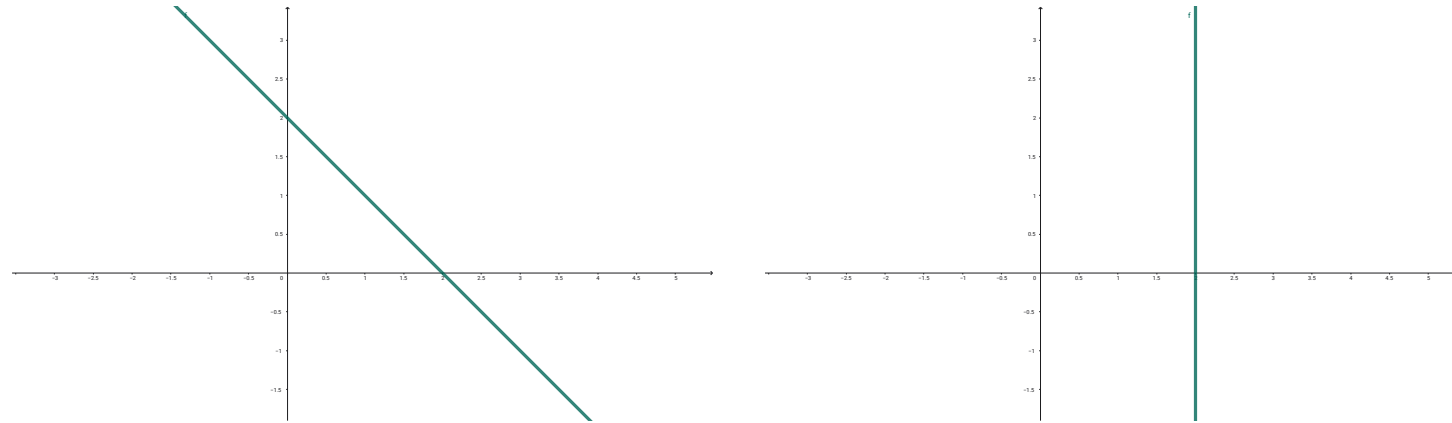
The domain of definition is such a set of real numbers that can be substituted into the function formula. If f is the function, the domain of definition is denoted by $D(f)$.

The range of values is such a set of all real numbers y that we obtain as the output of f provided we evaluate all x from the domain of definition. If f is the function, the range of values is denoted by $H(f)$.

We say that functions f and g are equal, if their domains of definition are equal (i.e. $D(f) = D(g)$) and at the same time if for all $x \in D(f)$ there is $f(x) = g(x)$, i.e. if the function values of both functions are the same at all points.

Graph of the function

By the graph of the function we mean a curve that graphically describes the behaviour of the function. The graph is a curve positioned in the coordinate system in such a way that the horizontal axis x displays the input values while the vertical axis y displays the output values. By the graph of the function f we mean the set of points $[x, f(x)]$. Since to each $x \in D(f)$ there exists a unique output $f(x)$, the graph may not have two or more different values at any point $x \in D(f)$. In the figure below two curves are depicted. On the left we can see a graph of a function, while what we see on the right is not a graph of any function.



Properties of functions

Monotonicity and strict monotonicity:

We say that a function is increasing if the increase of the input values x results in the increase of the output values $f(x)$. We write this formally as:

The function is increasing on M , if $\forall x_1, x_2 \in M; x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$.

We say that a function is decreasing if the increase of the input values x results in the decrease of the output values $f(x)$. We write this formally as:

The function is decreasing on M , if $\forall x_1, x_2 \in M; x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$.

We say that a function is **strictly monotonous** if it is increasing or decreasing everywhere on its domain of definition.

We say that a function is non-increasing if the increase of input values x results in the decrease of the output values $f(x)$ or if the output values remain the same. We write this formally as:

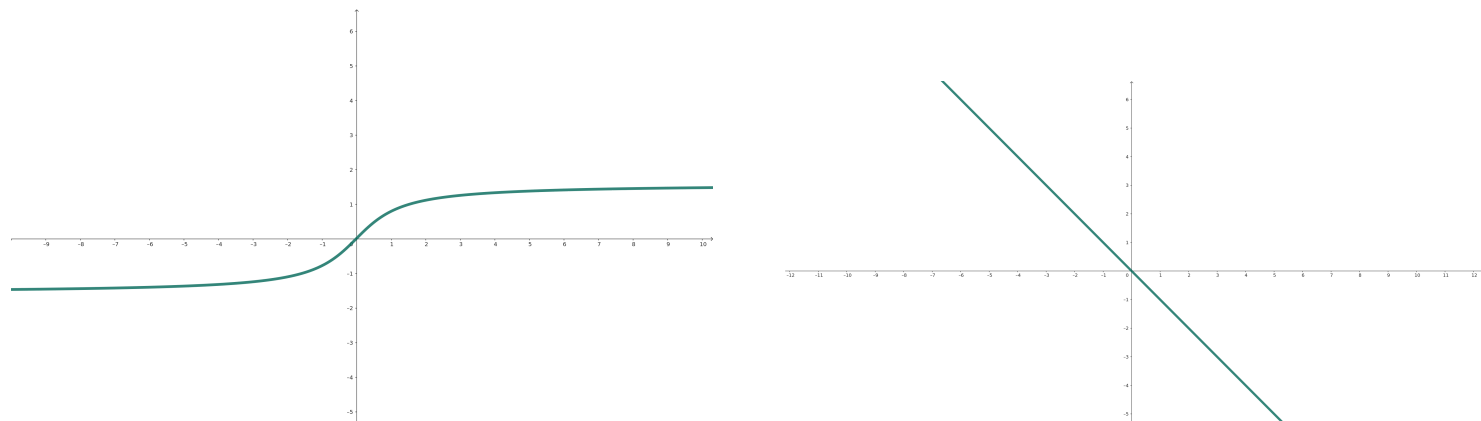
The function is non-increasing on M , if $\forall x_1, x_2 \in M; x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$.

We say that a function is non-decreasing if the increase of input values x results in the increase of the output values $f(x)$ or if the output values remain the same. We write this formally as:

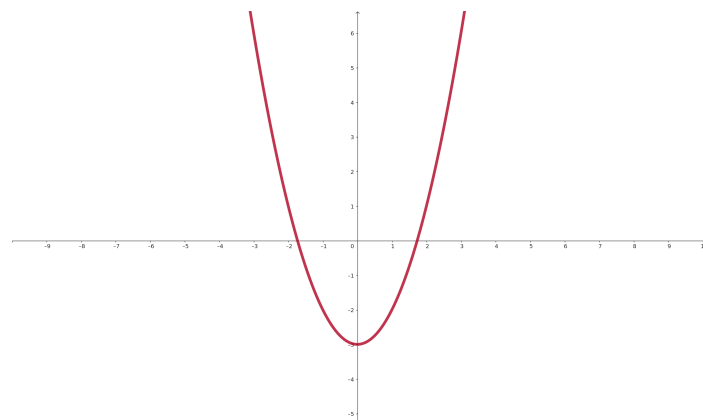
The function is non-decreasing on M , if $\forall x_1, x_2 \in M; x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$.

We say that a function is **monotonous** if it is non-increasing or non-decreasing everywhere on its domain of definition.

The function in the left figure is an increasing one while the function in the figure on the right is a decreasing one.



The function in the figure below is neither increasing nor decreasing – on condition that we regard the set of all real numbers. However, if we restrict ourselves on the input values from e.g. $M = [1, 2]$, the function is increasing on M . Analogously, it is decreasing in the interval $[-1, 0]$.



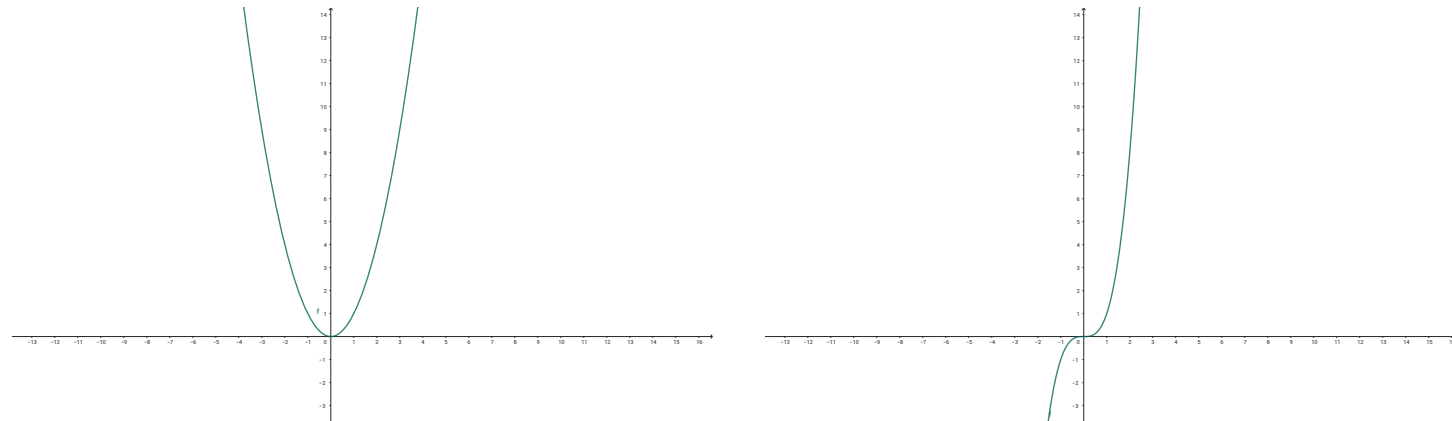
Properties of functions

Injective functions:

When there is

$\forall a, b \in D(f); a \neq b \Rightarrow f(a) \neq f(b)$, i.e. when different input values map to different output values,

for some function $f(x)$, we say that this function is injective. The function on the left is not injective because its values at $x = -1$ and $x = 1$ are the same. The function on the right is injective. If we draw lines parallel to the x axis, each of them will intersect the graph of the function not more than once.



Remark 1. Since the implication $A \Rightarrow B$ is equivalent with the implication $\neg B \Rightarrow \neg A$, the definition of an injective function can be written in an equivalent way as:

$$\forall a, b \in D(f); f(a) = f(b) \Rightarrow a = b.$$

We will make use of this when deciding whether a given function is injective.

Properties of functions

Parity:

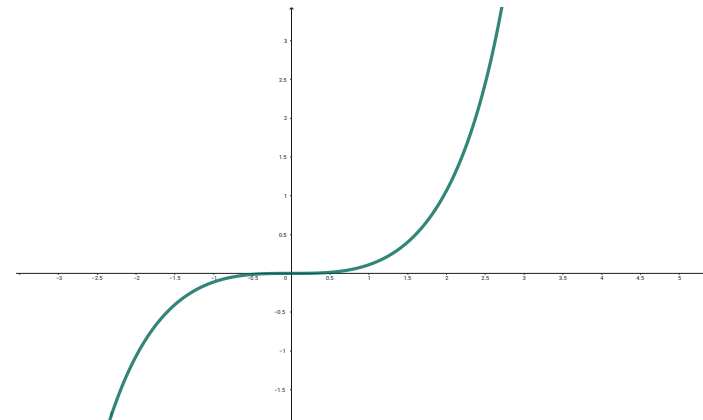
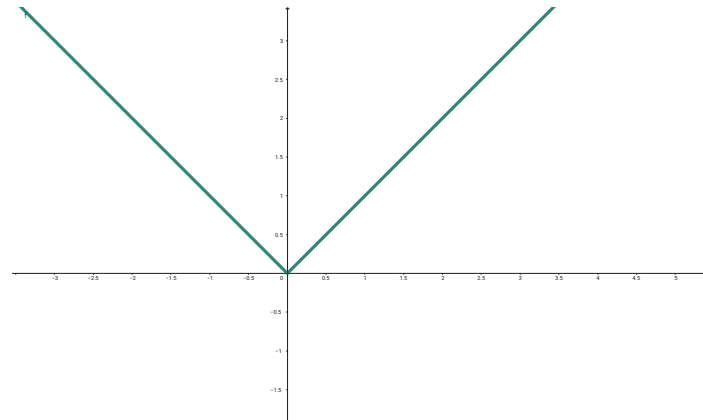
Whether a function is even or odd can be told by looking at the graph – naturally, on conditions that we are able to draw it and that the graph is of some use. If the graph is symmetric with respect to the axis y , the function is even. We write this formally as:

$$\text{The function is even if } \forall x \in D_f; f(-x) = f(x).$$

If the graph is symmetric with respect to origin of the coordinate system, the function is odd. We write this formally as:

$$\text{The function is odd if } \forall x \in D_f; f(-x) = -f(x).$$

On the left there is an even function while the function on the right is odd.



Whenever we want to decide the parity of functions defined by complicated formulas such as e.g. $f(x) = x^7 + 2x^3 - x^2 + 4$, we cannot rely on graphs and have to use the above formal formulas.

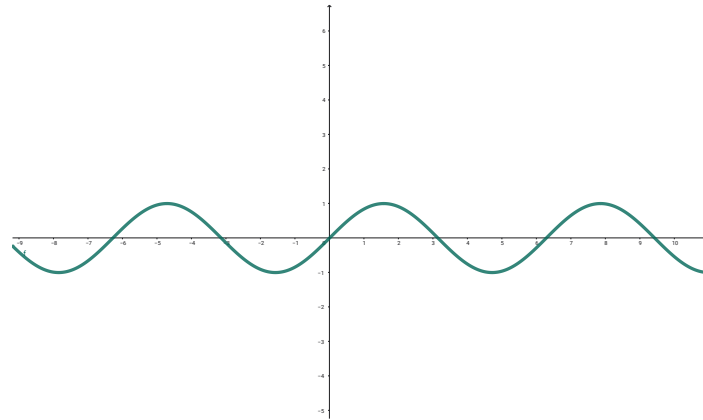
Properties of functions

Periodicity:

Periodicity of a function can be told from the graph. If the whole graph is given by its part which keeps repeating itself, it is a graph of a periodic function. We write this formally as:

$$\exists p \in \mathbb{R} \setminus \{0\} \forall x \in D_f: f(x \pm p) = f(x).$$

The figure below depicts an example of a periodic function.



Why do we have to consider $p \neq 0$?

Properties of functions

Boundedness:

The fact that a function is bounded from above means that its function values do not exceed a certain limit. The fact that it is bounded from below means that its function values do not fall below a certain limit. When a function is bounded both from above and from below, we say that it is bounded. We write this formally as:

- A function f is bounded from above if

$$\exists c \in \mathbb{R} \forall x \in D(f): f(x) \leq c.$$

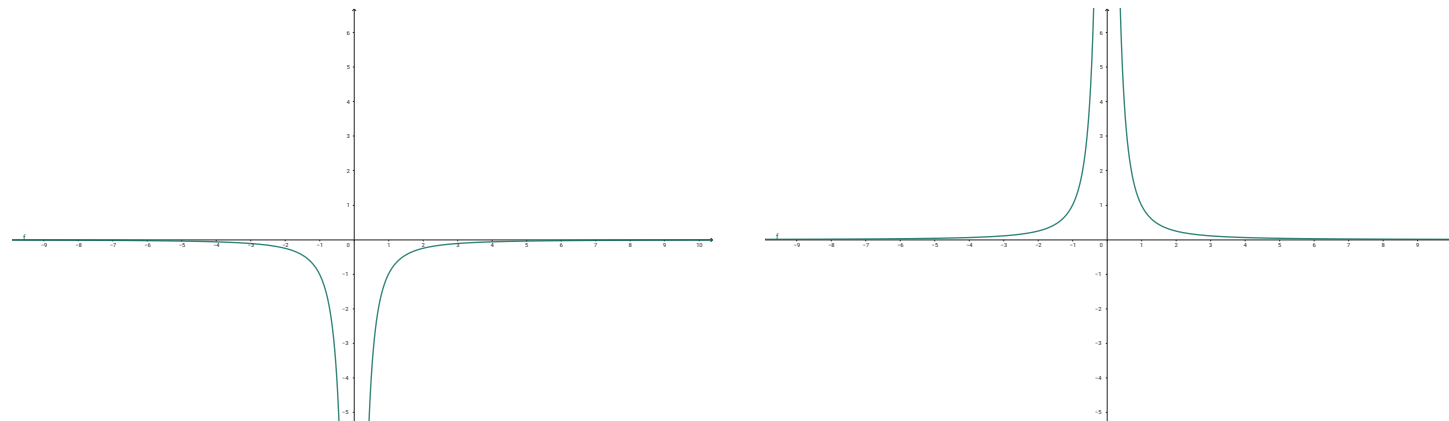
- A function f is bounded from below if

$$\exists d \in \mathbb{R} \forall x \in D(f): f(x) \geq d.$$

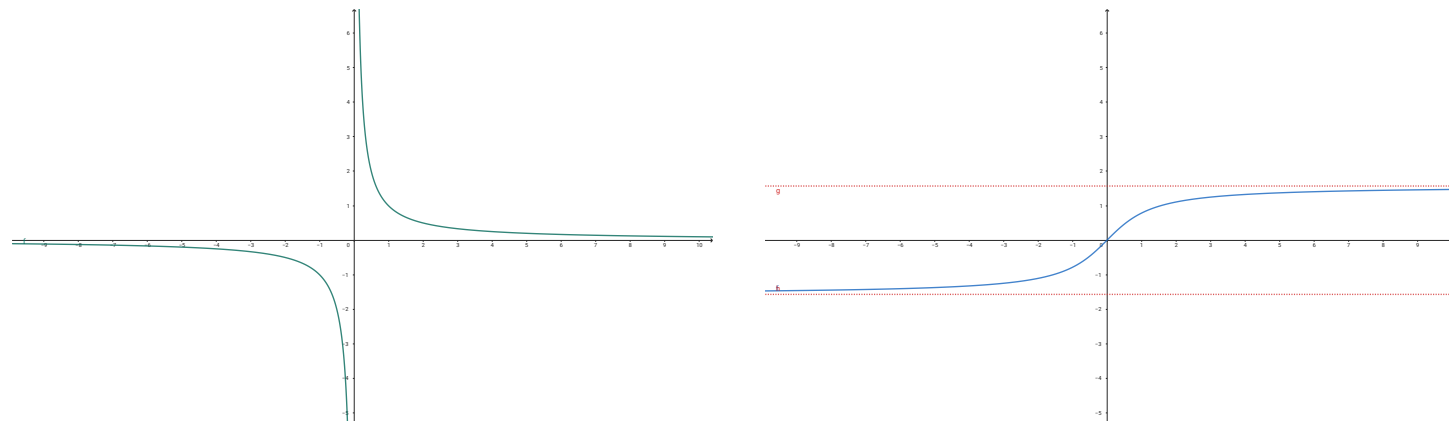
- A function f is bounded if

$$\exists c, d \in \mathbb{R} \forall x \in D(f): d \leq f(x) \leq c.$$

The function on the left is bounded from above because all its function values are greater than zero. The function on the right is bounded from below because all its function values are greater than zero.



Below, the function on the left is bounded neither from above nor from below while the function on the right is bounded from both above and below, i.e. it is a bounded function.



Linear functions

Linear functions are functions $f: \mathbb{R} \rightarrow \mathbb{R}$ which are defined by $f: y = ax + b$, where $a, b \in \mathbb{R}$. Graphs of linear functions are lines.

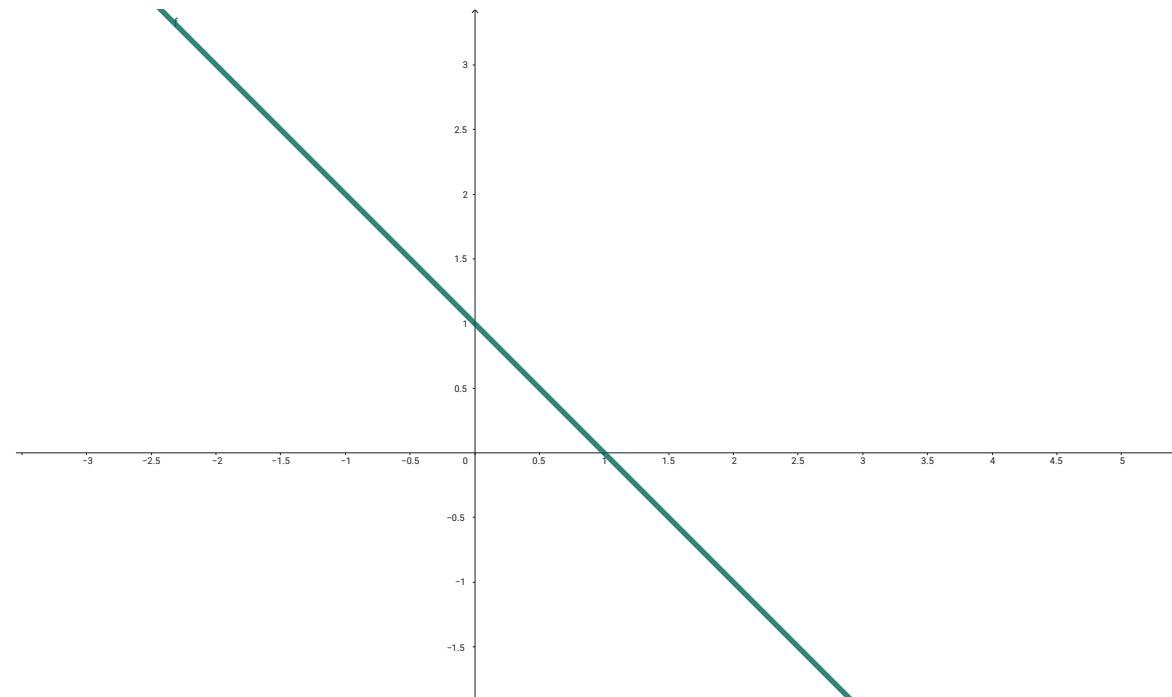
Basic properties of linear functions depending on the values of coefficients a, b :

- For $a = 0$ the function f reduces to $y = b$. Such functions are constant ones.
- For $a > 0$ the function f is increasing.
- For $a < 0$ the function f is decreasing.
- For $b = 0$ the function f passes through the origin of the coordinate system.

Exercise

Exercise 2. Sketch the graph of $f(x) = 1 - x$ and find out whether it is increasing or decreasing.

Solution. The graph of f is a line. Therefore, it is sufficient to find two points on it. The function value for $x = 0$ is $f(0) = 1$ while for $x = 1$ the function value is $f(x) = 1 - x = 0$. Therefore, the function passes through points $A = [1, 0]$ and $B = [0, 1]$ and its graph is as follows:



The function is a decreasing one, which can be told either from its graph or from the fact that the coefficient a is negative.

Inverse functions

When talking about function inverses, we obviously need to have the original function f . The sufficient condition for the existence of the function inverse is that the original function is injective. The original function $f(x)$ maps the elements of $D(f)$ into the set $H(f)$ while its inverse f^{-1} maps the elements of $H(f)$ into $D(f)$. This can be graphically depicted in such a way that we take the original function and draw its image symmetric with respect to the line $y = x$. In order to obtain the definition of the function inverse, we express x as a function of y . The inverse of a linear function is again a linear function.

Exercise

Exercise 3. Find the inverse of $f(x) = 3x - 4$.

Solution. The function f is linear, which means that it is injective. As a result, looking for the inverse makes sense. There is

$$y = 3x - 4.$$

Therefore, for its inverse f^{-1} , there must be:

$$x = 3y - 4,$$

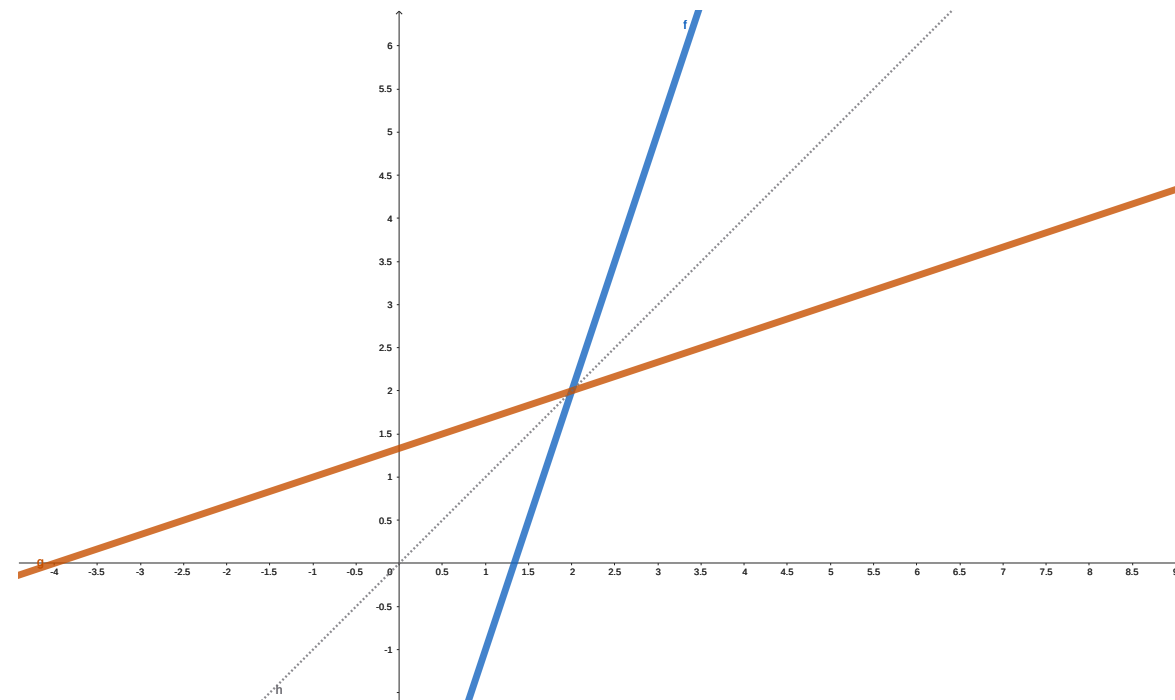
i.e.

$$x + 4 = 3y.$$

Therefore,

$$y = \frac{1}{3}x + \frac{4}{3}.$$

Both functions are depicted below: the original function f is highlighted in blue, the orange line is its inverse f^{-1} . Their symmetry axis is represented by the dotted line. The domain of definitions as well as ranges of values of both functions are in all cases the set of all real numbers.

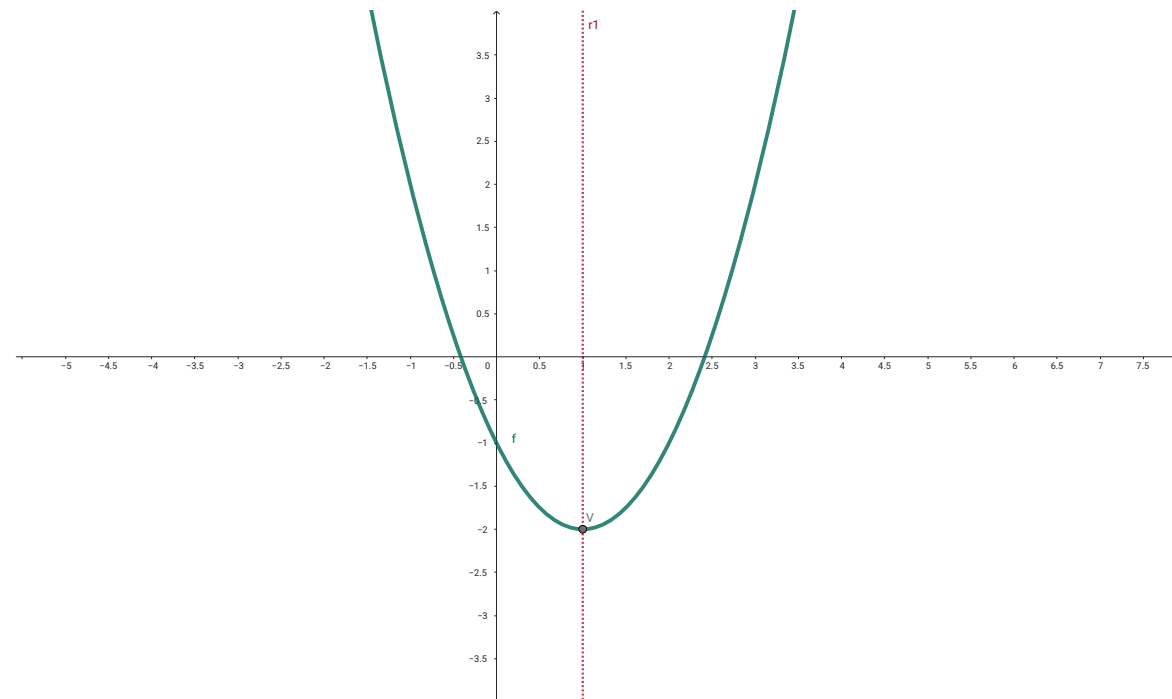


Quadratic functions

Quadratic functions are functions $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f: y = ax^2 + bx + c: a, b, c \in \mathbb{R}, a \neq 0$. Their graphs are parabolas.

Parabola

Graphs of quadratic functions are parabolas. One such curve can be seen in the figure. Parabola is an axially symmetric curve, its axis of symmetry is parallel to the y axis of the coordinate system. The axis of symmetry of a parabola intersects the y axis in a special point V called vertex.



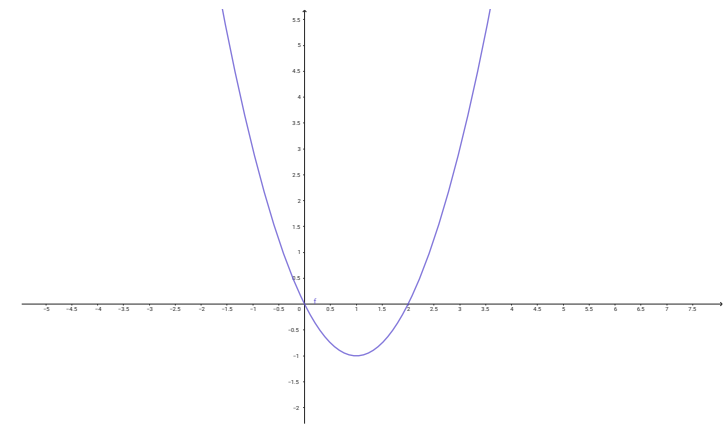
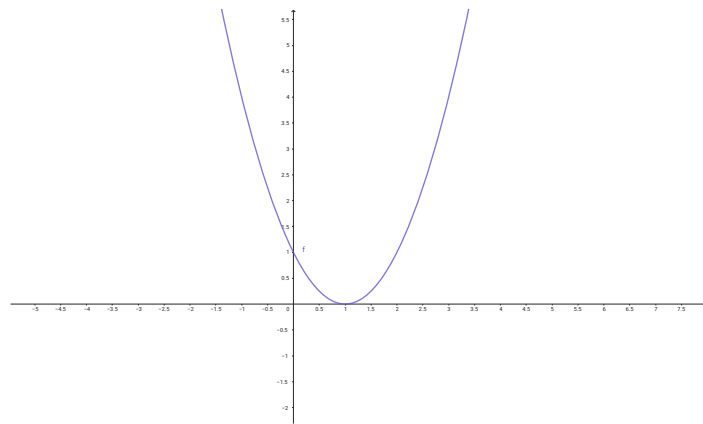
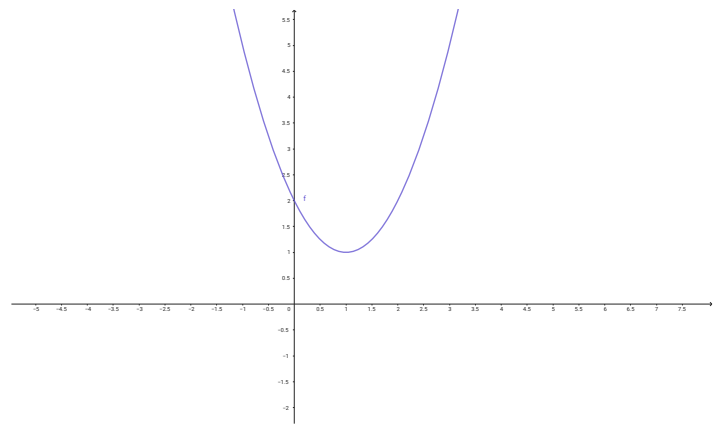
Properties of quadratic functions

When examining properties of quadratic functions, we apply our knowledge regarding the solutions of quadratic equations. For each quadratic function, the domain of definition is the whole set R . The exact course of the function depends on the parameter a :

- **Case $a > 0$:**

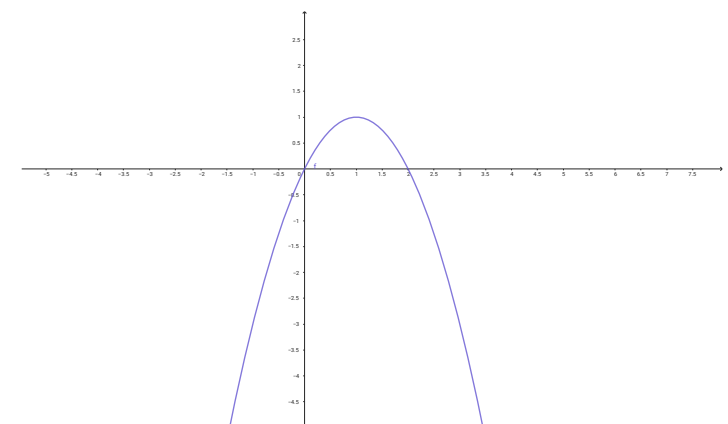
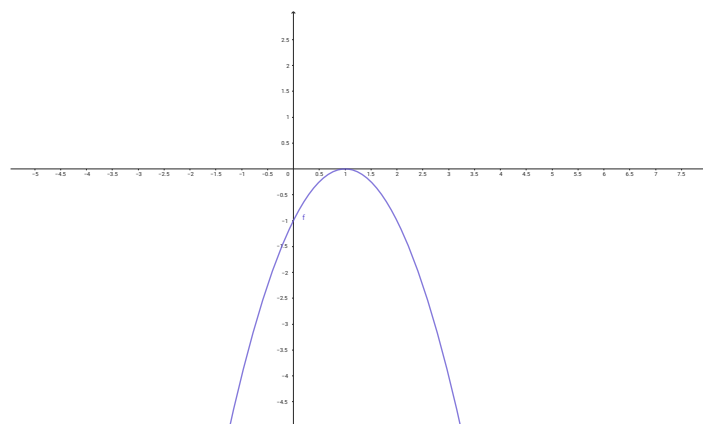
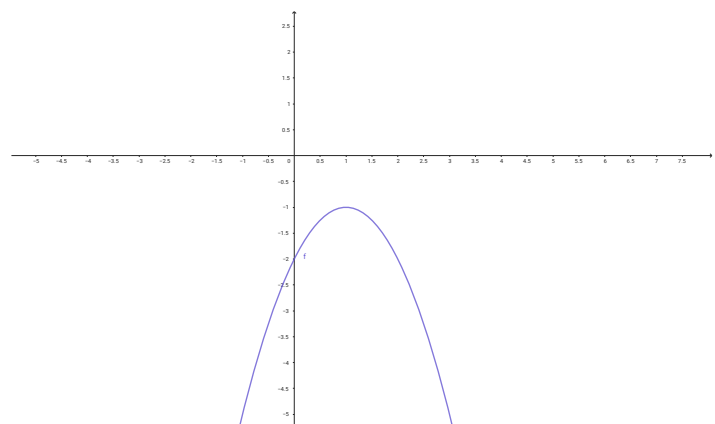
In this case the parabola looks upwards, it is bounded below and intersects the x -axis at most twice.

The number of intersections with the x -axis depends on the number of solutions of the quadratic equation $ax^2 + bx + c = 0$. If x_i is the solution of that equation, there is $f(x_i) = 0$, which means that x_i is either the intersection of the parabola and the x -axis, or it is a point in which the parabola touches the axis. In other words, the number of intersections is linked to the value of the discriminant. For $D < 0$ the function has no intersection with the x -axis, for $D = 0$ it touches it and for $D > 0$ it intersects it at two points.



- **Case $a < 0$:**

In this case the parabola looks downwards, it is bounded above and intersects the x -axis at most twice. The number of intersections can be determined in the same way as in the case of $a > 0$.



Vertex of the parabola

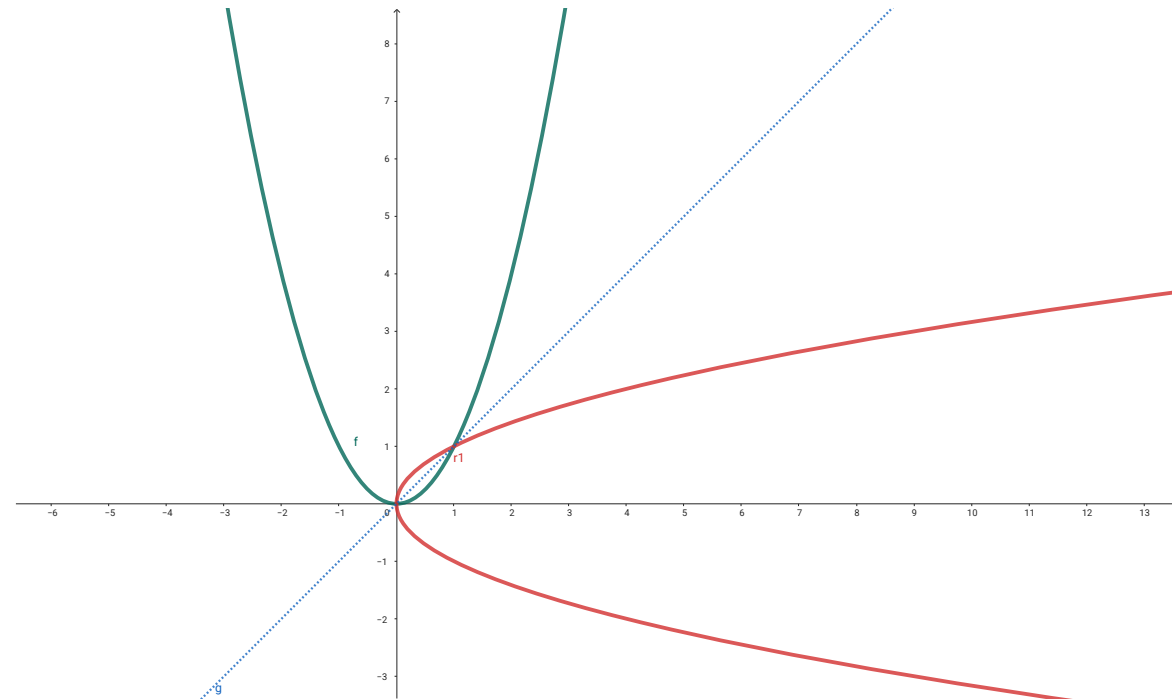
As the following example suggests, finding the coordinates of the vertex is based on the complete square transformation. Suppose that the function f is defined by $y = x^2 - 5x + 7$. We transform the quadratic trinomial $x^2 - 5x + 7$ to a complete square:

$$x^2 - 7x + 13 = \left(x - \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 13 = \left(x - \frac{7}{2}\right)^2 + \frac{3}{4}.$$

The function has its minimum at $x = \frac{7}{2}$; its value is exactly $f\left(\frac{7}{2}\right) = \frac{3}{4}$. As a result, the vertex of the parabola is $\left[\frac{7}{2}, \frac{3}{4}\right]$.

Inverse functions again

Why do only injective functions have inverses? The answer can be seen in the figure. The green color shows the graph of the function $f(x) = x^2$. Obviously there is $f(1) = f(-1) = 1$. Therefore, f is not an injective function. If its inverse existed, there would have to be $f^{-1}(1) = 1$ and at the same time also $f^{-1}(1) = -1$, which is not possible because f^{-1} must be a function. The red curve in the figure is a parabola symmetric to the function $f(x) = x^2$ along the line $y = x$. However, this curve is not a function.



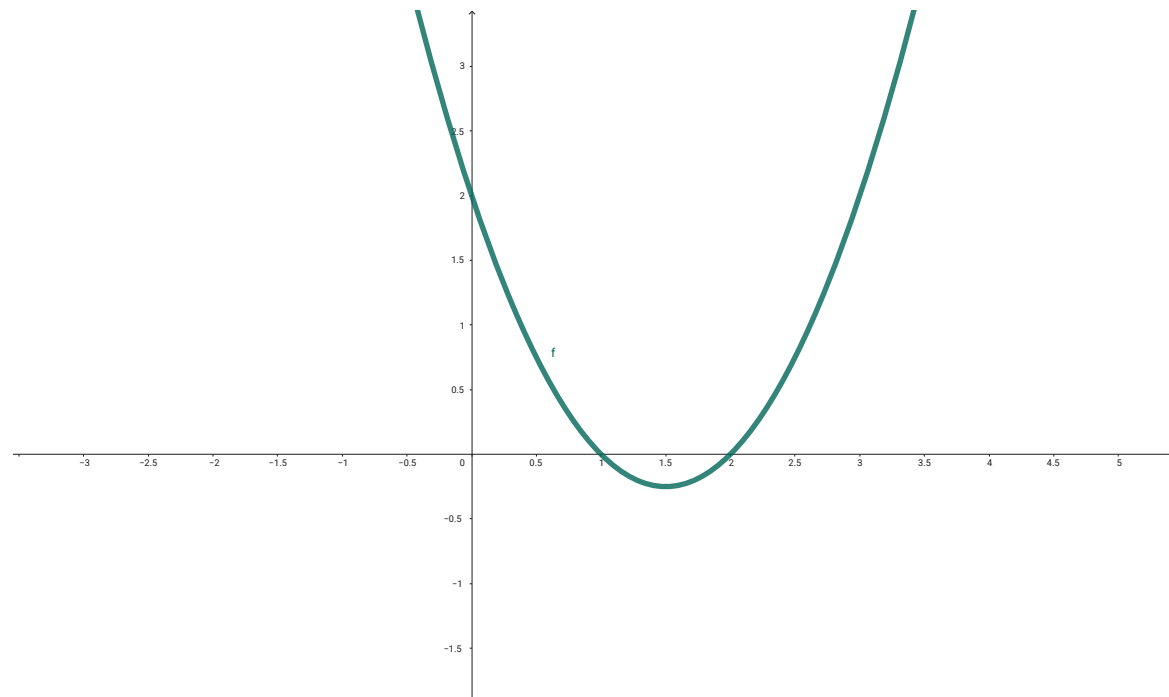
Exercise

Exercise 4. Find intervals in which the function $f(x) = x^2 - 3x + 2$ is increasing / decreasing.

Solution. The function f is quadratic, which means that its graph has the shape of a parabola. We either factorize the trinomial

$$x^2 - 3x + 2 = (x - 1) \cdot (x - 2)$$

using Viète formulas or we solve the equation $x^2 - 3x + 2 = 0$ and we write it as $(x - x_1) \cdot (x - x_2) = 0$, where x_1, x_2 are the solutions of the equation. The graph of the function f intersects the x -axis at points $x_1 = 1$ and $x_2 = 2$. Since the coefficient at x^2 is positive, we know how to sketch the graph:



In order to find the respective intervals we need to find the x -coordinate of the vertex $V = [x_v, y_v]$ of this parabola. It is useful to recall that parabolas are symmetric along the line $x = x_v$. As a result, x_v is exactly in the middle between the two intersections with the x -axis. Therefore, $x_v = \frac{3}{2}$. The function is decreasing in $(-\infty, \frac{3}{2})$ and increasing in $(\frac{3}{2}, \infty)$.

Exercise

Exercise 5. Find the coordinates of the vertex of the parabola $y = x^2 - 4x + 5$.

Solution. We transform the quadratic trinomial $x^2 - 4x + 5$ to a complete square in the following way::

$$\begin{aligned}x^2 - 4x + 5 &= x^2 - 2 \cdot 2 \cdot x + 2^2 + 1 \\&= (x^2 - 4x + 4) + 1 \\&= (x - 2)^2 + 1\end{aligned}$$

Now we can see that the function has its smallest value at $x = 2$. This value is $(2 - 2)^2 + 1 = 1$, which means that the coordinates of the vertex V are $[2, 1]$.

Polynomial functions

Polynomial functions, or polynomials are functions $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f: y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_1 x + a_0$, where $a_i \in \mathbb{R}, a_n \neq 0$. The number n is called degree. Linear and quadratic functions are special cases of polynomials.

If the function value of a polynomial at x_0 equals zero, i.e. if there is

$$a_n x_0^n + a_{n-1} x_0^{n-1} + \dots + a_1 x_0 + a_0 = 0,$$

we call x_0 the root of the polynomial. The geometrical interpretation of polynomial roots is simple. Graphs of polynomials either intersect or touch the x -axis at points which are polynomial roots. Each polynomial of degree n can have at most n roots.

If x_i is a root of the polynomial $p(x)$ of degree $n \geq 1$, then there is

$$p(x) = (x - x_i) \cdot g(x),$$

where $g(x)$ is a polynomial of degree $n - 1$. Polynomials $p(x)$ of degree $n \geq 1$ can be written as

$$p(x) = a_n (x - x_1)(x - x_2) \dots (x - x_n),$$

where x_i are the polynomial roots and terms $(x - x_i)$ are called factors. If some factors appear in the factorization of some polynomial more than once, we call the respective roots multiple roots.

Rational polynomial roots

Finding rational roots of polynomials with integral coefficients is rather simple. We apply the following theorem:

If a rational number $\frac{p}{q}$ is a root of

$$a_n x^n + a_{n-1} x^{n-1} + \cdots a_1 x + a_0,$$

then a_0 is a integral multiple of p and a_n is a natural multiple of q .

Remark 6. Recall that each rational number can be written in the form of a fraction $\frac{p}{q}$, where p is an integer, q is a natural number and p, q are coprime.

Horner scheme

Horner scheme is a nice tool that can help us find roots of polynomials of higher degrees. The polynomial of degree 2 can be written as:

$$a_2x^2 + a_1x + a_0 = (a_2x + a_1) \cdot x + a_0,$$

the polynomial of degree 3 can be written as:

$$a_3x^3 + a_2x^2 + a_1x + a_0 = (a_3x^2 + a_2x + a_1) \cdot x + a_0 = ((a_3x + a_2) \cdot x + a_1) \cdot x + a_0$$

and, in general, the polynomial of degree n can be written as:

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0 = \left(\left(\cdots ((a_nx + a_{n-1})x + a_{n-2})x + \cdots + a_2 \right)x + a_1 \right)x + a_0.$$

This form suggests a manual to compute function values of polynomials. Suppose we have a polynomial $p_3(x) = x^3 - 2x^2 + 3x - 7$. We write it as

$$x^3 - 2x^2 + 3x - 7 = ((x - 2)x + 3)x - 7.$$

If we now want to evaluate $p_3(3)$, we can substitute 3 into the original and get: $3^3 - 2 \cdot 3^2 + 3 \cdot 3 - 7 = 27 - 18 + 9 - 7 = 11$. Alternatively, we can substitute 3 into the new form. The computations will be as follows:

1. We substitute $x = 3$ into the internal brackets $(x - 2)$ and get $3 - 2 = 1$.
2. We take this result, multiply it by $x = 3$ and get $1 \cdot 3 = 3$.
3. We add 3 to the result 3 and get $3 + 3 = 6$.
4. We multiply the result by $x = 3$ and get $6 \cdot 3 = 18$.
5. Finally, we add -7 to the result and get $18 - 7 = 11$.

This algorithm can be performed in a purely mechanical way with the help of a table called “Horner scheme”.

Horner scheme – the table

Now we show how to assemble the table of the Horner scheme:

1. The first row of the table is reserved for the coefficients of the polynomial. We start with the coefficient at the highest power and write all other coefficients at the remaining powers, including their signs, in the ascending order (including the absolute term). If some power is missing in the polynomial, we write 0 because the respective coefficient is 0. Therefore e.g. for a polynomial $p_3(x) = x^3 - 2x^2 + 3x - 7$ we write:

$$\begin{array}{c|cccc} & 1 & -2 & 3 & -7 \\ \hline \end{array}$$

2. We start the second row with the input value we are interested in. In our case we write e.g. $x = 3$.

$$\begin{array}{c|cccc} & 1 & -2 & 3 & -7 \\ \hline 3 & & & & \end{array}$$

3. In the third row (to the left of the vertical line) we copy the coefficient at the greatest power of x :

$$\begin{array}{c|cccc} & 1 & -2 & 3 & -7 \\ \hline 3 & - & & & \\ & 1 & & & \end{array}$$

It helps if we continue the second row with $-$ which suggests that nothing will be written at this place.

4. We multiply the input value (in our case 3) by the rightmost number of the third row (in our case 1 which is $-$ for the time being $-$ the only, i.e. the rightmost, number in the third row) and we write the result in the first empty position in the second row:

$$\begin{array}{c|cccc} & 1 & -2 & 3 & -7 \\ \hline 3 & - & & 3 & \\ & 1 & & & \end{array}$$

5. We make a sum of the first and second row and write the sum in the third row:

$$\begin{array}{c|cccc} & 1 & -2 & 3 & -7 \\ \hline 3 & - & & 3 & \\ & 1 & & 1 & \end{array}$$

6. We repeat steps 4 and 5 until the whole table is filled in. The rightmost number of the third column is the output value.

$$\begin{array}{c|cccc} & 1 & -2 & 3 & -7 \\ \hline 3 & - & 3 & 3 & 18 \\ & 1 & 1 & 6 & \mathbf{11} \end{array}$$

Important. If we end up with the output value 0, the input $x = c$ is the root of the given polynomial. Moreover, the coefficients in the third row of the table are the coefficients of a polynomial $q_{n-1}(x)$ which we obtain after we divide $p_n(x)$ by $(x - c)$.

Remark 7. Since the values in the second row of the table are interim calculations only, the second row can be omitted.

Exercise

Exercise 8. Factorize polynomial $p(x) = x^5 + x^4 - 4x^2 - x + 3$ and find $p(2)$.

Solution. The set of all integral divisors of the absolute term of the polynomial p is $\{-1, 1, -3, 3\}$ while the set of all natural divisors of the coefficient at the highest power of x is $\{1\}$. Therefore all rational roots of the polynomial are included in the set $\{-1, 1, -3, 3\}$. The Horner scheme gives the following results:

We test $x = 1$:

$$\begin{array}{r|rrrrrr} & 1 & 1 & 0 & -4 & -1 & 3 \\ 1 & & 1 & 2 & 2 & -2 & -3 \\ & & 1 & 2 & 2 & -3 & \mathbf{0} \end{array}$$

This means that $p(x) = (x - 1) \cdot (x^4 + 2x^3 + 2x^2 - 2x - 3)$. The set of rational roots remains unchanged. We test $x = 1$ once more:

$$\begin{array}{r|rrrrr} & 1 & 2 & 2 & -2 & -3 \\ 1 & & 1 & 3 & 5 & 3 \\ & & 1 & 3 & 5 & 3 & \mathbf{0} \end{array}$$

This means that $p(x) = (x - 1)^2 \cdot (x^3 + 3x^2 + 5x + 3)$. The set of all rational roots reduces to $\{-1, -3\}$. We can see (consider this carefully!) that substituting an arbitrary positive number for x into $(x^3 + 3x^2 + 5x + 3)$ always results in a positive output value. Therefore we test $x = -1$:

$$\begin{array}{r|rrrr} & 1 & 3 & 5 & 3 \\ -1 & & -1 & -2 & -3 \\ & & 1 & 2 & 3 & \mathbf{0} \end{array}$$

As a result $p(x) = (x - 1)^2 \cdot (x + 1) \cdot (x^2 + 2x + 3)$. The quadratic trinomial $(x^2 + 2x + 3)$ is irreducible in the real domain (consider why). Thus we have completed the task. Now we need to find the value $p(2)$. Once again, we make use of the Horner scheme:

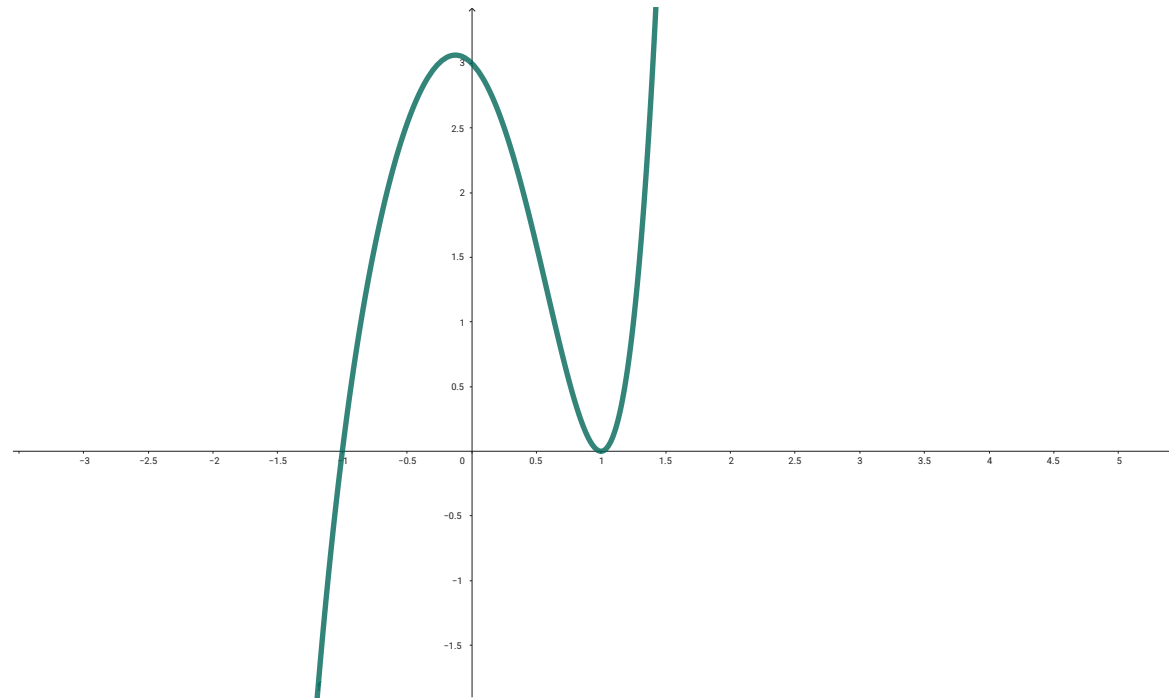
$$\begin{array}{r|rrrrrr} & 1 & 1 & 0 & -4 & -1 & 3 \\ 2 & & 2 & 6 & 12 & 16 & 30 \\ & & 1 & 3 & 6 & 8 & 15 & \mathbf{33} \end{array}$$

This mean that there is $p(2) = 33$.

Exercise

Exercise 9. Find all $x \in \mathbb{R}$ for which the values of $f(x) = x^5 + x^4 - 4x^2 - x + 3$ are positive.

Solution. We will make use of the factorization of polynomial p from the previous exercise because the polynomial p is exactly our given function f . The graph of the function intersects the x axis at the roots of p , i.e. at points $x_1 = -1$ and $x_2 = 1$. The sign of $(x - 1)^2$ is positive for all $x \in \mathbb{R} \setminus \{1\}$ while the sign of $(x^2 + 2x + 3)$ is positive for all $x \in \mathbb{R}$. As a result, what influences the sign of f is the expression $(x + 1)$. This is positive for all $x \in (-1, \infty)$ and negative for all $x \in (-\infty, -1)$. Therefore, the function f is positive for all $x \in (-1, 1) \cup (1, \infty)$. Its graph is given below:



Exercise

Exercise 10. Factorize polynomial $p(x) = 4x^6 - x^5 + 8x^3 - 38x^2 + 33x - 6$ and find $p(0)$.

Solution. The set of all integral divisors of the absolute term of p is $\{-1, 1, -2, 2, -3, 3, -6, 6\}$ while the set of natural divisors of the coefficients at the greatest power of x is $\{1, 2, 4\}$. Therefore all rational roots of our polynomial are included in the set $M = \{-1, 1, -2, 2, -3, 3, -6, 6, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, \frac{3}{4}\}$. We will make use of the Horner scheme. First, we test $x = 1$:

$$\begin{array}{r|rrrrrrrr} & 4 & -1 & 0 & 8 & -38 & 33 & -6 \\ 1 & & 4 & 3 & 3 & 11 & -27 & 6 \\ & 4 & 3 & 3 & 11 & -27 & 6 & \mathbf{0} \end{array}$$

Therefore, there is $p(x) = (x - 1) \cdot (4x^5 + 3x^4 + 3x^3 + 11x^2 - 27x + 6)$. The set M remains unchanged. Therefore, we test $x = 1$ again:

$$\begin{array}{r|rrrrrr} & 4 & 3 & 3 & 11 & -27 & 6 \\ 1 & & 4 & 7 & 10 & 21 & -6 \\ & 4 & 7 & 10 & 21 & -6 & \mathbf{0} \end{array}$$

Therefore, there is $p(x) = (x - 1)^2 \cdot (4x^4 + 7x^3 + 10x^2 + 21x - 6)$. The set M remains unchanged yet the substitution of $x = 1$ into $4x^4 + 7x^3 + 10x^2 + 21x - 6$ results in a number greater than 0. Therefore, we test e.g. $x = -2$:

$$\begin{array}{r|rrrrrr} & 4 & 7 & 10 & 21 & -6 \\ -2 & & -8 & 2 & -24 & 6 \\ & 4 & -1 & 12 & -3 & \mathbf{0} \end{array}$$

Therefore, there is $p(x) = (x - 1)^2 \cdot (x + 2) \cdot (4x^3 - x^2 + 12x - 3)$. The root $x = 2$ is not a multiple one (test this fact yourselves by substituting the root into $(4x^3 - x^2 + 12x - 3)$). Therefore, values 1 and -2 can be taken out of the set M . Moreover, the set will be reduced due to the change of the absolute term. As a result, we get $M^* = \{-1, 2, -3, 3, -\frac{1}{2}, \frac{1}{2}, -\frac{3}{2}, \frac{3}{2}, -\frac{1}{4}, \frac{1}{4}, -\frac{3}{4}, \frac{3}{4}\}$. Test e.g. $x = \frac{1}{4}$:

$$\begin{array}{r|rrrr} & 4 & -1 & 12 & -3 \\ \frac{1}{4} & & 1 & 0 & 3 \\ & 4 & 0 & 12 & \mathbf{0} \end{array}$$

Therefore, there is $p(x) = (x - 1)^2 \cdot (x + 2) \cdot (x - \frac{1}{4}) \cdot (4x^2 + 12) = (x - 1)^2 \cdot (x + 2) \cdot (4x - 1) \cdot (x^2 + 3)$.

Since the quadratic binomial $(x^2 + 3)$ cannot be factorized in the set of real numbers, we can stop the algorithm.

Regarding the value $p(0)$, there is obviously $p(0) = -6$ as it is sufficient to substitute $x = 0$ in the polynomial.

1 TEST

1. The function $f(x) = 3x - 1$ intersects the x -axis at:
 - (a) $x = 0$
 - (b) $x = \frac{1}{3}$
 - (c) $x = -\frac{1}{3}$
 - (d) $x = 3$
2. The function $f(x) = 2 - x$ is
 - (a) increasing in $\mathbb{R}$
 - (b) decreasing in $\mathbb{R}$
 - (c) constant in $\mathbb{R}$
 - (d) increasing in $(-\infty, 2)$ and decreasing in $(2, \infty)$.
3. The inverse of $f(x) = 3x - 1$ is
 - (a) $f^{-1}(x) = \frac{1}{3}x + \frac{1}{3}$
 - (b) $f^{-1}(x) = \frac{1}{3}x - \frac{1}{3}$
 - (c) $f^{-1}(x) = 3x + 1$
 - (d) $f^{-1}(x) = -3x + 1$
4. The function $f(x) = x^2 - 2x - 3$ intersects the x -axis at:
 - (a) $x_1 = 3, x_2 = -1$
 - (b) $x_1 = -3, x_2 = 1$
 - (c) it does not intersect the x -axis
 - (d) $x_1 = x_2 = 0$
5. The vertex of parabola $y = 9 - x^2$ is:
 - (a) $[0, 3]$
 - (b) $[0, 9]$
 - (c) $[0, -9]$
 - (d) $[9, 0]$
6. The factorization of $p(x) = 2x^5 + 9x^4 + 16x^3 + 14x^2 + 6x + 1$ is:
 - (a) $(2x + 1) \cdot (x + 1)^4$
 - (b) $(2x - 1) \cdot (x + 1)^4$
 - (c) $(2x + 1) \cdot (x - 1)^4$
 - (d) $\left(x + \frac{1}{2}\right) \cdot (x + 1)^4$
7. The set of roots of the polynomial $p(x) = 2x^5 + 9x^4 + 16x^3 + 14x^2 + 6x + 1$ includes:

- (a) $x = 1$
- (b) $x = -1$
- (c) $x = \frac{1}{2}$
- (d) $x = 0$

8. The function $f(x) = x^2 - 2x + 5$ is increasing in:

- (a) $(1, \infty)$
- (b) $(-\infty, 1)$
- (c) $[1, 2]$
- (d) $(-\infty, \infty)$

9. The function $f(x) = x^2 - 2x + 5$ is positive in:

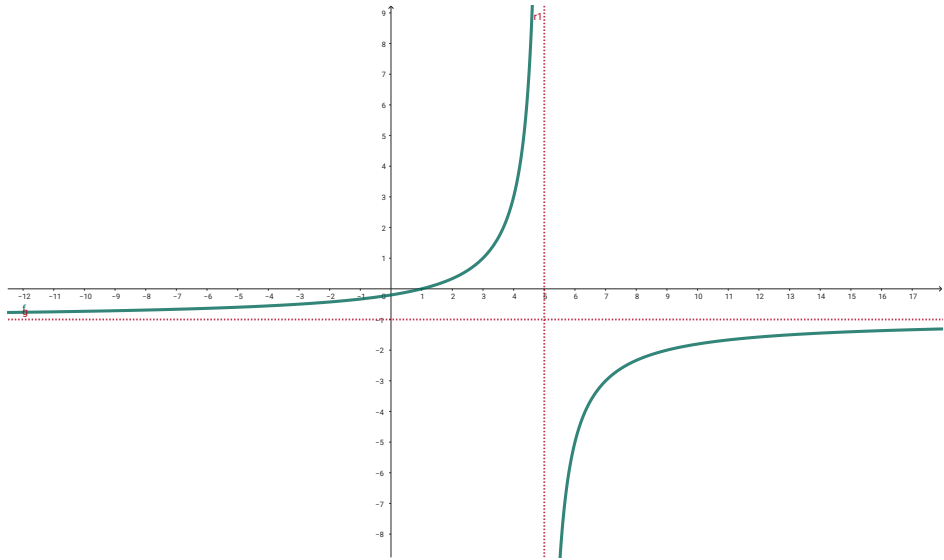
- (a) $(0, \infty)$
- (b) $(-\infty, \infty)$
- (c) $[0, \infty)$
- (d) $[-2, 5]$

10. The function $f(x) = x^2 - 2x + 5$ intersects the y axis in :

- (a) $[0, 5]$
- (b) $[0, -5]$
- (c) $\{0, 5\}$
- (d) $[5, 0]$

4. RATIONAL FUNCTIONS

Rational functions



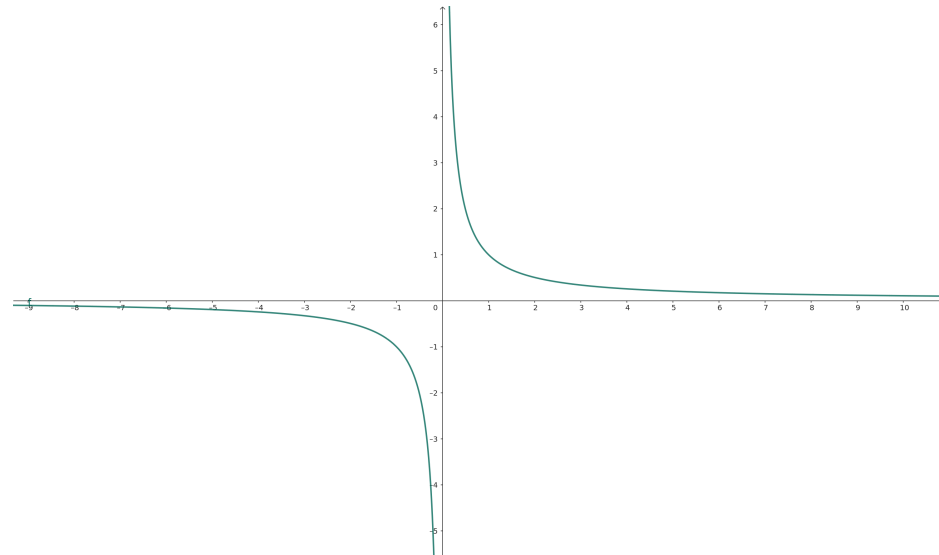
Rational functions

By a rational function we mean a function f defined, for all $a \in \mathbb{R}$, by

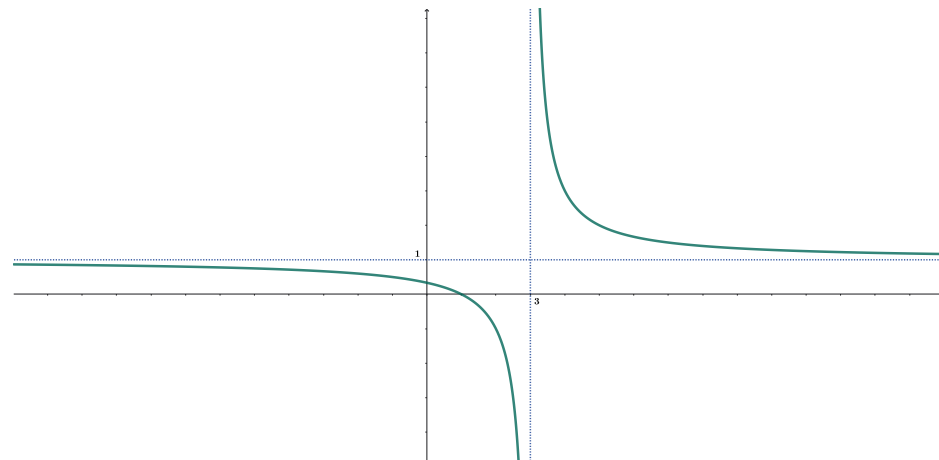
$$f(x) = \frac{P_n(x)}{Q_m(x)},$$

where P_n and Q_m are polynomials of degree n and m , respectively. In general, we can say that the domain of definition of rational functions is the whole real domain with the exception of all real roots of the polynomial Q_m , i.e. $D(f) = \{x \in \mathbb{R} : Q_m(x) \neq 0\}$.

We will focus on a subclass of rational functions such that $P(x) = ax + b$ and $Q(x) = cx + d$, i.e. such that both of these are linear polynomials, where $P(x)$ and $Q(x)$ do not share their roots. The graph of such functions is called a hyperbola. An example of a hyperbola in its normal form (here the graph of $y = \frac{1}{x}$) is given below. The domain of definition of the function below is $\mathbb{R} \setminus \{0\}$. Its range of values is $\mathbb{R} \setminus \{0\}$ as well.



The graph of $g(x) = \frac{x-1}{x-3}$ is a hyperbola as well. However, this one is shifted. The domain of definition follows either from the graph or from the definition of the function. It is $D(g) = \mathbb{R} \setminus \{3\}$. The range of values of g can be given either from the graph or from the definition of the function (we will show this in Exercise 1). From the graph we can see that there is $H(g) = \mathbb{R} \setminus \{1\}$.



The dashed lines are called asymptotes. The line $x = 3$ is called a vertical asymptote while the line $y = 1$ is called a horizontal asymptote. Vertical asymptotes are linked to the domain of definition while horizontal asymptotes are linked to the range of values.

Exercise

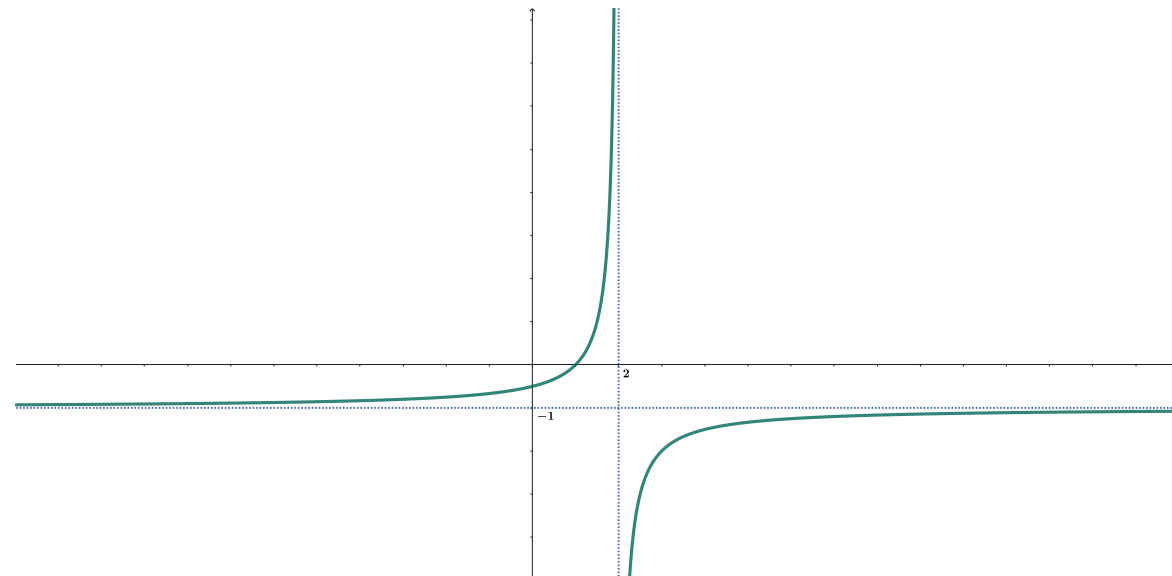
Exercise 1. Sketch the graph of $f(x) = \frac{x-1}{2-x}$, determine $D(f)$ and $H(f)$.

Solution. In order to find the domain of definition it is sufficient to find out such $x \in \mathbb{R}$ that $2 - x \neq 0$. This is because the denominator of f is $2 - x$. Therefore, there is $D(f) = \mathbb{R} \setminus \{2\}$. This means that the vertical asymptote is $x = 2$.

In order to find the range of values, we transform the function as follows:

$$f(x) = \frac{x-1}{2-x} = \frac{-(1-x)}{2-x} = -\frac{(2-x)+1}{2-x} = -\frac{2-x}{2-x} - \frac{1}{2-x} = -1 - \frac{1}{2-x}.$$

Since there is $\frac{1}{2-x} \neq 0$ for all $x \in D(f)$, there is $f(x) \neq -1$ for all $x \in D(f)$. As a result we get that $H(f) = \mathbb{R} \setminus \{-1\}$. This means that the horizontal asymptote is $y = -1$. In order to sketch the graph we first sketch the vertical and horizontal asymptotes. After this we find the intersections of the function with the coordinate axes. Finally, we sketch a hyperbola which approaches the asymptotes and passes through the intersections. In our case, the intersection with the x -axis is $[1, 0]$. We obtain it from the equation $f(x) = 0$. The intersection with the y -axis is $[0, -\frac{1}{2}]$ and we obtain it by evaluating the function at $x = 0$.



Obviously, the function f is increasing in $(-\infty, 2)$ as well as in $(2, \infty)$. However, this does not mean that it is increasing in their union. Why? The answer is simple. There is e.g. $1 < 3$ yet $f(1) > f(3)$. Find another counterexample yourselves.

Exercise

Exercise 2. Find the inverse of $f(x) = \frac{1+x}{3+x}$. Then sketch both functions.

Solution. First of all we find out whether f is an injective function. This means that we have to test the validity of the implication

$$\forall a, b \in D(f): f(a) = f(b) \Rightarrow a = b.$$

Obviously, there is $D(f) = \mathbb{R} \setminus \{-3\}$. Let $a, b \in \mathbb{R} \setminus \{-3\}$. Suppose that $f(a) = f(b)$. We substitute in the function relation and get:

$$\frac{1+a}{3+a} = \frac{1+b}{3+b},$$

which results in:

$$(1+a)(3+b) = (1+b)(3+a)$$

$$3 + b + 3a + ab = 3 + a + 3b + ab$$

$$b + 3a = a + 3b$$

$$a = b$$

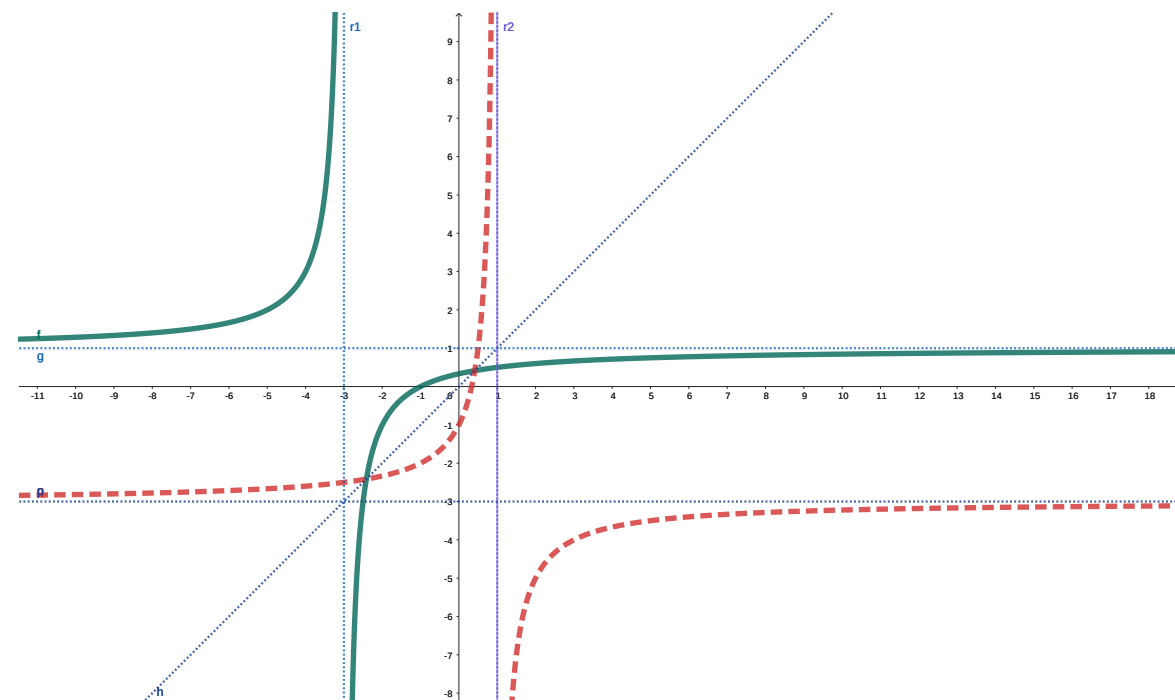
Since a, b are arbitrary elements of $D(f)$, the implication holds for f . The function is thus injective in all its domain of definition, which means that its inverse exists in all $D(f)$. We get the definition of f^{-1} by swapping x and y and expressing y .

$$x = \frac{1+y}{3+y} \Rightarrow x \cdot (3+y) = 1+y \Rightarrow 3x + xy = 1+y \Rightarrow xy - y = 1 - 3x,$$

i.e.

$$y = \frac{1-3x}{x-1}.$$

The graphs of both functions including their asymptotes are given in the figure. The original function f is green, its inverse f^{-1} is red.



Exercise

Exercise 3. Find the domain of definition of $f(x) = \frac{1}{x^2-3x+2}$.

Solution. In order to find the domain of definition we need to find out all $x \in \mathbb{R}$ such that $x^2 - 3x + 2 \neq 0$. We solve the quadratic equation $x^2 - 3x + 2 = 0$ and we exclude its roots from the domain of definition. We transform the quadratic trinomial on the left-hand side and get

$$(x - 2) \cdot (x - 1) = 0.$$

The solutions of the equation are $x_1 = 2, x_2 = 1$, which means that $D(f) = \mathbb{R} \setminus \{1, 2\}$.

Exercise

Exercise 4. Find the intersections of graphs of functions $f(x) = \frac{1+x}{3+x}$ and $g(x) = \frac{2+x}{x+6}$.

Solution. In order to solve this task it is sufficient to find all $x \in \mathbb{R}$ such that the function values of both functions are the same. In other words, we will find the solution of the equation

$$\frac{1+x}{3+x} = \frac{2+x}{x+6}.$$

We will solve this equation in $\mathbb{R} \setminus \{-6, -3\}$ because the expression on the right-hand side is not defined for $x = -6$ while the expression on the left-hand side is not defined for $x = -3$. We multiply the equation by

$$(1+x) \cdot (x+6) = (2+x) \cdot (3+x)$$

and get

$$x^2 + 7x + 6 = x^2 + 5x + 6,$$

which means that there is

$$2x = 0.$$

Therefore, functions f and g have the same function values $f(0) = g(0) = \frac{1}{3}$ at $x = 0$, which means that their graphs intersect at $A = \left[0, \frac{1}{3}\right]$.

Exercise

Exercise 5. Sketch the graph of $f(x) = \frac{|x-1|}{2-x}$. Then find $D(f)$ and $H(f)$.

Solution. In order to find the domain of definition we need to find all $x \in \mathbb{R}$ such that $2 - x \neq 0$. Therefore, there is $D(f) = \mathbb{R} \setminus \{2\}$ and the graph will have the vertical asymptote at $x = 2$. In order to find the range of values it will be best if we sketch the graph first. The expression in the nominator contains an absolute value, which means that we must write the function separately for the interval in which there is $x - 1 < 0$ and for the interval in which there is $x - 1 \geq 0$. We get

$$f(x) = \begin{cases} \frac{1-x}{2-x} & x \in (-\infty, 1) \\ \frac{x-1}{2-x} & x \in [1, 2) \cup (2, \infty) \end{cases}.$$

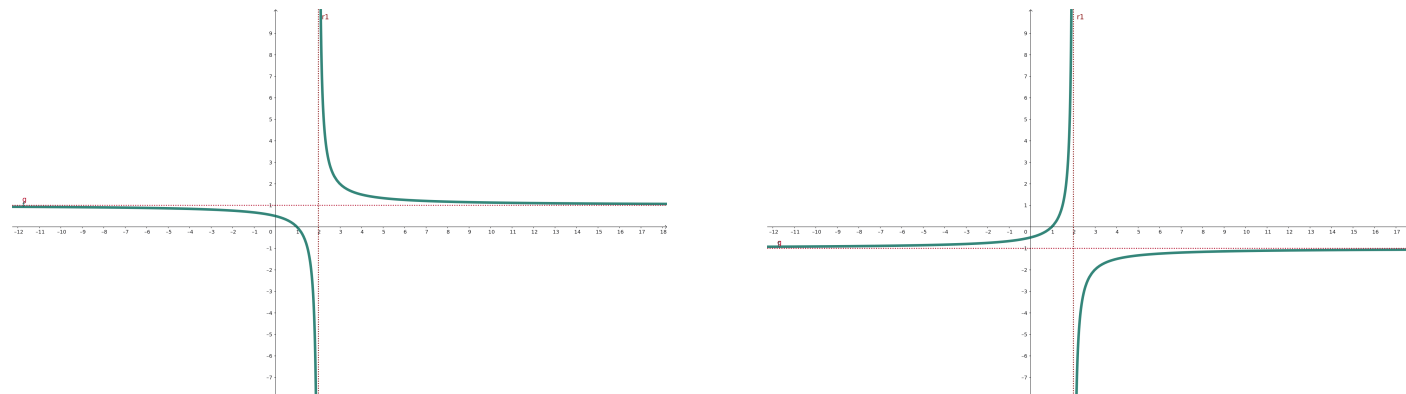
We sketch the graphs separately. The function definition in $(-\infty, 1)$ can be transformed in the following way:

$$\frac{1-x}{2-x} = \frac{2-x-1}{2-x} = 1 - \frac{1}{2-x}.$$

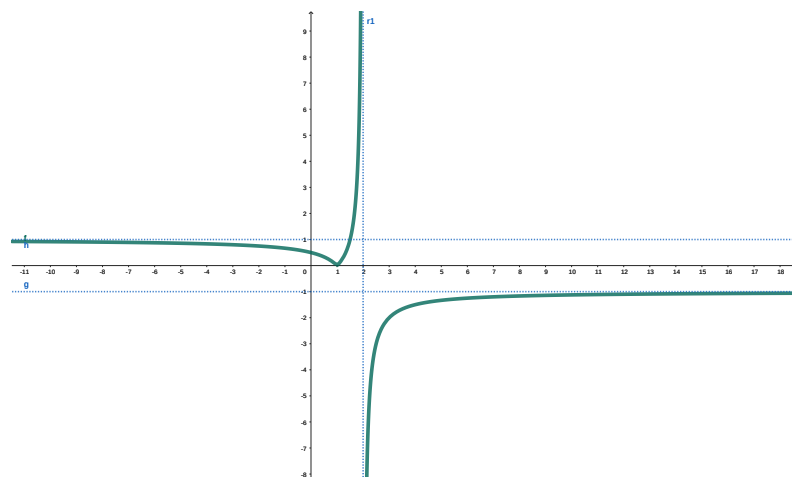
Since there is $\frac{1}{2-x} \neq 0$, the function f cannot attain value 1 in this interval. As a result, the horizontal asymptote will be $y = 1$. The manipulation with the function in the union of intervals $[1, 2) \cup (2, \infty)$ is analogous

$$\frac{x-1}{2-x} = \frac{x-2+1}{2-x} = -1 + \frac{1}{2-x}.$$

Obviously, there is $\frac{1}{2-x} \neq 0$, which means that the function in this union of intervals cannot attain value -1 . Therefore, $y = -1$ will be the horizontal asymptote. We sketch these two graphs



and we obtain the graph of our function by sketching the graph on the left only in $(-\infty, 1)$ and the graph on the right in the union of intervals $[1, 2) \cup (2, \infty)$:



The graph tells us that the range of values is $H(f) = (-\infty, -1) \cup [0, \infty)$.

Exercise

Exercise 6. Find the domain of definition of $f(x) = \frac{1}{x^2 - 2|x| - 3}$.

Solution. In order to find the domain of definition we need to find all $x \in \mathbb{R}$ such that $x^2 - 2|x| - 3 \neq 0$. We solve the quadratic equation $x^2 - 2|x| - 3 = 0$ and we exclude its roots from the domain of definition. Since the quadratic trinomial on the left contains an absolute value, we need to write it separately for the interval in which there is $x < 0$ and for the interval in which there is $x \geq 0$. In $(-\infty, 0)$ we get

$$x^2 + 2x - 3 = 0 \iff (x + 3) \cdot (x - 1) = 0 \iff x = -3 \vee x = 1.$$

Since we focus on the interval $(-\infty, 0)$, the only solution is $x = -3$. In $[0, \infty)$ we get that

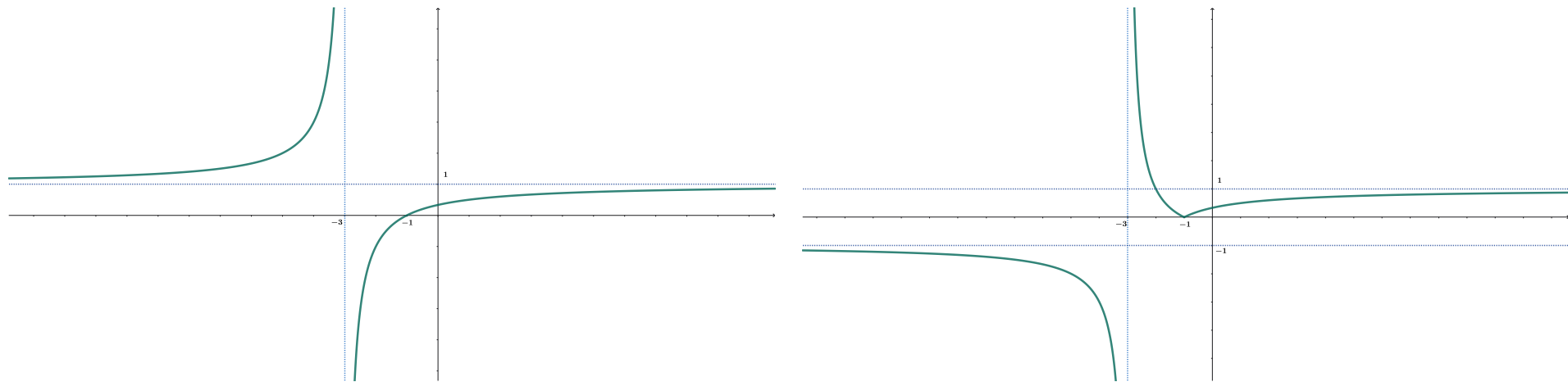
$$x^2 - 2x - 3 = 0 \iff (x - 3) \cdot (x + 1) = 0 \iff x = 3 \vee x = -1.$$

Since we focus on the interval $[0, \infty)$, the only solution is $x = 3$. The solutions of the original equation are thus $x_1 = -3$ and $x_2 = 3$, which means that $D(f) = \mathbb{R} \setminus \{-3, 3\}$.

Exercise

Exercise 7. Suppose we have $f(x) = \frac{1+x}{3+x}$, $g(x) = \frac{|1+x|}{3+x}$. Find all $x \in \mathbb{R}$ such that there is $f(x) = g(x)$.

Solution. We can solve the exercise graphically. The following figures depict graphs of f and g and it is obvious that these functions are equal in $[-1, \infty)$.



Alternatively, it is enough to realize that the function g can be written also as

$$g(x) = \begin{cases} \frac{-1-x}{3+x} & x \in (-\infty, -1), \\ \frac{1+x}{3+x} & x \in [-1, \infty). \end{cases}$$

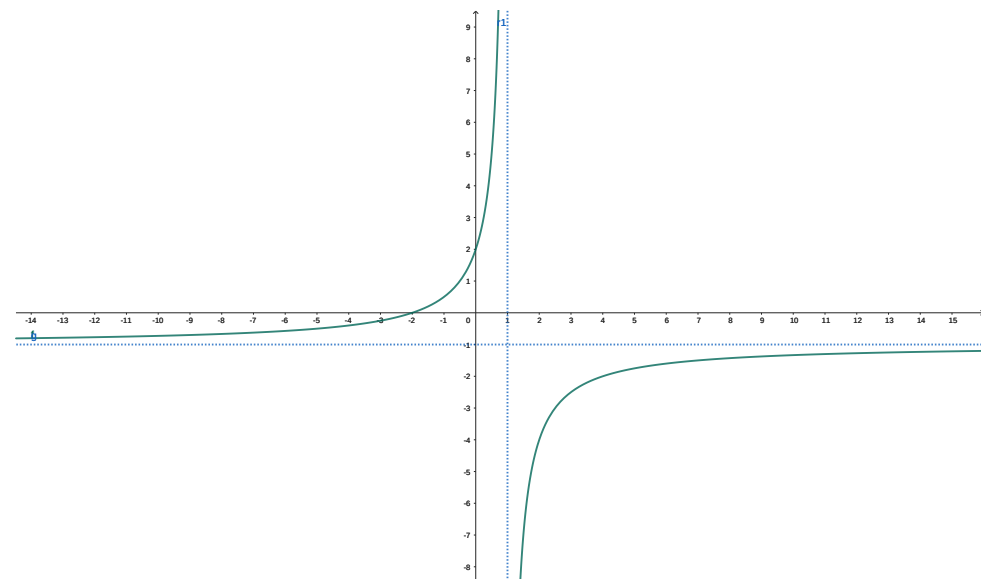
(i.e. without the absolute value). Therefore, there is $f(x) = g(x)$ for all $x \in [-1, \infty)$.

1 TEST

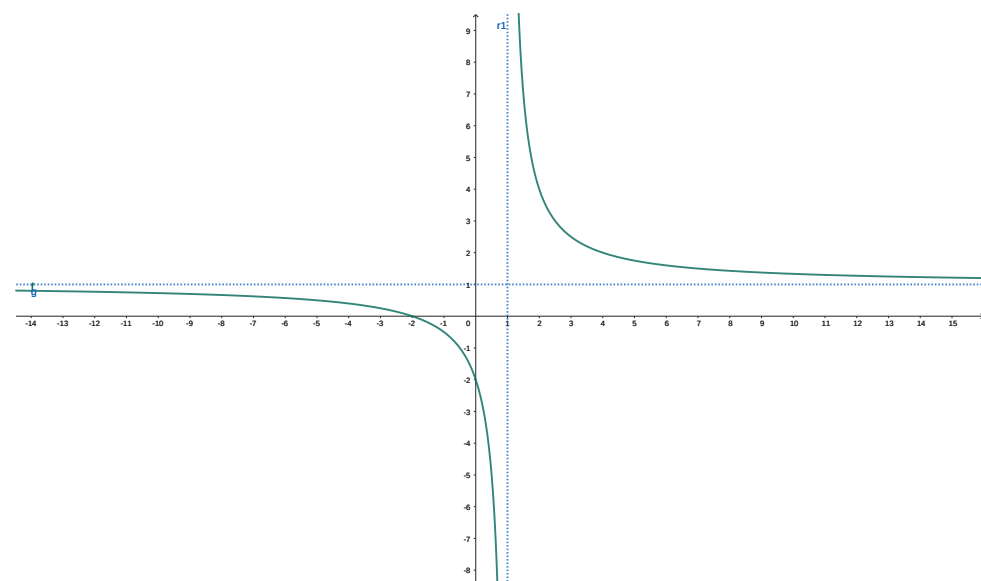
1. The domain of definition of $f(x) = \frac{x+3}{x+2}$ is
 - (a) $\mathbb{R} \setminus \{-3\}$
 - (b) $\mathbb{R} \setminus \{-2\}$
 - (c) $\mathbb{R} \setminus \{-3, -2\}$
 - (d) $\mathbb{R} \setminus \{2\}$
2. The inverse of $f(x) = \frac{x+3}{x+2}$ is
 - (a) $f^{-1}(x) = \frac{x+2}{x+3}$
 - (b) $f^{-1}(x) = \frac{3-2x}{x-1}$
 - (c) $f^{-1}(x) = \frac{2-3x}{x-1}$
 - (d) $f^{-1}(x) = \frac{3-2x}{x+1}$
3. The function $f(x) = \frac{1-x}{x-3}$ intersects the x -axis at
 - (a) $A = [1, 0]$
 - (b) $B = [0, 1]$
 - (c) $C = [0, \frac{1}{3}]$
 - (d) $D = [0, -\frac{1}{3}]$
4. The domain of definition of $f(x) = \frac{x+3}{x^2+2x-3}$ is
 - (a) $\mathbb{R} \setminus \{-1, 3\}$
 - (b) $\mathbb{R} \setminus \{-3, 1\}$
 - (c) $\mathbb{R} \setminus \{-3\}$
 - (d) $\mathbb{R} \setminus \{1\}$
5. The domain of definition of $f(x) = \frac{x+3}{x^2-2|x|-3}$ is
 - (a) $\mathbb{R} \setminus \{-3, 3\}$
 - (b) $\mathbb{R} \setminus \{-3, -1, 1, 3\}$
 - (c) $\mathbb{R} \setminus \{-1, 1\}$
 - (d) $\mathbb{R} \setminus \{0\}$
6. The functions $f(x) = \frac{x+1}{|x-2|}$, $g(x) = \frac{|x+1|}{x-2}$ are equal for all x in
 - (a) $(2, \infty)$
 - (b) $(-\infty, -1]$
 - (c) $[2, \infty)$
 - (d) $(-1, 2)$

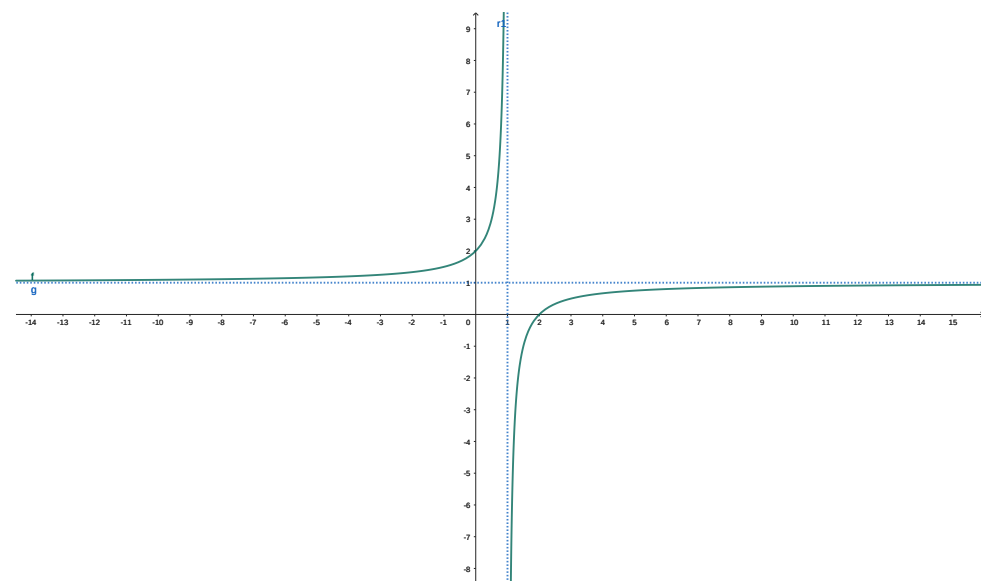
7. The graph of $f(x) = \frac{x+2}{1-x}$ is

(a)

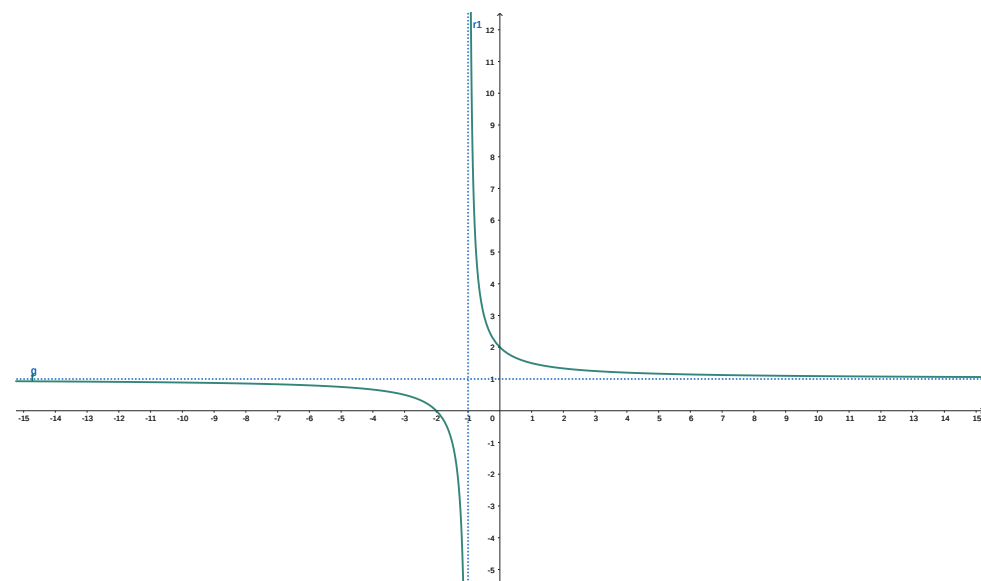


(b)



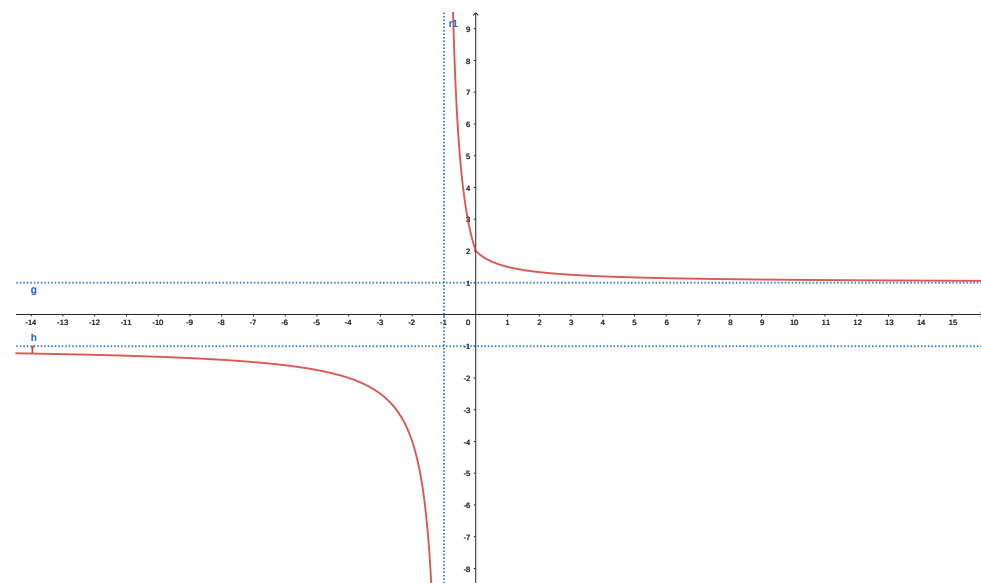


(c)

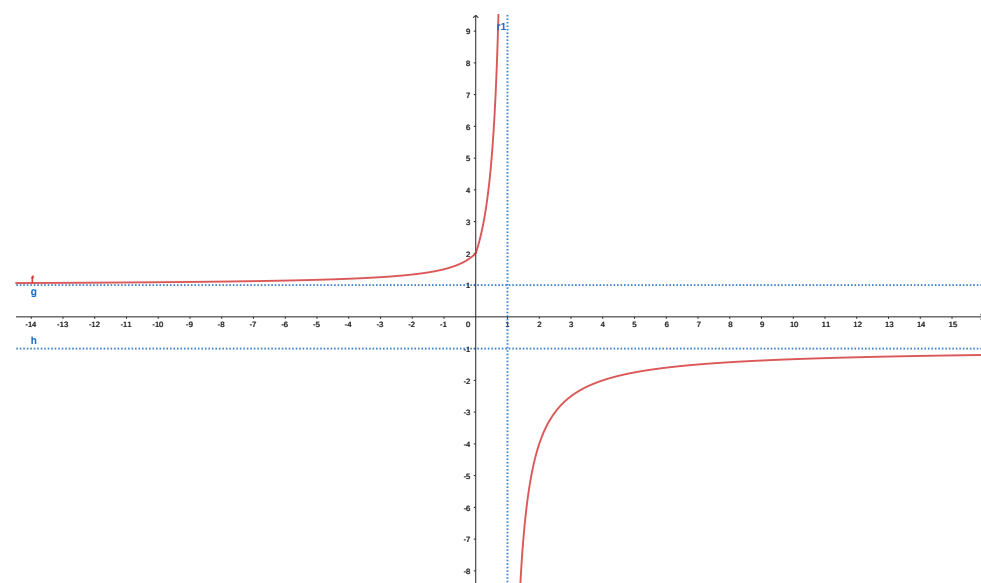


(d)

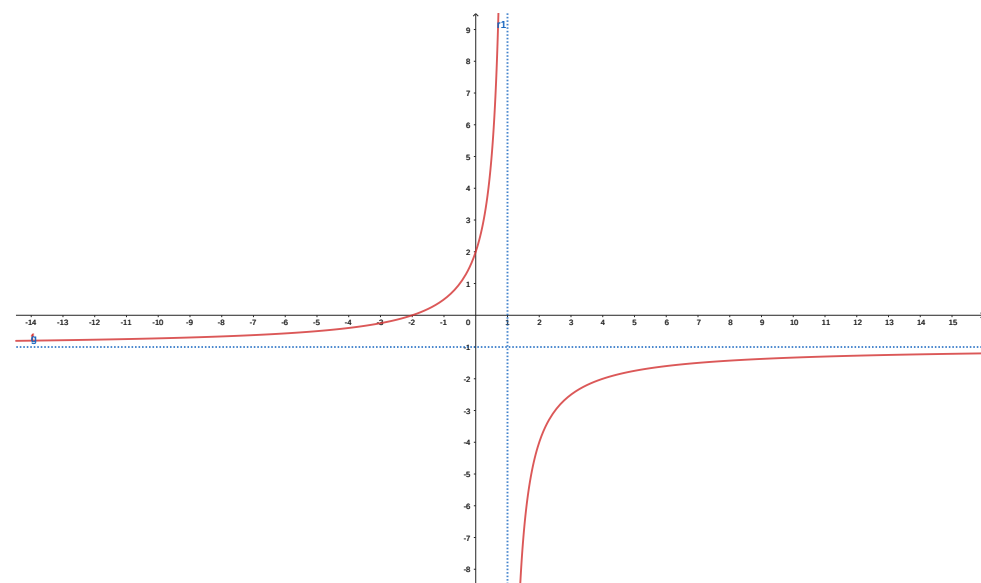
8. The graph of $f(x) = \frac{|x|+2}{1-x}$ is



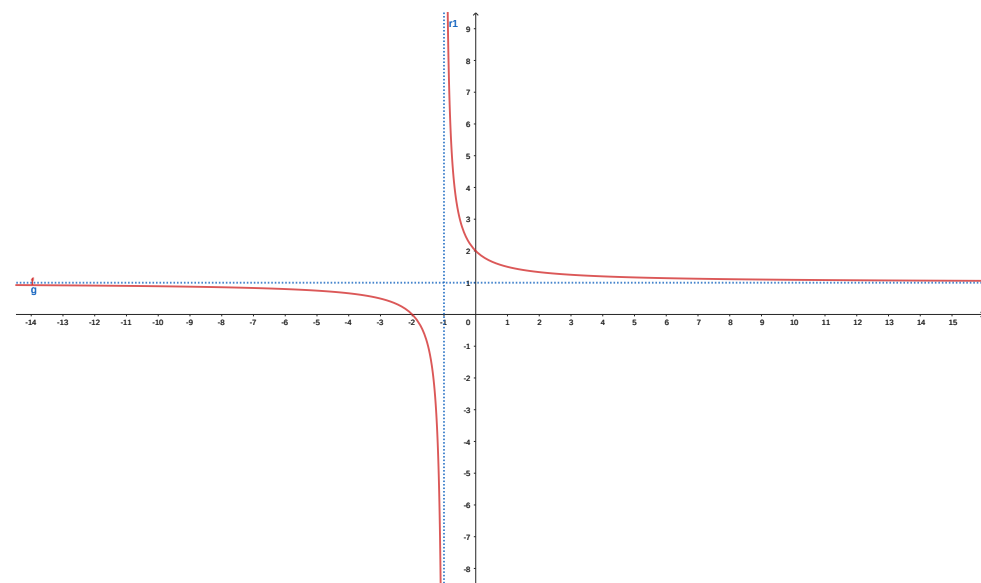
(a)



(b)

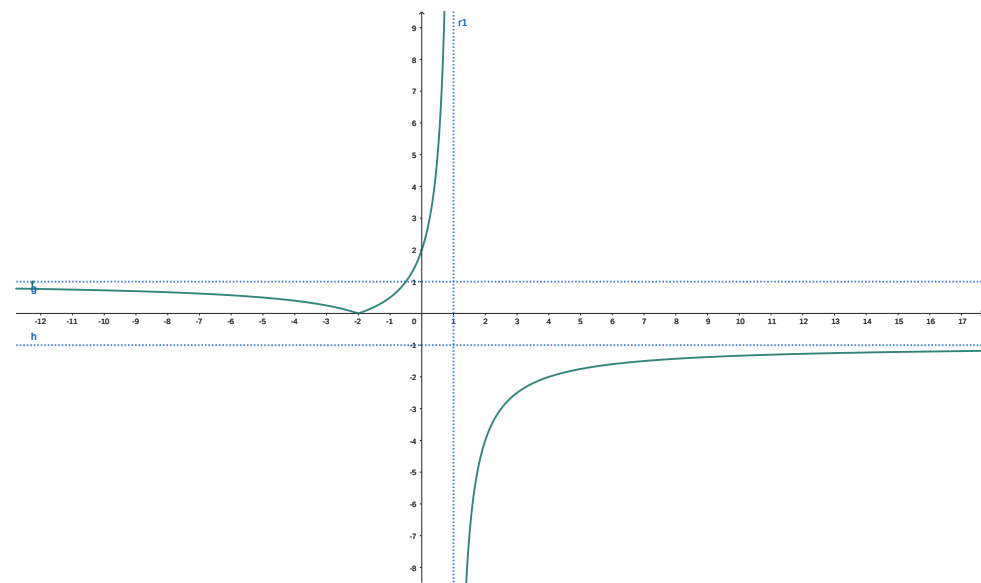


(c)

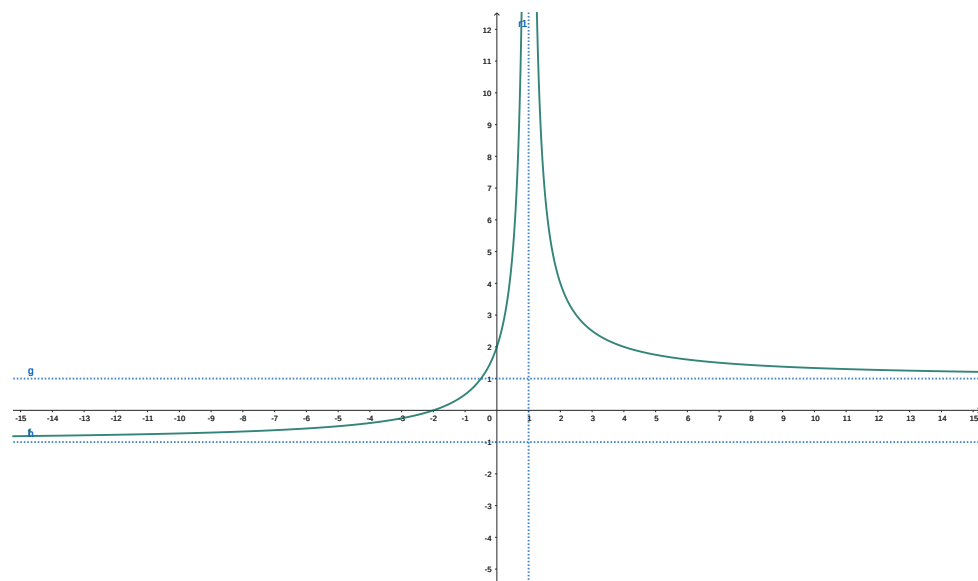


(d)

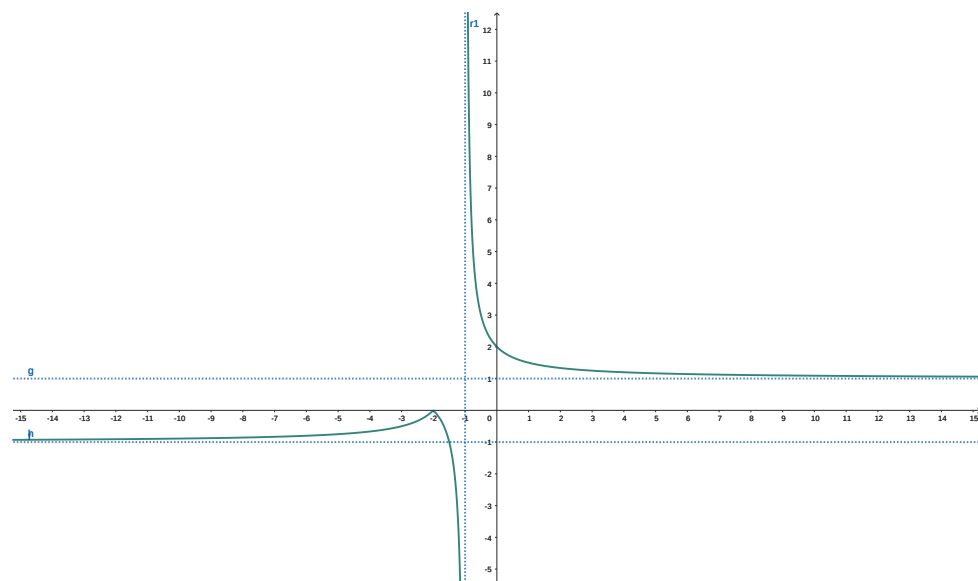
9. The graph of $f(x) = \frac{|x+2|}{1-x}$ is



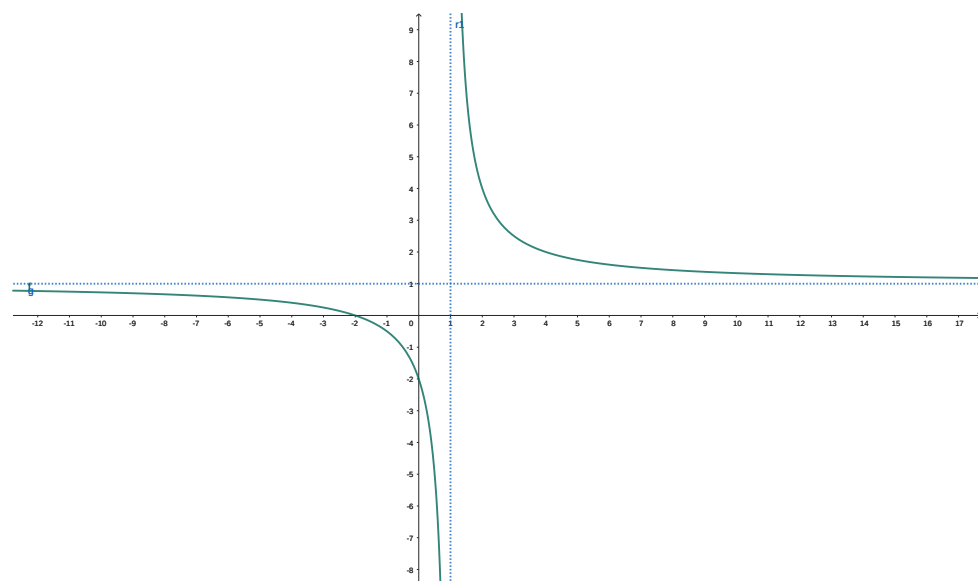
(a)



(b)

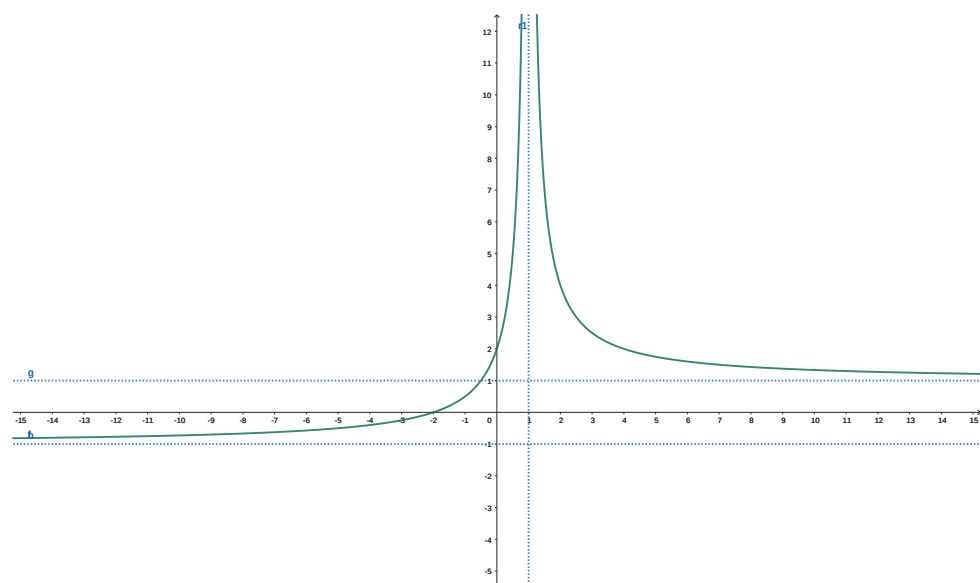


(c)

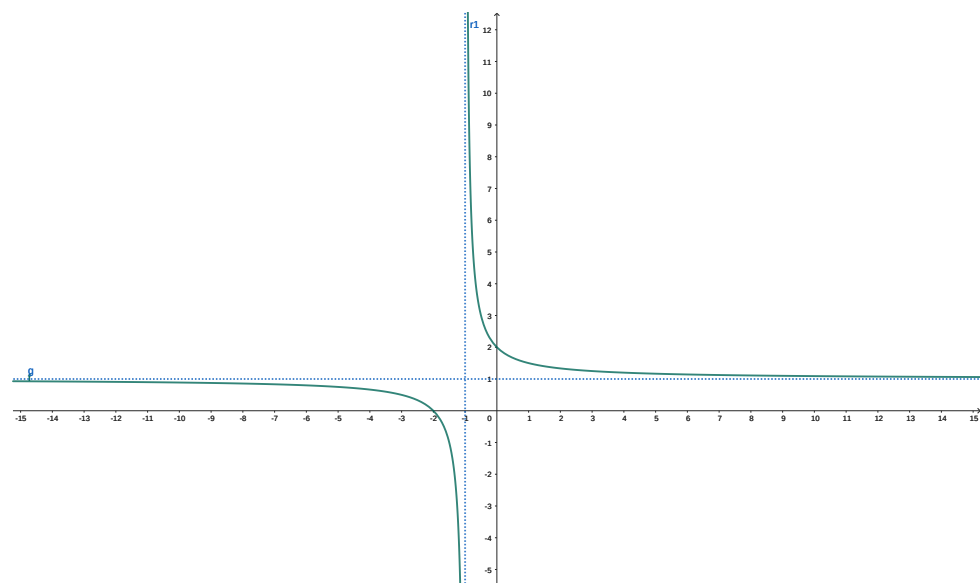


(d)

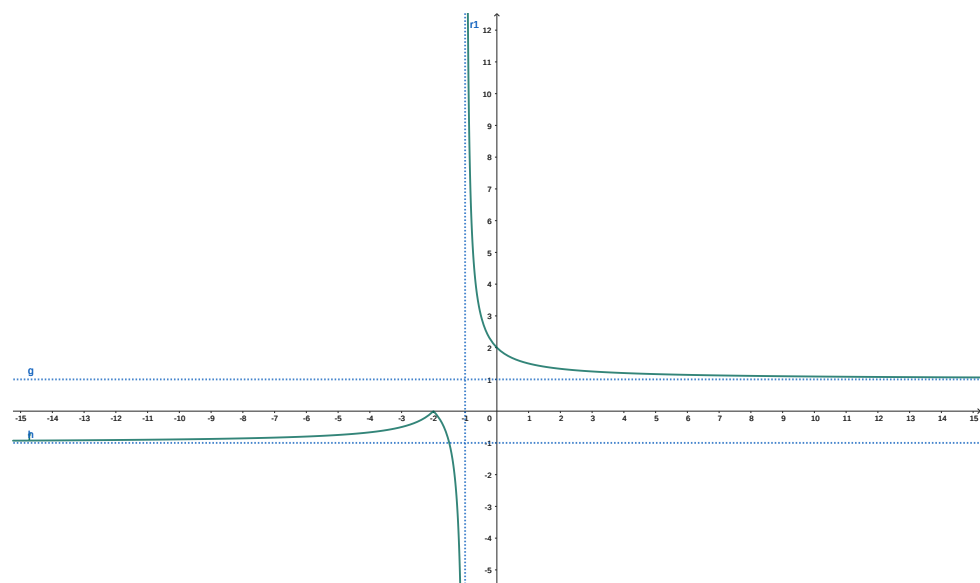
10. The graph of $f(x) = \frac{x+2}{|1-x|}$ is



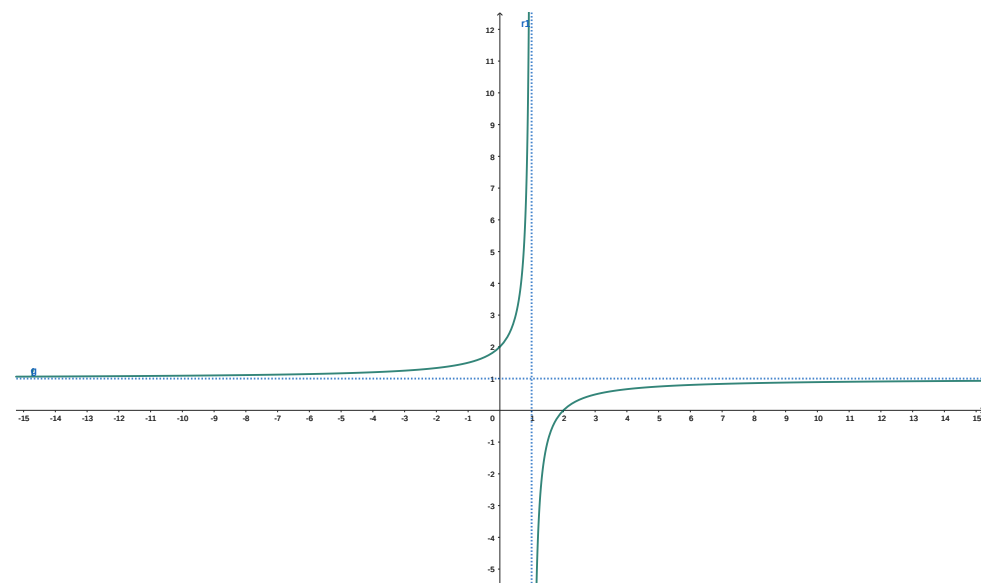
(a)



(b)



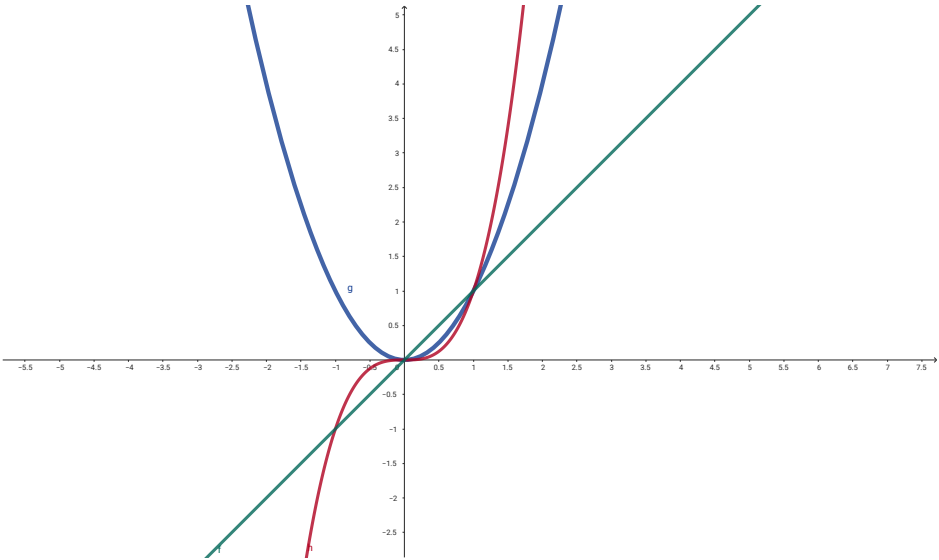
(c)



(d)

5. POWER FUNCTIONS

Power functions



Power functions

By a power function we mean a function f defined, for all $a \in \mathbb{R}$, by

$$f(x) = x^a.$$

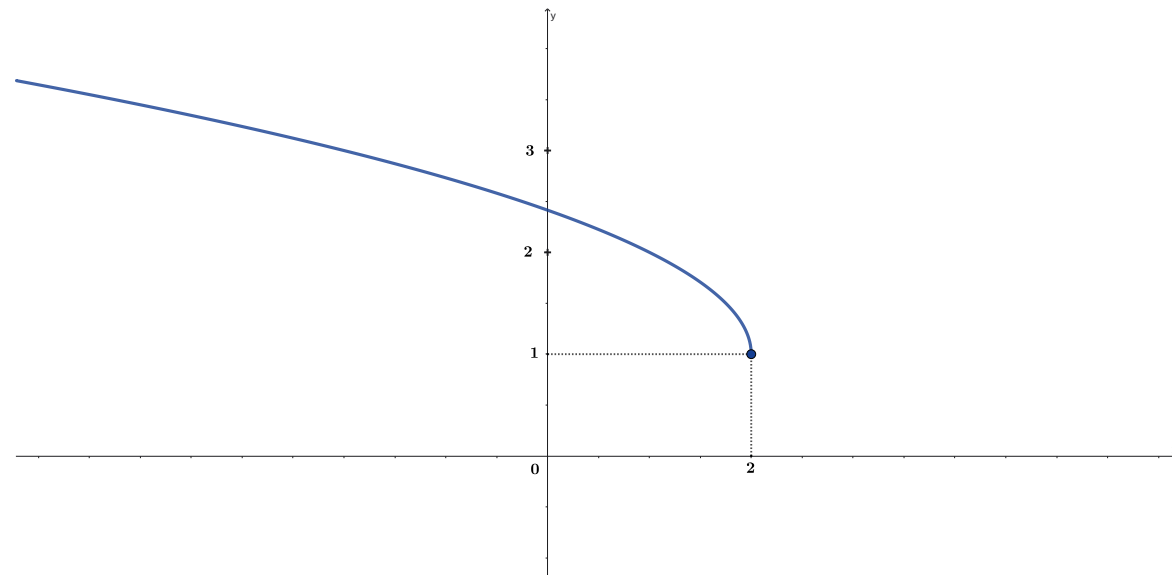
Depending on the value of a we distinguish the following cases:

- $a = 0$. This is a constant function $f(x) = 1$ for $x \neq 0$.
- $a \in \mathbb{N} \setminus \{0\}$. The power function with a natural exponent is defined $\forall x \in \mathbb{R}$. If a is an even number, it is an even function, which is decreasing in $(-\infty, 0)$ and increasing in $(0, \infty)$. If a is odd, it is an odd and increasing function.
- $a = -r: r \in \mathbb{N}$. Then there is $f(x) = \frac{1}{x^r}$. The function is defined for $x \neq 0$.
- $a = \frac{1}{r}: r \in \mathbb{N}$. Then there is $f(x) = x^{\frac{1}{r}} = \sqrt[r]{x}$. The function is defined in the interval $[0, \infty)$ for r even and in the interval $(-\infty, \infty)$ for r odd. It is an increasing function.
- $a \in \mathbb{Q}$ and at the same time it is none of the cases above. Then $f(x) = x^{\frac{p}{q}} = (x^p)^{\frac{1}{q}} = \sqrt[q]{x^p}$. For $\frac{p}{q} > 0$, and q even the function f is defined for $x \in [0, \infty)$, for $\frac{p}{q} > 0$, and q odd, the function f is defined for $x \in (-\infty, \infty)$. For $\frac{p}{q} < 0$ and q even the function f is defined for $x \in (0, \infty)$; for $\frac{p}{q} > 0$ and q odd the function f is defined for $x \in (-\infty, 0) \cup (0, \infty)$.
- For a irrational is the power function defined in the interval $(0, \infty)$ for $a > 0$ and in the interval $(0, \infty)$ for $a < 0$.

Exercise

Exercise 1. Find the domain of definition and range of values of $f(x) = \sqrt{2-x} + 1$.

Solution. In order to find the domain of definition it is enough to find out for which x there is $2-x \geq 0$ because the expression $2-x$ is under the square root. There is $D(f) = (-\infty, 2]$. Since the definition of the function is the sum of a square root and number 1, the function values are at least 1. Therefore there is $H(f) = [1, \infty)$. The graph of the function is given in the figure below:



The dot at the point $[2, 1]$ suggests that $x = 2$ belongs to the domain of definition of the function. If there was $2 \notin D(f)$, we would use the symbol $\circ$ at $[2, 1]$.

Exercise

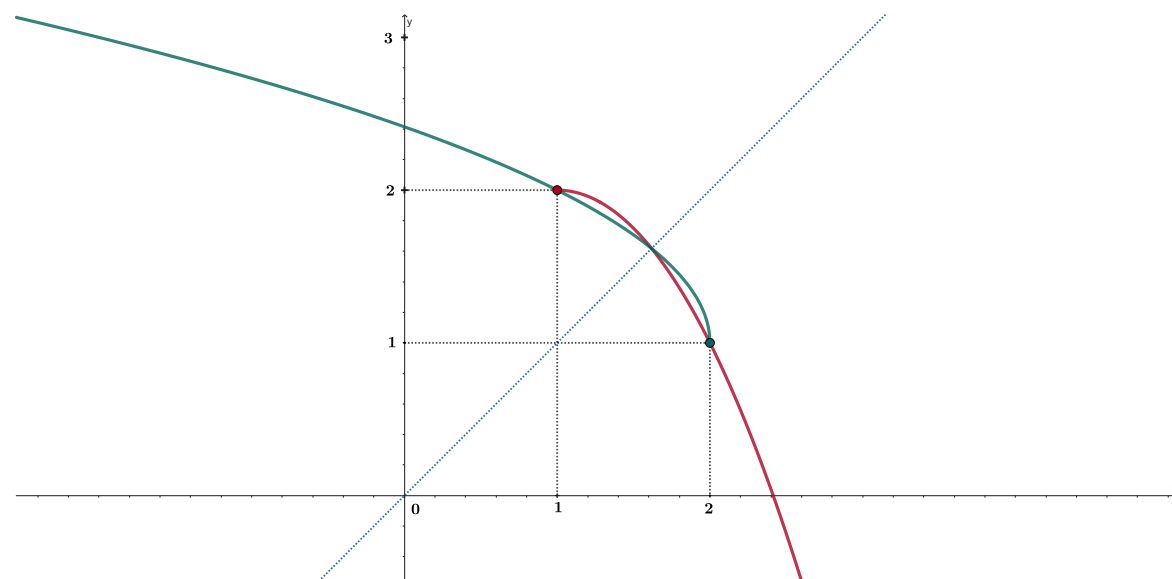
Exercise 2. Find the inverse function f^{-1} to the function $f(x) = \sqrt{2-x} + 1$. Sketch both functions in the same figure.

Solution. From the previous exercise we know that f is injective. Therefore, its inverse does exist. In order to obtain its definition, we swap x and y in the definition of the function f and express y .

$$x = \sqrt{2-y} + 1 \Rightarrow x - 1 = \sqrt{2-y} \Rightarrow (x - 1)^2 = 2 - y \Rightarrow y = 2 - (x - 1)^2.$$

As we can see, we obtained a quadratic function, which means that the function is not injective. Therefore, we have to return to the previous exercise and realize that $D(f) = (-\infty, 2] = H(f^{-1})$ and $H(f) = [1, \infty) = D(f^{-1})$. This restriction of the domain of definition of the quadratic function provides that the function f^{-1} will be injective and therefore inverse to f . Therefore, the definition of the inverse function is: $f^{-1}(x) = 2 - (x - 1)^2 \wedge x \in [1, \infty)$. Without this restriction, the function would not be inverse to f .

The graphs of both functions including the symmetry axis can be seen in the figure. The function f is green, its inverse f^{-1} is red.



Exercise

Exercise 3. Find the domain of definition of $f(x) = \sqrt{\frac{x-1}{2-x}}$.

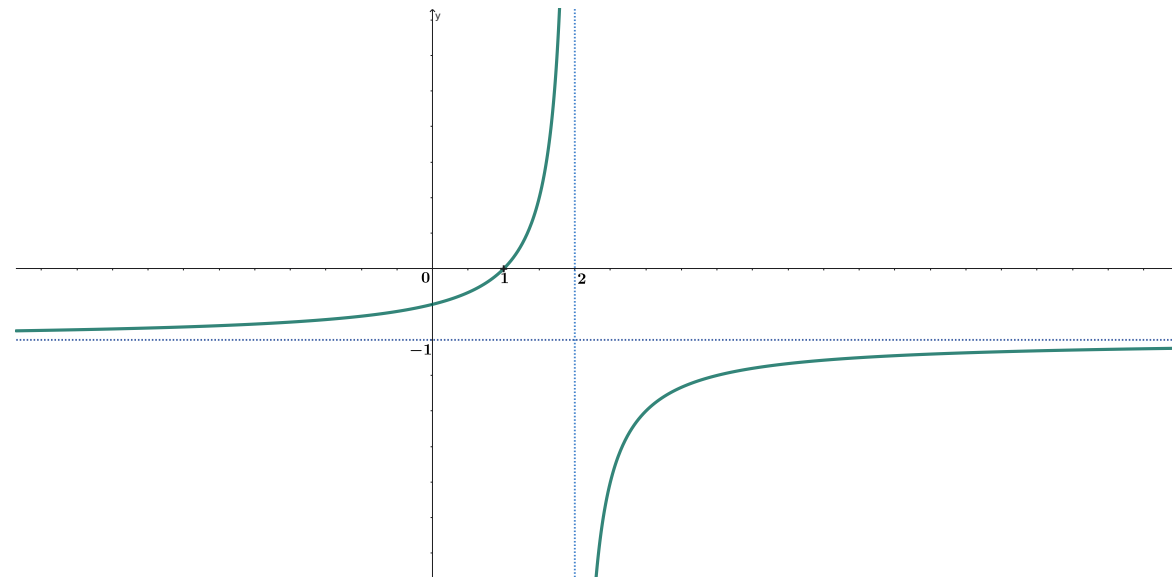
Solution. In order to find the domain of definition it is enough to find out for which x there is $\frac{x-1}{2-x} \geq 0$ because the expression $\frac{x-1}{2-x}$ is under the square root.

$$\frac{x-1}{2-x} \geq 0 \iff (x-1 \geq 0 \wedge 2-x > 0) \vee (x-1 \leq 0 \wedge 2-x < 0).$$

We discuss both conditions.

- $x-1 \geq 0 \wedge 2-x > 0 \iff x \geq 1 \wedge x < 2 \iff x \in [1, \infty) \wedge x \in (-\infty, 2) \iff x \in [1, 2).$
- $x-1 \leq 0 \wedge 2-x < 0 \iff x \leq 1 \wedge x > 2 \iff x \in (-\infty, 1] \wedge x \in (2, \infty) \iff x \in \emptyset.$

After uniting the sets $[1, 2)$ and $\emptyset$, we get the set $[1, 2)$, which means that $D(f) = [1, 2)$. We could have found the domain of definition also graphically, with the help of the graph of $g(x) = \frac{x-1}{2-x}$, which is the internal component of f . The graph is as follows:



The function values of this function are non-negative only in $[1, 2)$. As a result, the function f is defined only for $x \in [1, 2)$.

Exercise

Exercise 4. Find the domain of definition for $g(x) = \frac{\sqrt{x-1}}{\sqrt{2-x}}$.

Solution. In order to find the domain of definition it is enough to find out for which x there is $x-1 \geq 0 \wedge 2-x > 0$ because both expressions are under the square root.

$$x-1 \geq 0 \wedge 2-x > 0 \iff x \geq 1 \wedge x < 2 \iff x \in [1, \infty) \wedge x \in (-\infty, 2) \iff x \in [1, 2).$$

As a result, the domain of definition is $D(g) = [1, 2)$.

Remark 5. The functions g and f from the previous exercise have the same domain of definition. Moreover, for all $x \in D(f)$, i.e. also for all $x \in D(g)$, there is $f(x) = g(x)$. This means that the two functions are equal.

Exercise

Exercise 6. Find the domain of definition of $f(x) = \sqrt{\frac{|x|-1}{2-x}}$.

Solution. In order to find the domain of definition it is enough to find out for which x there is $\frac{|x|-1}{2-x} \geq 0$ because the expression $\frac{|x|-1}{2-x}$ is under the square root.

$$\frac{|x|-1}{2-x} \geq 0 \iff (|x|-1 \geq 0 \wedge 2-x > 0) \vee (|x|-1 \leq 0 \wedge 2-x < 0).$$

We discuss both conditions:

$$\bullet \quad |x|-1 \geq 0 \wedge 2-x > 0 \iff |x| \geq 1 \wedge x < 2 \iff (x \in (-\infty, -1] \cup [1, \infty)) \wedge (x \in (-\infty, 2)), \text{ i.e.}$$

$$x \in (-\infty, -1] \cup [1, 2).$$

$$\bullet \quad |x|-1 \leq 0 \wedge 2-x < 0 \iff |x| \leq 1 \wedge x > 2 \iff x \in [-1, 1] \wedge x \in (2, \infty) \iff x \in \emptyset.$$

After uniting the sets $(-\infty, -1] \cup [1, 2)$ and $\emptyset$, we get the set $(-\infty, -1] \cup [1, 2)$. Therefore, $D(f) = (-\infty, -1] \cup [1, 2)$.

Exercise

Exercise 7. Find the domain of definition of $f(x) = \sqrt{\frac{x^2-3x+2}{x+2}}$.

Solution. In order to find the domain of definition it is enough to find out, for which x there is $\frac{x^2-3x+2}{x+2} \geq 0$ because the expression $\frac{x^2-3x+2}{x+2}$ is under the square root.

$$\frac{x^2-3x+2}{x+2} \geq 0 \iff (x^2-3x+2 \geq 0 \wedge x+2 > 0) \vee (x^2-3x+2 \leq 0 \wedge x+2 < 0).$$

We discuss both conditions:

$$(x^2-3x+2 \geq 0 \wedge x+2 > 0) \iff (x-1) \cdot (x-2) \geq 0 \wedge x+2 > 0,$$

which means that

$$x \in (-\infty, 1] \cup [2, \infty) \wedge x \in (-2, \infty),$$

i.e.

$$x \in (-2, 1] \cup [2, \infty).$$

The second condition:

$$(x^2-3x+2 \leq 0 \wedge x+2 < 0) \iff (x-1) \cdot (x-2) \leq 0 \wedge x+2 < 0,$$

which means that

$$x \in [-1, 1] \wedge x \in (-\infty, -2) \iff x \in \emptyset.$$

After uniting the sets $(-2, 1] \cup [2, \infty)$ and $\emptyset$, we get the set $(-2, 1] \cup [2, \infty)$, which means that $D(f) = (-2, 1] \cup [2, \infty)$.

Equations with (square) roots

When solving equations with (square) roots, it is imperative to keep in mind that taking roots is not an equivalent transformation, which means that false solutions may appear. For example, when we want to remove the square root in the equation

$$\sqrt{x-2} = x-4,$$

we write the second powers of both sides:

$$x-2 = (x-4)^2 \Rightarrow x-2 = x^2-8x+16 \Rightarrow x^2-9x+18 = 0 \Rightarrow (x-3)(x-6) = 0.$$

The solutions of the last equation are $x_1 = 3$ and $x_2 = 6$. However, after substituting $x_1 = 3$ for x in the original equation, we get $\sqrt{3-2} = 1$ on the left-hand side and $3-4 = -1$ on the right-hand side, which is not true. As a result, $x_1 = 3$ is not a solution of the original equation. *Why has this false solution appeared?*

It is writing the second powers that caused the problem. Obviously, there is $1 \neq -1$. Yet the second powers of -1 and 1 are the same. We could have identified this problem if we had discussed the solvability conditions. We can take the second root of non-negative numbers only, which results in condition $x-2 \geq 0 \iff x \in [2, \infty)$. Since second roots are always nonnegative, the right-hand side must be nonnegative as well, and there is $x-4 \geq 0 \iff x \in [4, \infty)$. Both conditions must hold simultaneously. Therefore, there is $x \in [4, \infty)$. As a result, $x_1 = -3$ is a false root.

Therefore, it is necessary to either verify the results by substitution or clarify the conditions of solvability in advance and check the solutions against those conditions.

Exercise

Exercise 8. Solve the equation $2\sqrt{x+5} = x+2$.

Solution. We will show two ways of finding the solution:

- We raise both sides of the equation to power two and get:

$$4(x+5) = (x+2)^2,$$

which is equivalent to

$$4x + 20 = x^2 + 4x + 4,$$

$$x^2 - 16 = 0 \iff |x| = 4 \iff x = 4 \vee x = -4.$$

Since powering is not an equivalent transformation, we need to perform a verification test for $x = 4$ and for $x = -4$.

For $x = 4$ we get:

$$L = 2\sqrt{4+5} = 2 \cdot 3 = 6 \wedge R = 4 + 2 = 6,$$

which means that $L = R$.

For $x = -4$ we get:

$$L = 2\sqrt{-4+5} = 2 \cdot 1 = 2 \wedge R = -4 + 2 = -2,$$

which means that $L \neq R$.

Therefore, the only solution of the equation is $x = 4$. Why has $x = -4$ failed the test? The reason is simple: the second powers of -2 and 2 are the same yet there is $2 \neq -2$.

- We start by finding conditions under which raising the equation to the second power is an equivalent transformation. Since there is $x+5$ under the square root, there must be $x \in [-5, \infty)$. Therefore, the left-hand side of the equation is nonnegative for all $x \in [-5, \infty)$. As a result, the right-hand side, i.e. the expression $x+2$, must be nonnegative as well. This means that $x \in [-2, \infty)$. Intersecting these two sets results in the interval $[-2, \infty)$. Only now can we raise the equation to the second power:

$$4(x+5) = (x+2)^2.$$

This is equivalent to

$$4x + 20 = x^2 + 4x + 4,$$

$$x^2 - 16 = 0 \iff |x| = 4 \iff x = 4 \vee x = -4.$$

Since we work in interval $[-2, \infty)$ and since there is $-4 \notin [-2, \infty)$, the only solution of the equation is $x = 4$. Thus we have obtained the same result as above.

Inequations with (square) roots

When solving inequations of this type we must keep in mind that the expression under the (square) root must be nonnegative. We must also pay attention to raising to the (second) power because false solutions may appear as a result.

For example, the inequation $\sqrt{x-2} < -1$ has no solution because the square root is always non-negative. However, after raising the inequation to the second power we would get $x-2 < 1$, i.e. $x < 3$ which would – with respect to the condition $x-2 \geq 0$ – imply the existence of a (wrong) solution $x \in [2, 3)$. Therefore, it is necessary to clarify the signs of both sides of the inequation. For example, in the inequation

$$\sqrt{x-2} < x-4,$$

we conclude that there must be $x-2 \geq 0$, which means that $x \in [2, \infty)$. The expression on the right-hand may be nonnegative as well as negative. If there is $x-4 < 0$, i.e. $x \in (-\infty, 4)$, the inequation has no solution. For $x \in [4, \infty)$ there is $x-4 \geq 0$. For all $x \in [4, \infty)$ the expression on the left-hand side is defined as well, which means that for such x we can raise the inequation to the second power. Thus we get

$$x-2 < (x-4)^2 \Rightarrow x-2 < x^2-8x+16 \Rightarrow x^2-9x+18 > 0 \Rightarrow (x-3) \cdot (x-6) > 0.$$

The solution of the last inequation is $x \in (-\infty, 3) \cup (6, \infty)$. However, since we raised the inequation for all $x \geq 4$, the solution of the inequation is $x \in [(-\infty, 3) \cup (6, \infty)] \cap [4, \infty)$, i.e. $x \in (6, \infty)$.

Exercise

Exercise 9. Find the domain of definition of $f(x) = \sqrt{x + \sqrt{x + 2}}$.

Solution. First of all we find conditions for both square roots:

$$(x + 2 \geq 0) \wedge (x + \sqrt{x + 2} \geq 0).$$

From the first condition there follows: $x \geq -2 \Rightarrow x \in [-2, \infty)$.

The second condition will be somewhat more interesting:

$$x + \sqrt{x + 2} \geq 0$$

If we raised the inequation to the second power in this form, we would not manage to remove the square root. Therefore, we adjust it: $x \geq -\sqrt{x + 2}$.

Obviously, there is $-\sqrt{x + 2} \leq 0$ for all $x \in [-2, \infty)$. On the left-hand side there is x . Therefore, before we raise the inequation to the second power, we must discuss two cases: $x \geq 0$ or $x < 0$.

- Suppose that $x \geq 0$:

$$(x \geq 0) \wedge (-\sqrt{x + 2} \leq 0).$$

Then

$$x \geq -\sqrt{x + 2} \text{ for } \forall x \in [0, \infty).$$

- Suppose that $x < 0$. Then the raising to the second power turns the inequation

$$x \geq -\sqrt{x + 2}$$

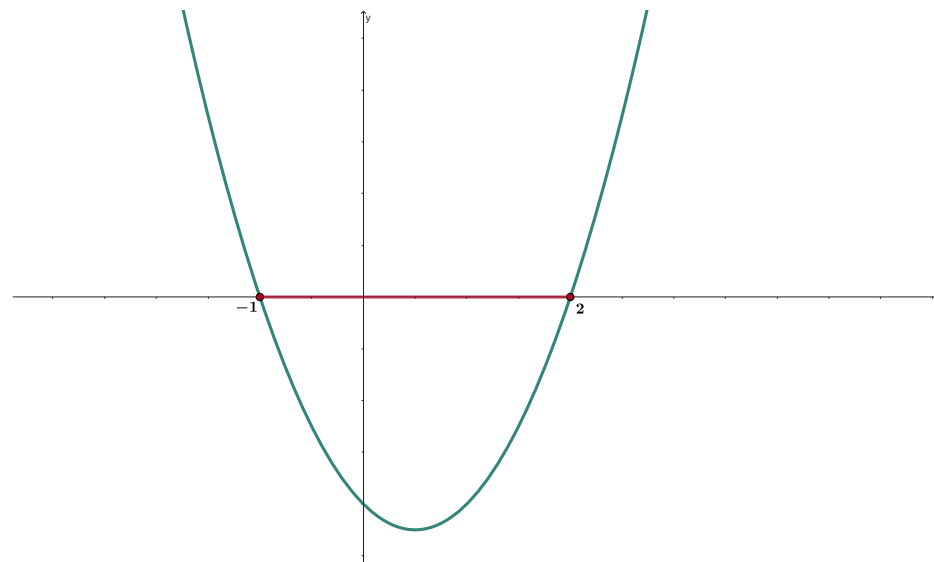
into

$$x^2 \leq x + 2,$$

(Why?) Then

$$(x - 2)(x + 1) \leq 0 \iff x \in [-1, 2]$$

and the interval can be read from the figure:



Since there is $x < 0$, we get: $x \in [-1, 0)$.

- The condition $x + \sqrt{x + 2} \geq 0$ results into the interval $[0, \infty) \cup [-1, 0) = [-1, \infty)$.

Finally, we make the intersection of our partial results and get:

$$x \in [-2, \infty) \cap [-1, \infty) \Rightarrow x \in [-1, \infty).$$

Exercise

Exercise 10. Find the domain of definition of $f(x) = \sqrt{x - \sqrt{x+2}}$.

Solution. First of all we find conditions for both square roots:

$$(x+2 \geq 0) \wedge (x - \sqrt{x+2} \geq 0).$$

From the first condition there follows: $x \geq -2 \Rightarrow x \in [-2, \infty)$.

The second condition will be somewhat more interesting:

$$x - \sqrt{x+2} \geq 0.$$

If we raised the inequation to the second power in this form, we would not manage to remove the square root. Therefore, we adjust it to $x \geq \sqrt{x+2}$.

Obviously, there is $\sqrt{x+2} \geq 0$ for all $x \in [-2, \infty)$. On the left-hand side there is x . Therefore, before we raise the inequation to the second power, we must discuss two cases: $x \geq 0$ or $x < 0$.

- Suppose that $x < 0$:

$$(x < 0) \wedge (\sqrt{x+2} \geq 0),$$

which means that the condition $x \geq \sqrt{x+2}$ will not hold for any $x < 0$.

- Suppose that $x \geq 0$: Raising

$$x \geq \sqrt{x+2}$$

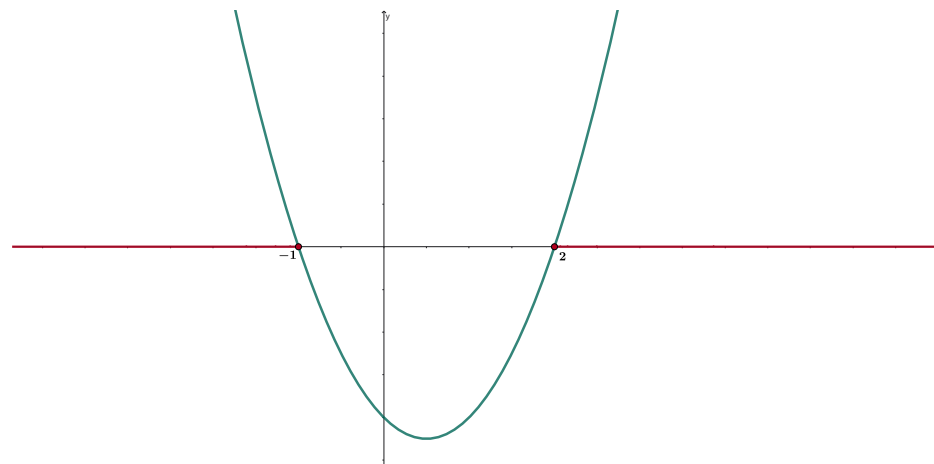
to the second power results in

$$x^2 \geq x+2.$$

Then

$$(x-2)(x+1) \geq 0 \iff x \in (-\infty, -1] \cup [2, \infty),$$

and the interval can be read from the figure:



Since there is $x \geq 0$, we get: $x \in [2, \infty)$.

- The condition $(x - \sqrt{x+2} \geq 0)$ results into the interval

$$[2, \infty) \cup \emptyset = [2, \infty).$$

Finally, we make the intersection of our partial results and get:

$$x \in [-2, \infty) \cap [2, \infty) \Rightarrow x \in [2, \infty).$$

1 TEST

1. The domain of definition of $f(x) = 1 - \sqrt{3 - x}$ is:
 - (a) $(-\infty, 3)$
 - (b) $(-\infty, 3]$
 - (c) $[3, \infty)$
 - (d) $(-\infty, -3)$
2. The inverse of $f(x) = 1 - \sqrt{3 - x}$ is defined by:
 - (a) $y = 3 - (1 - x)^2$
 - (b) $y = 3 - (1 - x)^2$ for $x \in (-\infty, -1]$
 - (c) $y = 3 - (1 - x)^2$ for $x \in [0, \infty)$
 - (d) the inverse does not exist
3. The domain of definition of $f(x) = \sqrt{x - \sqrt{1 - x}}$ is:
 - (a) $\left[\frac{\sqrt{5}-1}{2}, 1\right]$
 - (b) $\left[\frac{\sqrt{5}-1}{2}, \infty\right)$
 - (c) $\left[-\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}\right]$
 - (d) $\emptyset$
4. The domain of definition of $f(x) = \sqrt{1 - x} + \sqrt{2x - 3}$ is:
 - (a) $\left[1, \frac{3}{2}\right]$
 - (b) $\emptyset$
 - (c) $\{1\}$
 - (d) $\left[\frac{2}{3}, 1\right]$
5. The domain of definition of $f(x) = \sqrt{\frac{3-x}{2x+5}}$ is:
 - (a) $\left[-\frac{5}{2}, 3\right]$
 - (b) $[3, \infty)$
 - (c) $\left(-\frac{5}{2}, 3\right]$
 - (d) $\left(-\frac{5}{2}, \infty\right)$
6. The functions $f(x) = \sqrt{\frac{1-x}{2+x}}$ and $g(x) = \frac{\sqrt{1-x}}{\sqrt{2+x}}$ are equal for:
 - (a) $(-2, 1)$
 - (b) $(-2, 1)$
 - (c) $[-2, 1)$

(d) $[-2, 1]$

7. The functions $f(x) = \sqrt{\frac{1-x}{x+2}}$ and $g(x) = \sqrt{\frac{1-x}{|x+2|}}$ are equal for:

(a) $(-\infty, 1)$

(b) $(-\infty, 1]$

(c) $(-\infty, 1] \setminus \{-2\}$

(d) $(-\infty, -2)$

8. The domain of definition of $f(x) = \sqrt{x + \sqrt{1-x}}$ is:

(a) $\left[\frac{-1-\sqrt{5}}{2}, 1\right]$

(b) $\left[\frac{-1-\sqrt{5}}{2}, \infty\right)$

(c) $\emptyset$

(d) $\left[\frac{-1-\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right]$

9. The domain of definition of $f(x) = \sqrt{\frac{x^2-x-2}{x-2}}$ is:

(a) $(-1, 2) \cup (2, \infty)$

(b) $(-1, \infty)$

(c) $(2, \infty)$

(d) $[2, \infty)$

10. The solution of $4 + 2\sqrt{x-4} = x$ is:

(a) $\{8\}$

(b) $\{4, 8\}$

(c) $\{0, 4\}$

(d) $\{4\}$

6. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

Exponencial functions

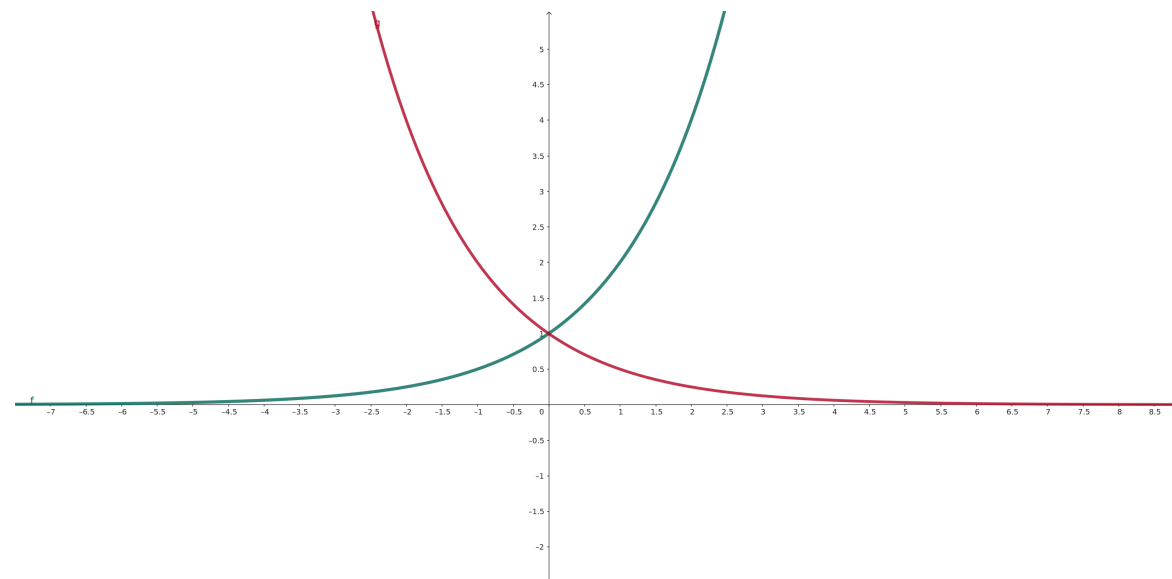
By an exponential function we mean each function f defined, for all $a \in \mathbb{R}$, by

$$f(x) = a^x,$$

where $a, x \in \mathbb{R}, a > 0, a \neq 1$. Number a is called the base.

Why do we impose some restricting conditions on the value of the base? If there was $a = 1$, we would get a constant function, which we do not regard as an exponential one. Therefore, we assume that $a \neq 1$. For the same reason there must be $a \neq 0$. Since exponential functions are defined for all real numbers and since non-integer exponents may be applied without restrictions on positive numbers only, the base must be positive.

If the value of a is such that $a \in (0, 1)$, the function f is a decreasing one. If there is $a \in (1, \infty)$, the function f is an increasing one. In both cases, the functions are injective. The graph of an exponential function is called the exponential. In the figure you can see the graph of $f(x) = 2^x$ (in green) and the graph of $g(x) = \left(\frac{1}{2}\right)^x$ in red.



The set of all real numbers is the domain of definition of all exponential functions while the interval $(0, \infty)$ is their range of values. Recall the basic relations between powers

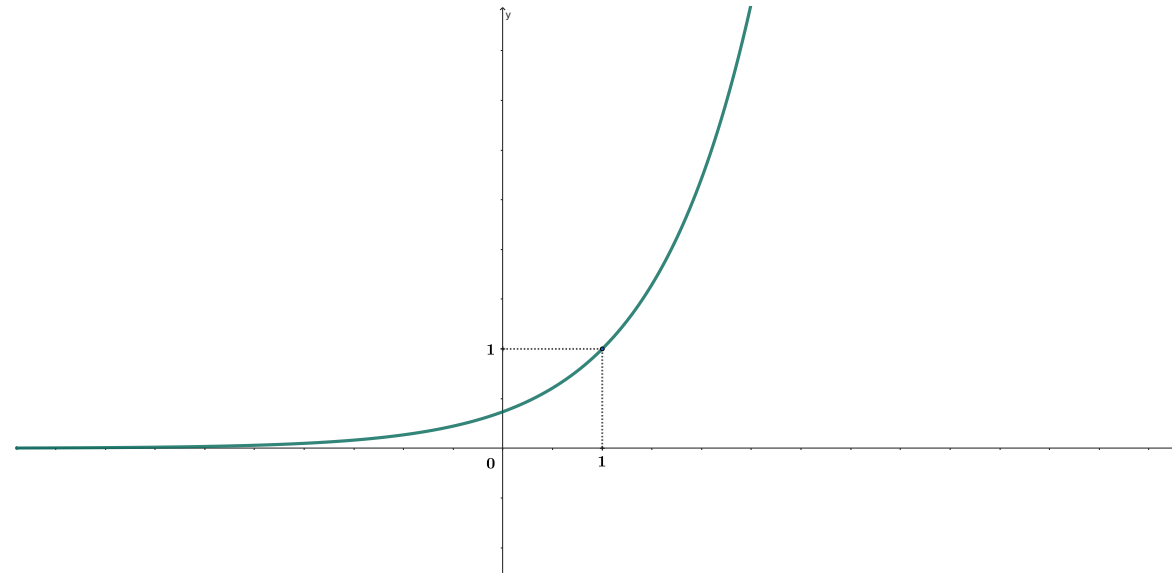
$$\begin{aligned} a^{x+y} &= a^x \cdot a^y, \\ (a^x)^y &= a^{x \cdot y}, \end{aligned}$$

which we will be using during the process of solution of exponential equations and inequations. Moreover, properties of exponential functions will be useful as well.

Exercise

Exercise 1. Sketch the graphs of $f(x) = e^{x-1}$ and $g(x) = e^{|x-1|}$.

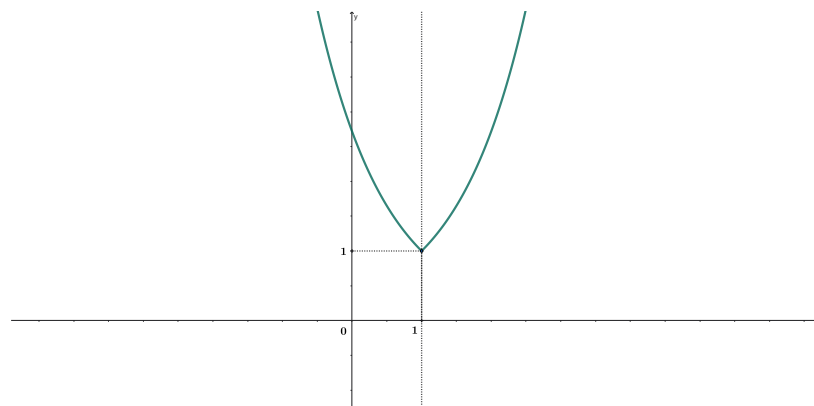
Solution. The domain of definition of both functions is $D(f) = D(g) = \mathbb{R}$. We know how to sketch the graph of $h(x) = e^x$. The function f is shifted to the right. Its value at $x = 1$ is the same as the value of h at $x = 0$. The graph of f is:



When sketching the graph of the function g , we must keep in mind that, in the interval $[1, \infty)$, it is identical with the function f . After we remove the absolute value, the function g is defined, in the interval $(-\infty, 1)$, as follows:

$$g(x) = e^{1-x} : x \in (-\infty, 1).$$

Therefore, its graph must be symmetric to its graph in $[1, \infty)$. The graph of the function g is given below:



Exercise

Exercise 2. Find the domain of definition and range of values of $f(x) = 2\frac{1}{\sqrt{x-2}}$.

Solution. In order to find the domain of definition it is sufficient to find all $x \in \mathbb{R}$ such that $x - 2 > 0$ because the expression $x - 2$ is under the square root and in the denominator. We solve the inequation

$$x - 2 > 0 \iff x > 2$$

Therefore, there is

$$x \in (2, \infty).$$

As a result $D(f) = (2, \infty)$. Identifying the range of values is not so straightforward. The expression $\sqrt{x-2}$ attains all positive real values for all $x \in D(f)$. This means that its inverted value, i.e. $\frac{1}{\sqrt{x-2}}$ attains all positive values as well. Therefore, the values of f include all values greater than 1. As a result, there is $H(f) = (1, \infty)$.

Exercise

Exercise 3. In the set of real numbers solve the equation $2^{x-1} + 2^{x-2} + 2^{x-3} = 224$.

Solution. Since all the expressions on both the left and right-hand sides are defined for all real numbers, we will solve the equation without any restricting conditions. When transforming the equation, we will make use of the following formula:

$$2^{x-1} = 2 \cdot 2^{x-2}, \quad 2^{x-1} = 2^2 \cdot 2^{x-3}.$$

There is

$$2^{x-1} + 2^{x-2} + 2^{x-3} = 224 \iff 2^{x-3}(4 + 2 + 1) = 224,$$

which is equivalent to

$$2^{x-3} \cdot 7 = 224 \iff 2^{x-3} \cdot 7 = 32 \cdot 7 \iff 2^{x-3} = 32 \iff 2^{x-3} = 2^5.$$

Obviously, there is $x = 8$, which is the only solution of the equation.

Exercise

Exercise 4. In the set of real numbers solve the inequation $\frac{2^{x-1}-1}{2^{x+1}+1} < 0$.

Solution. Since the inequation contains a fraction, we need to verify conditions under which it is defined. The values of the expression 2^{x+1} are positive for all real x , which means that there is $2^{x+1} + 1 > 0$. Therefore, we can solve the inequation on the whole set of real numbers. Since the right-hand side of the inequation is 0, we write the solution of the inequation in the following way:

$$\frac{2^{x-1}-1}{2^{x+1}+1} < 0 \iff (2^{x-1}-1 > 0 \wedge 2^{x+1}+1 < 0) \vee (2^{x-1}-1 < 0 \wedge 2^{x+1}+1 > 0).$$

The conjunction in the first brackets is impossible because there is $2^{x+1} + 1 > 0$ for all real numbers. Therefore, we focus on the second brackets:

$$(2^{x-1}-1 < 0 \wedge 2^{x+1}+1 > 0) \iff (2^{x-1} < 1) \iff (2^{x-1} < 2^0).$$

The monotonicity of the exponential function results in

$$x-1 < 0 \iff x < 1.$$

Therefore, the solutions of the inequation is the interval $(-\infty, 1)$.

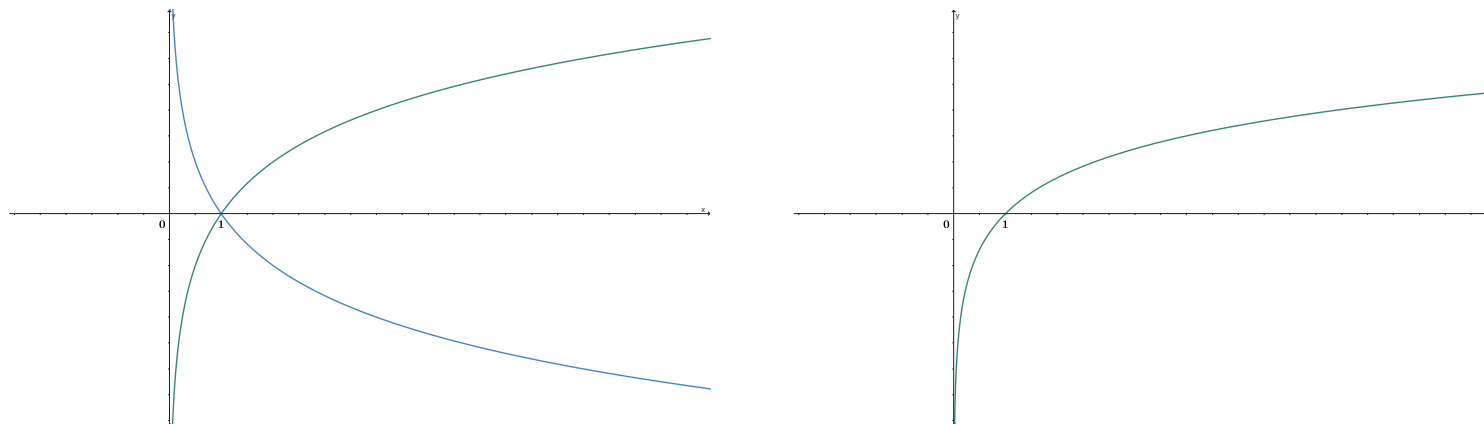
Logarithmic functions

By a logarithmic function f with the base a we mean the inverse of an exponential function $g(x) = a^x$. We write

$$f(x) = \log_a x,$$

where a is a positive number different from 1.

Given the facts that logarithmic functions are inverses of exponential functions and that the domain of definition of exponential functions is the set of all real numbers while their range of values is the interval $(0, \infty)$, we easily conclude that the domain of definition of logarithmic functions is the interval $(0, \infty)$ while their range of values is the set of all real numbers. Monotonicity of logarithmic functions is the same as monotonicity of exponential functions. They decrease for $a \in (0, 1)$ and increase for $a \in (1, \infty)$. In both cases, the functions are always injective. In the left figure you can see the graph of $f(x) = \log_2 x$ (green) and $g(x) = \log_{\frac{1}{2}} x$ (red). In the right figure there is the graph of logarithmic function with base e, called the Euler's number. Such a logarithm is called the natural logarithm. We denote it $\ln$.



For $x, x_1, x_2 \in (0, \infty)$, $r \in \mathbb{R}$, $a, b \in (0, \infty) \setminus \{1\}$ there is

$$\log_a (x_1 x_2) = \log_a x_1 + \log_a x_2,$$

$$\log_a \left(\frac{x_1}{x_2} \right) = \log_a x_1 - \log_a x_2,$$

$$\log_a x^r = r \log_a x,$$

$$\log_a a = 1,$$

$$\log_a 1 = 0.$$

We will use these relations, together with the properties of logarithmic functions, when looking for solutions of logarithmic equations and inequations.

Exercise

Exercise 5. Find the domain of definition, range of values and the inverse of $f(x) = 2e^{x-2}$.

Solution. Finding the domain of definition and the range of values of the function f is simple. The expression $2e^{x-2}$ is defined for all real numbers, which means that $D(f) = \mathbb{R}$. Since the function $y = e^x$ attains values from the set $(0, \infty)$ and our function f is only its double shifted to the right, there is $H(f) = (0, \infty)$ as well. Further, we need to find out whether f is an injective function. As we already know, the definition of the injective function can be written as:

$$\forall a, b \in D(f): f(a) = f(b) \Rightarrow a = b.$$

We will assume that for arbitrary $a, b \in D(f)$ there is $f(a) = f(b)$, which means that

$$2e^{a-2} = 2e^{b-2} \iff e^{a-2} = e^{b-2} \iff a - 2 = b - 2 \iff a = b.$$

As a result, f is injective and it has an inverse. For this inverse f^{-1} there is $D(f^{-1}) = (0, \infty)$ and $H(f^{-1}) = \mathbb{R}$. Finally, we find the definition of the inverse. As usually, we first swap x and y in the definition of f

$$x = 2e^{y-2}.$$

After applying logarithms to both sides, the equality will remain valid:

$$\ln x = \ln 2e^{y-2}.$$

We apply the fact that there is $\ln xy = \ln x + \ln y$ and get

$$\ln x = \ln 2 + \ln e^{y-2}.$$

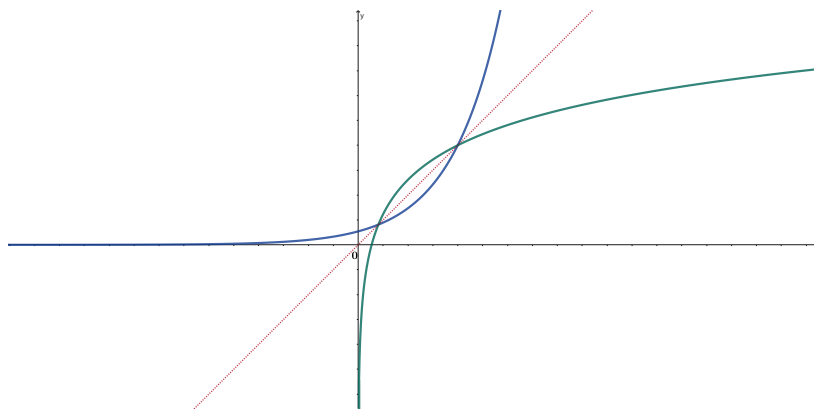
Since there is $\ln x^\alpha = \alpha \cdot \ln x$ and $\ln e = 1$, we get that

$$\ln x = \ln 2 + (y - 2) \cdot \ln e \iff \ln x = \ln 2 + y - 2 \iff \ln x - \ln 2 + 2 = y.$$

As a result

$$y = \ln \frac{x}{2} + 2.$$

In the figure below you can see the original function f (blue), its inverse (green) and the symmetry axis $y = x$.



Exercise

Exercise 6. Sketch the graphs of $f(x) = \ln(x - 3)$ and $g(x) = \ln|x - 3|$.

Solution. In order to identify the domain of definition of the function f it is sufficient to find all $x \in \mathbb{R}$ such that $x - 3 > 0$ because $x - 3$ is the argument of a logarithm. Therefore, there is $D(f) = (3, \infty)$. It is also useful to know the exact intersections with the coordinate axes x and y . Given the domain of definition, the function will have no intersections with the y -axis. In order to find the intersection of the function with the x -axis we need to solve the equation

$$\ln(x - 3) = 0,$$

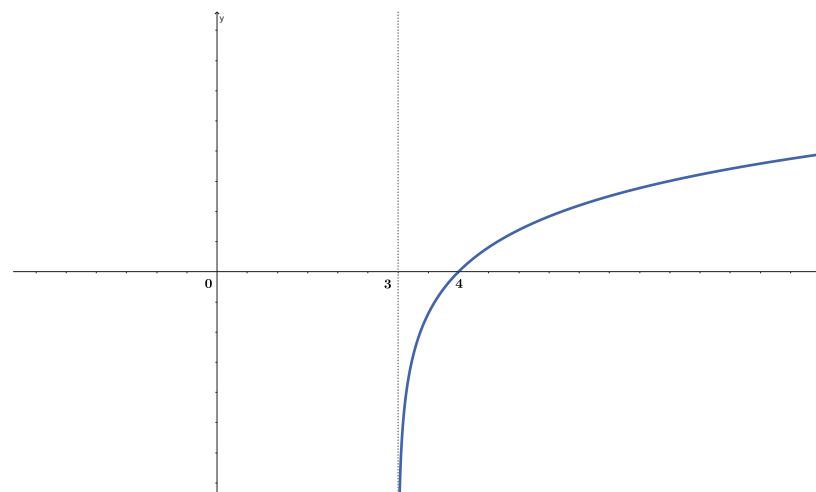
which is equivalent to

$$x - 3 = 1.$$

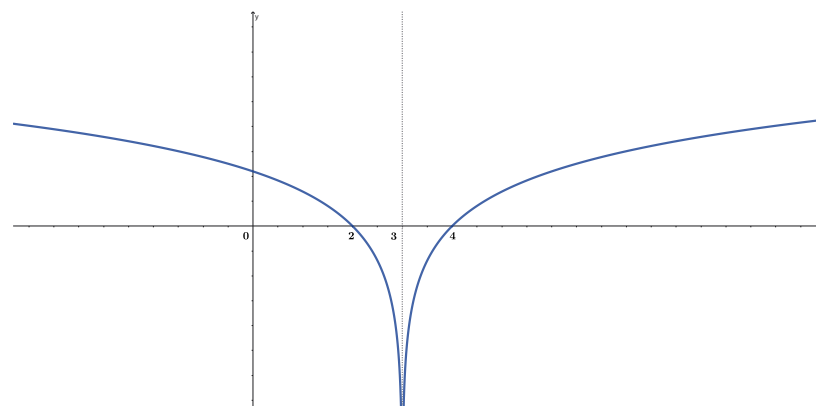
Therefore, there is

$$x = 4.$$

The graph of the function f is



Regarding the function g we must keep in mind that in the interval $(3, \infty)$ it is identical to the function f . However, the function g is defined also in $(-\infty, 3)$. The line $x = 3$ is a vertical asymptote of both functions as well as the symmetry axis for the function g . The graph of the function g is



Exercise

Exercise 7. Find the domain of definition and the range of values of $f(x) = \ln \frac{x}{x-1}$.

Solution. In order to identify the domain of definition of the function f it is sufficient to find all $x \in \mathbb{R}$ such that $\frac{x}{x-1} > 0$ because $\frac{x}{x-1}$ is the argument of a logarithm. We solve the inequation:

$$\frac{x}{x-1} > 0,$$

which is equivalent to

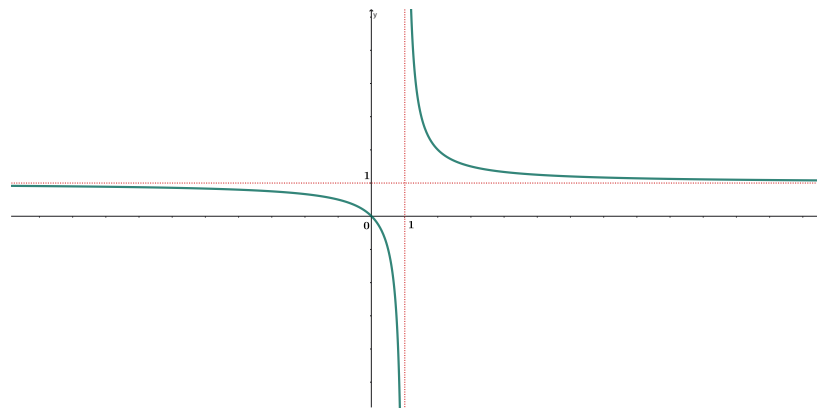
$$(x > 0 \wedge x - 1 > 0) \vee (x < 0 \wedge x - 1 < 0),$$

$$(x > 0 \wedge x > 1) \vee (x < 0 \wedge x < 1).$$

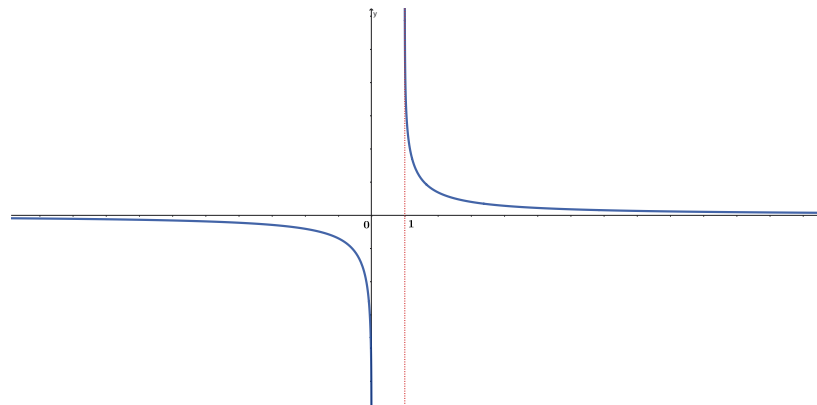
Therefore, there is

$$x \in (1, \infty) \vee x \in (-\infty, 0) \iff x \in (1, \infty) \cup (-\infty, 0).$$

This means that $D(f) = (1, \infty) \cup (-\infty, 0)$. Finding the range of values is not so straightforward. First, we sketch the graph of an auxiliary function $g(x) = \frac{x}{x-1}$ which is the internal component of f . The graph of the function g is given below:



The function g is positive in $(1, \infty) \cup (-\infty, 0)$ which is $D(f)$. Now we sketch the graph of the function f using graphs of g and of $h(x) = \ln x$. The function g has an asymptote in $+\infty$ (a line $y = 1$). The value of h at $x = 1$ is 0. This means that, at $+\infty$, the asymptote of f , which is a composition of functions g and h , will be the x -axis. In a similar way we find out that there will be a vertical asymptote at $x = 1$, there will be one more vertical asymptote at $x = 0$ and the asymptote at $-\infty$ will be the x -axis. As a result, the graph of f will be



Therefore, there is $H(f) = \mathbb{R} \setminus \{0\}$.

Exercise

Exercise 8. Find the domain of definition, range of values and the inverse of $f(x) = \ln \frac{x-2}{x+1}$.

Solution. In order to identify the domain of definition of the function f it is sufficient to find all $x \in \mathbb{R}$ such that $\frac{x-2}{x+1} > 0$ because $\frac{x-2}{x+1}$ is the argument of a logarithm. We solve the inequation:

$$\frac{x-2}{x+1} > 0,$$

which is equivalent to

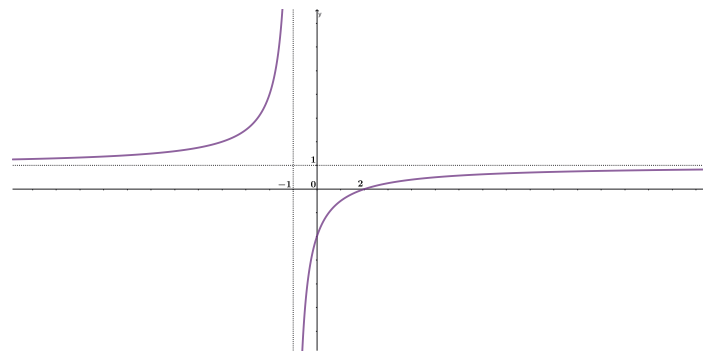
$$(x-2 > 0 \wedge x+1 > 0) \vee (x-2 < 0 \wedge x+1 < 0),$$

$$(x > 2 \wedge x > -1) \vee (x < 2 \wedge x < -1).$$

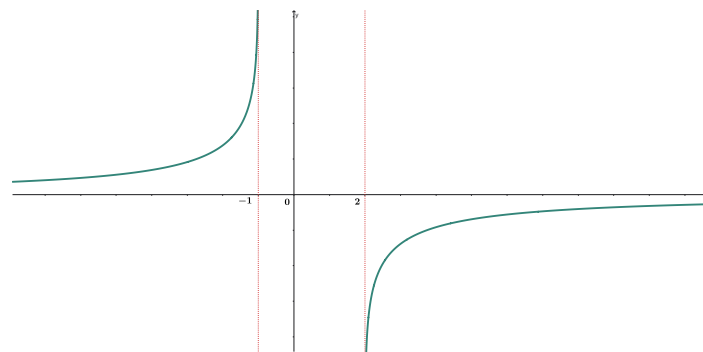
Therefore

$$x \in (2, \infty) \vee x \in (-\infty, -1) \iff x \in (2, \infty) \cup (-\infty, -1).$$

As a result $D(f) = (2, \infty) \cup (-\infty, -1)$. In order to find the range of values we can proceed in a way analogous to the previous exercise with the help of the internal function $g(x) = \frac{x-2}{x+1}$. The graph of the function g , which is



enables us to sketch the graph of the function f .



We can see that f is an injective function, which means that its inverse exists. We can also see that $H(f) = \mathbb{R} \setminus \{0\}$. Therefore, for the inverse of f , i.e. f^{-1} , we have that $D(f^{-1}) = \mathbb{R} \setminus \{0\}$ and $H(f^{-1}) = (2, \infty) \cup (-\infty, -1)$.

Exercise – continued

Now we will prove that f is indeed injective. The definition of injective functions is:

$$\forall a, b \in D(f): f(a) = f(b) \Rightarrow a = b.$$

We will assume that there is $f(a) = f(b)$ for arbitrary $a, b \in D(f)$. This means that

$$\ln \frac{a-2}{a+1} = \ln \frac{b-2}{b+1} \iff \frac{a-2}{a+1} = \frac{b-2}{b+1} \iff (a-2) \cdot (b+1) = (b-2) \cdot (a+1).$$

This expands to

$$ab + a - 2b - 2 = ab + b - 2a - 2 \iff a - 2b = b - 2a \iff 3a = 3b \iff a = b.$$

Finally, we find the definition of the inverse function in the usual way. First, we swap x and y in the definition of f

$$x = \ln \frac{y-2}{y+1}.$$

Then there obviously is:

$$e^x = e^{\ln \frac{y-2}{y+1}}.$$

Since there is $e^{\ln x} = x$, we get

$$e^x = \frac{y-2}{y+1}.$$

This expands to

$$e^x \cdot (y+1) = y-2 \iff e^x \cdot y + e^x = y-2 \iff e^x \cdot y - y = -2 - e^x.$$

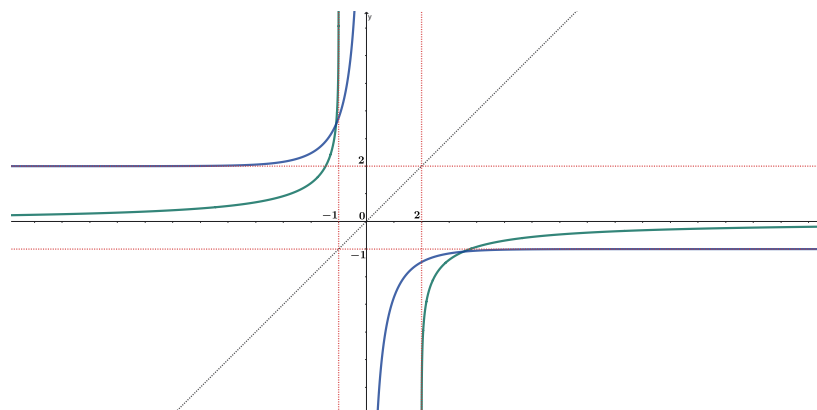
Consequently,

$$y = \frac{-2 - e^x}{e^x - 1},$$

which means that

$$y = \frac{2e^x}{1 - e^x}.$$

The figure depicts the function f (green), its inverse (blue) and their symmetry axis $y = x$.



Exercise

Exercise 9. In the real domain, find the solution of the equation $\ln(x-2) + \ln(x+5) = 2\ln x$.

Solution. Since the equation contains logarithms, we first of all find out for which $x \in \mathbb{R}$ the expressions on the left and right-hand sides are defined. Obviously, there is

$$x-2 > 0 \wedge x+5 > 0 \wedge x > 0 \iff x \in (2, \infty).$$

We will transform the equation and make use of the following property of logarithms:

$$\ln x + \ln y = \ln x \cdot y, \ln x^\alpha = \alpha \cdot \ln x.$$

Then there is

$$\ln(x-2) + \ln(x+5) = 2\ln x \iff \ln(x-2) \cdot (x+5) = \ln x^2.$$

Now we can compare arguments of logarithms on both sides of the equation:

$$(x-2) \cdot (x+5) = x^2 \iff x^2 + 3x - 10 = x^2 \iff 3x = 10 \iff x = \frac{10}{3}.$$

Obviously there is $\frac{10}{3} \in (2, \infty)$, which means that $x = \frac{10}{3}$ is the only solution.

Exercise

Exercise 10. In the real domain, find the solution of the inequation $\ln(x-2) - \ln(x+2) \leq 0$.

Solution. Since the equation contains logarithms, we first of all find out for which $x \in \mathbb{R}$ the expressions on the left and right-hand sides are defined. Obviously, there is

$$x-2 > 0 \wedge x+2 > 0 \iff x \in (2, \infty).$$

We will transform the inequation and make use of the following property of logarithms:

$$\ln x - \ln y = \ln \frac{x}{y}, \ln 1 = 0.$$

Then there is

$$\ln(x-2) - \ln(x+2) \leq 0 \iff \ln \frac{x-2}{x+2} \leq \ln 1.$$

Now we can apply the monotonicity of logarithmic functions and compare arguments of logarithms on both sides of the inequation:

$$\frac{x-2}{x+2} \leq 1 \iff \frac{x-2}{x+2} - 1 \leq 0 \iff \frac{x-2-x-2}{x+2} \leq 0 \iff -\frac{4}{x+2} \leq 0 \iff x+2 > 0.$$

The last inequality is equivalent to the fact that $x \in (-2, \infty)$. Obviously there is $(-2, \infty) \cap (2, \infty) = (2, \infty)$, which means that the interval $(2, \infty)$ is the solution.

1 TEST

1. The domain of definition of $f(x) = \ln \frac{1}{\sqrt{1-x}}$ is:
 - (a) $(-\infty, 1)$
 - (b) $(-\infty, 1]$
 - (c) $[1, \infty)$
 - (d) $(-\infty, 0) \cup (0, 1)$.
2. The definition of the inverse of $f(x) = 1 - \ln(3 - x)$ is:
 - (a) $y = 3 - \ln(1 - x)$
 - (b) $y = 3 + \ln(1 - x)$
 - (c) $y = 3 - e^{1-x}$
 - (d) the inverse does not exist
3. The domain of definition of $f(x) = \sqrt{\ln \frac{x-1}{x+2}}$ is:
 - (a) $(-\infty, -2]$
 - (b) $(-\infty, 2)$
 - (c) $(-\infty, -2)$
 - (d) $\emptyset$
4. The domain of definition of $f(x) = \ln(1 - x) - \ln(2x - 3)$ is:
 - (a) $\left[1, \frac{3}{2}\right]$
 - (b) $\emptyset$
 - (c) $\{1\}$
 - (d) $\left[\frac{2}{3}, 1\right]$
5. The domain of definition of $f(x) = \ln \frac{1-x}{2x-3}$ is:
 - (a) $\left(1, \frac{3}{2}\right)$
 - (b) $\emptyset$
 - (c) $\left[1, \frac{3}{2}\right]$
 - (d) $(-\infty, 1)$
6. The functions $f(x) = 0$ and $g(x) = \ln \frac{x-1}{|x-1|}$ are equal for:
 - (a) $x \in (1, \infty)$
 - (b) $x \in [1, \infty)$
 - (c) $x \in \emptyset$
 - (d) $x \in [-1, 1]$

7. The functions $f(x) = e^{3-x}$ and $g(x) = e^{|3-x|}$ are equal for:

- (a) $(-\infty, 3)$
- (b) $(-\infty, 3]$
- (c) $(-\infty, 3] \setminus \{0\}$
- (d) $\emptyset$

8. The domain of definition of $f(x) = e^{\sqrt{\frac{x-2}{3-x}}}$ is:

- (a) $(-\infty, 2] \cup (3, \infty)$
- (b) $(2, 3]$
- (c) $\emptyset$
- (d) $(2, 3)$

9. The inverse of $f(x) = 5e^{-x}$ is:

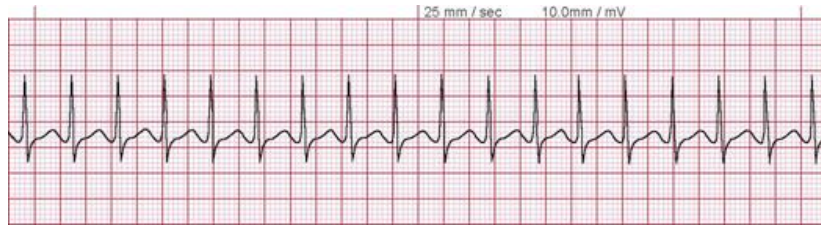
- (a) $y = -\ln \frac{x}{5}$
- (b) $y = e^{5x}$
- (c) $y = \frac{1}{5}e^x$
- (d) the inverse does not exist

10. The solution of $\ln(x^3 + 1) - \ln(2 + x) = 2 \ln x$ is:

- (a) $\{\frac{1}{2}\}$
- (b) $\{\frac{\sqrt{2}}{2}\}$
- (c) $\{-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\}$
- (d) $\emptyset$

7. TRIGONOMETRIC FUNCTIONS

Periodic functions



In mathematics, by periodic functions we mean functions values of which keep repeating with a certain interval. Formally, this is written as

$$f(x + p) = f(x) \text{ for all } x \in D(f),$$

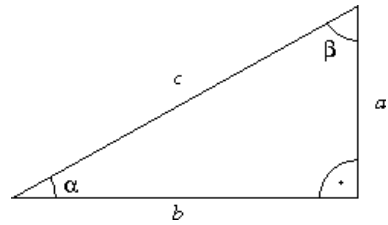
where p is called period. Graphs of periodic functions keep repeating after some intervals on the x -axis.

Trigonometric functions

Trigonometric functions describe relations between angles and lengths of sides in triangles. They are a tool useful for solving numerous tasks not only in geometry or physics. Trigonometric functions have far reaching practical applications in areas such as navigation, mechanics of space objects or in description of complex numbers which are the basic tool for electrical engineering. Most common trigonometric functions include *sine*, *cosine*, *tangent* and *cotangent*.

Trigonometric functions and the right-angled triangle

Graphs are not necessary to understanding what trigonometric functions mean. They can be easily explained using a right-angled triangle.



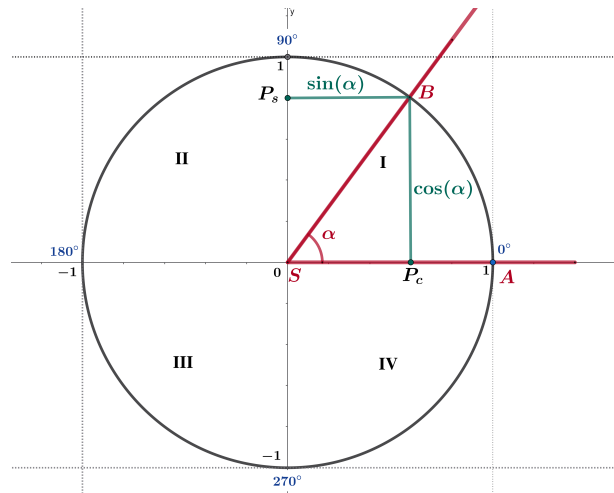
Trigonometric functions sine, cosine, tangent and cotangent are defined as ratios of lengths of a right-angled triangle. They are given for one of the non-right angles of the right-angled triangle in the following way:

$\sin \alpha = \frac{a}{c} = \frac{\text{opposite cathetus}}{\text{hypotenuse}}$	$\cos \alpha = \frac{b}{c} = \frac{\text{adjacent cathetus}}{\text{hypotenuse}}$
$\operatorname{tg} \alpha = \frac{a}{b} = \frac{\text{opposite cathetus}}{\text{adjacent cathetus}}$	$\operatorname{cotg} \alpha = \frac{b}{a} = \frac{\text{adjacent cathetus}}{\text{opposite cathetus}}$

Unit circle

Unit circle is a circle with radius of length 1. Trigonometric functions can be easily depicted using a unit circle. The circle is divided into four quarters called quadrants. The top right quarter is called the first quadrant, the top left quarter is the second quadrant, the bottom left quarter is the third quadrant and the bottom right quarter is the fourth quadrant. Since we will be using the circle to show angles, the figure below highlights important degrees.

Sine and cosine and the unit circle



As has been mentioned, trigonometric functions can be depicted using the unit circle. First we draw an angle, one side of which coincides with the positive semi-axis x such as the red angle $\angle ASB$ in the figure. We denote this angle α . The sides of the angle intersect the unit circle at points A and B . If we draw a line passing through B and parallel with the x -axis (see the dashed horizontal line in the figure), this line will intersect the y -axis in a point which we denote P_s . Then the length of the line segment SP_s (green) is $\sin \alpha$. Since our considerations happen in the unit circle centred in the origin of the Cartesian coordinate system, the length of the line segment SP_s is equal to the y -axis of the point P_s which is equivalent to the y -axis of the point B . The explanation follows from the right-angled triangle SP_sB . There is $|\angle ASB| = |\angle SBP_s| = \alpha$. By the definition of sine, there is

$$\sin \alpha = \frac{\text{opposite cathetus}}{\text{hypotenuse}}.$$

Obviously, the length of the opposite cathetus is $|SP_s|$ and the length of the hypotenuse equals the radius of the circle. As a result, there is

$$\sin \alpha = \frac{|SP_s|}{1} = |SP_s|.$$

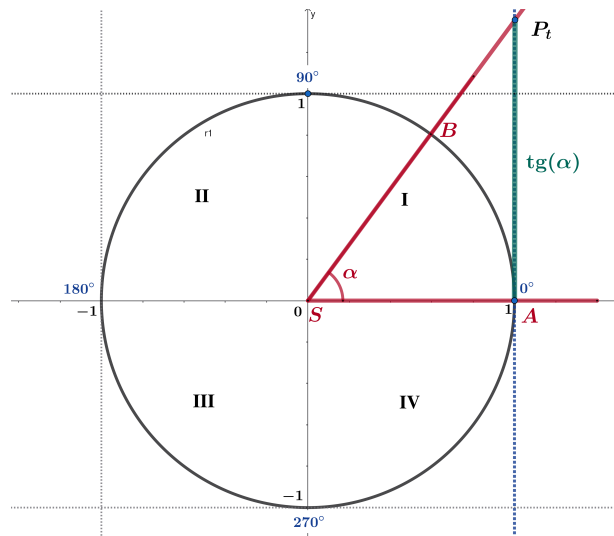
In a similar way, the unit circle can help us to read the value of $\cos \alpha$. Through point B we draw a line parallel with the y -axis. This line intersects the x -axis at point P_c . The length of the line segment SP_c is $\cos \alpha$. This value equals the x coordinate of the point P . This is justified by the fact that in a right-angled triangle SP_cB we have:

$$\cos \alpha = \frac{\text{adjacent cathetus}}{\text{hypotenuse}}.$$

After substitution we get

$$\cos \alpha = \frac{|SP_c|}{1} = |SP_c|.$$

Tangent and the unit circle



Finding the values of $\operatorname{tg}\alpha$ and $\operatorname{cotg}\alpha$ is similar to finding the values of $\sin\alpha$ and $\cos\alpha$. First, we show how to find the value of $\operatorname{tg}\alpha$. We need a line parallel with the y -axis and passing through the point A , i.e. through $[1, 0]$ (see the blue line in the figure). This line intersects the half-line SB at a point P_t . The length of the line segment AP_t is $\operatorname{tg}\alpha$. The explanation is simple and follows from the triangle SAP_t . We know that there is

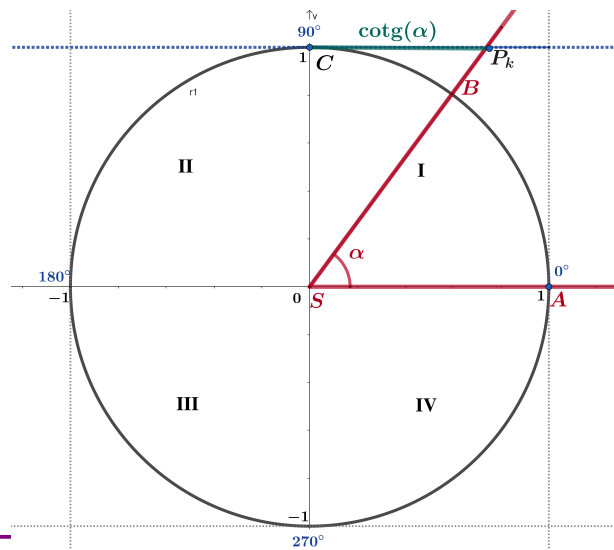
$$\operatorname{tg}\alpha = \frac{\text{opposite cathetus}}{\text{adjacent cathetus}}.$$

The length of the opposite cathetus is $|AP_t|$ while the length of the adjacent cathetus is the radius of the unit circle. Therefore there is

$$\operatorname{tg}\alpha = \frac{|AP_t|}{1} = |AP_t|.$$

Naturally, we need to ask what happens if there is $\alpha = 90^\circ$. Tangent of such an angle is not defined because the side of such an angle would be parallel with the x -axis, i.e. it would be parallel with a line it would be supposed to intersect.

Cotangent and the unit circle



In order to find the value of $\cot \alpha$ we need a line parallel with the x -axis and passing through the point $[0, 1]$ (see the green line in the figure). This line intersects the half-line SB at a point P_k . The length of the line segment CP_k is $\cot \alpha$.

The explanation follows from the triangle SP_kC . By the definition of cotangent there is

$$\cot \alpha = \frac{\text{adjacent cathetus}}{\text{opposite cathetus}}.$$

The length of the adjacent cathetus is $|CP_k|$ while length of the opposite cathetus is the radius of the unit circle, which means that

$$\tan \alpha = \frac{|CP_k|}{1} = |CP_k|.$$

If there is $\alpha = 180^\circ$, cotangent is not defined. Given the unit circle, the reason is obvious: the line p and the side of the angle will never intersect as they are parallel with each other.

Degrees and radians

Traditionally, and worldwide, angles are measured in degrees. One degree is defined as $\frac{1}{360}$ of the full circle. As a result, the right angle measures 90° , half of the right angle is 45° , etc.

Alternatively, especially in science and technology, angles are measured in radians. Radians are defined as ratios of arc lengths and arc radii. Full circle equals 2π . Angles are then measured in as parts of full circle, in radians. For example, the right angle measures $\frac{\pi}{2}$ radians, half of the right angle measures $\frac{\pi}{4}$ radians, etc.

Conversion between degrees and radians follows simple formulas. When converting from degrees to radians we multiply the angle size in degrees by $\frac{\pi}{180}$. Conversely, when converting radians to degrees, we multiply the angle size in radians by $\frac{180}{\pi}$.

Oriented angle

Angle $\sphericalangle AVB$ is a section of plane bordered by two half-lines (or rays) VA and VB . We call these half-lines sides of the angle. By an oriented angle we mean such an angle whose sides are not interchangeable, meaning that one of them is the starting side while the other the ending side.

The orientation of the angle $\sphericalangle AVB$ is given by the order of letters. In the oriented angle $\sphericalangle AVB$, VA is the starting side while VB is the ending side.

Sides of the oriented angle can be viewed as clock hands. The ending side of the angle rotates around the angle vertex either clockwise or counter-clockwise. Thus we speak of positive or negative revolution. The positive revolution is counter-clockwise while the negative revolution is a clockwise one. The idea of analog clock helps to understand the concept of the size of the basic oriented angle $\sphericalangle AVB$. By this we mean the size of the angle $\sphericalangle AVB$ created by rotating the half-line VA counter-clockwise until it coincides with the half-line VB .

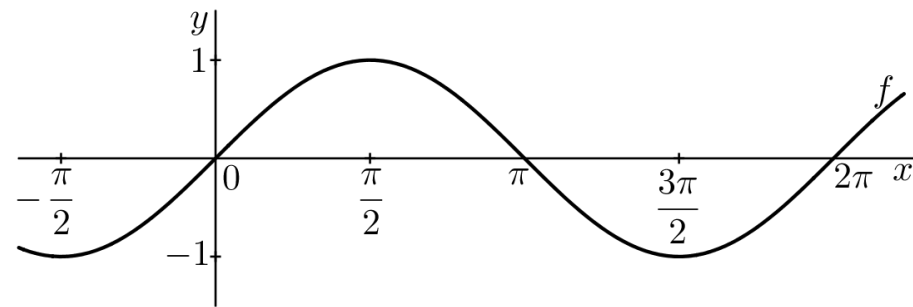
The basic size of the oriented angle is always a number from the interval $(0^\circ, 360^\circ)$ (measured in degrees) or from the interval $(0, 2\pi)$ (measured in radians). If the two sides of an oriented angle coincide, the angle is called a zero oriented angle.

We need to talk about the basic size of oriented angles because the size of the oriented angle $\sphericalangle AVB$ can be an arbitrary value $\alpha + k \cdot 360^\circ$ or $\alpha + k \cdot 2\pi$, where α is the basic size of $\sphericalangle AVB$ and $k \in \mathbb{Z}$. For example, the fact that the size of the oriented angle $\sphericalangle AVB$ is 420° can be written as $60^\circ + 1 \cdot 360^\circ$ or $\frac{\pi}{6} + 1 \cdot 2\pi$. Given the unit circle, the angle is constructed by one full revolution followed by a revolution by 60° or $\frac{\pi}{6}$ radian. As a result, the angle 60° (or $\frac{\pi}{6}$) is the basic size of the angle 420° , which means that in the unit circle the same angles are displayed in both cases.

We apply this idea for expanding the values of trigonometric functions beyond 360° or 2π radians. Since the oriented angle $\alpha + 2k\pi$ is the same as the oriented angle α , their values of the respective functions are the same. Therefore, trigonometric functions are periodic.

The graph of sine

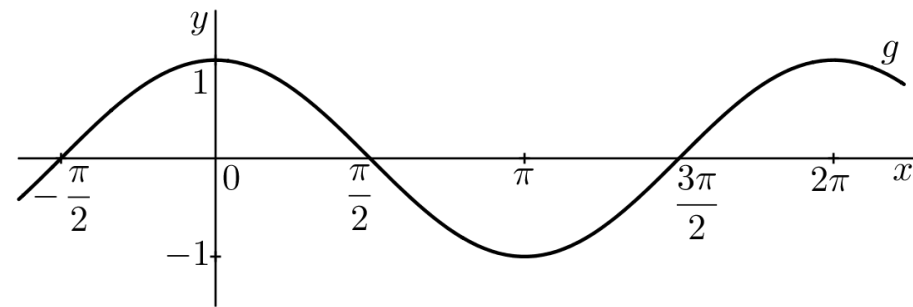
The graph of sine is called sinusoid. It can be seen in the figure below:



Sine is an odd bounded periodic function with a period 2π . Its domain of definition and range of values are: $D(f) = \mathbb{R}$, $H(f) = [-1, 1]$.

The graph of cosine

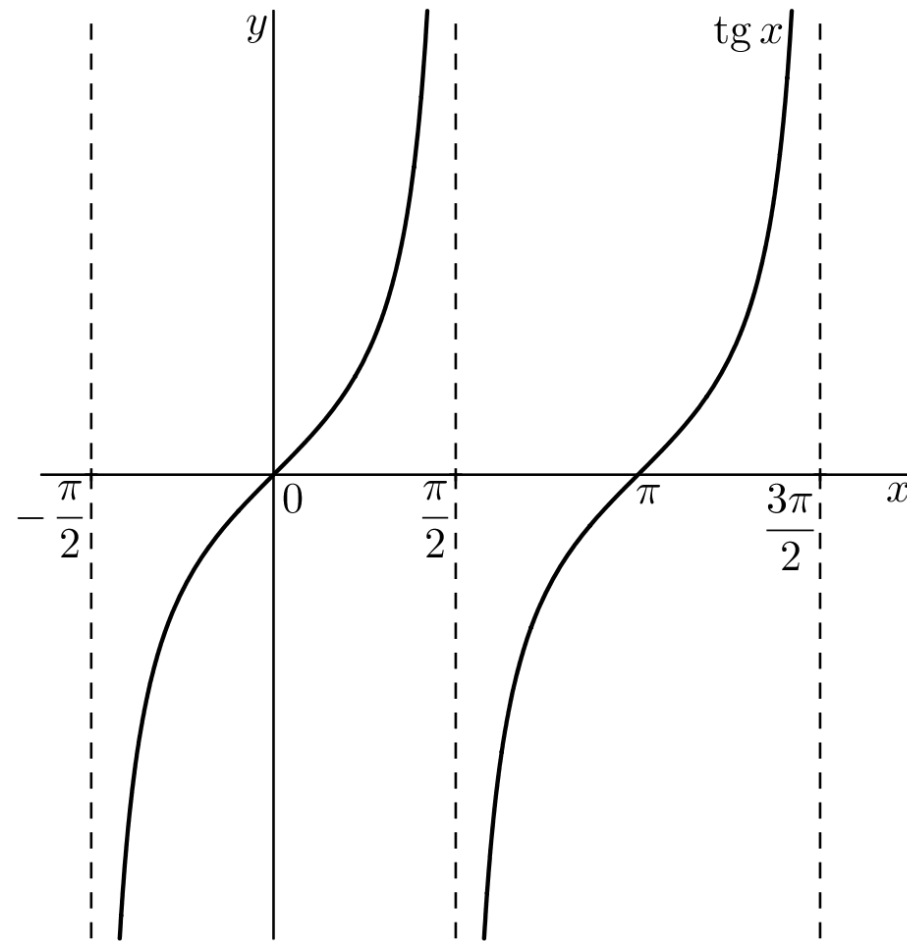
The graph of cosine is called cosinusoid. It can be seen in the figure below:



Cosine is an even bounded periodic function with a period 2π . Its domain of definition and range of values are: $D(f) = \mathbb{R}$, $H(f) = [-1, 1]$.

The graph of tangent

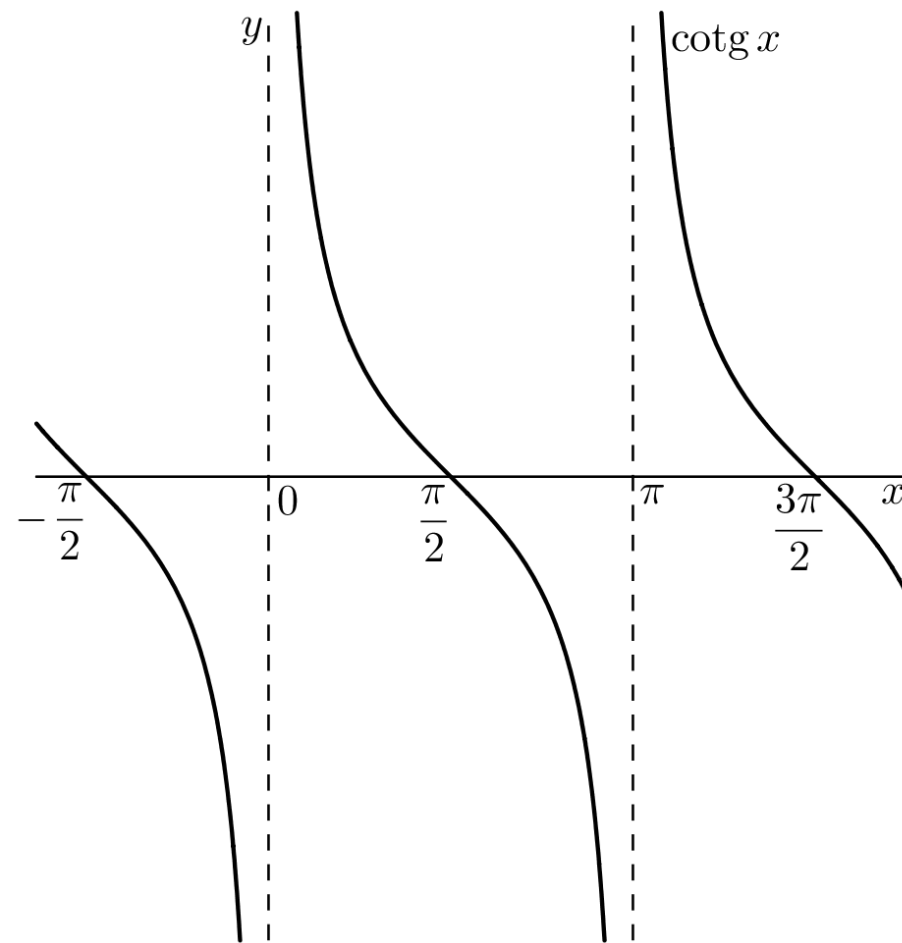
The graph of tangent is called tangentoid. It can be seen in the figure below:



Tangent is an odd unbounded periodic function with a period π . Its domain of definition and range of values are: $D(f) = \mathbb{R} \setminus \left\{ \frac{\pi}{2} + k\pi : k \in \mathbb{Z} \right\}$, $H(f) = \mathbb{R}$.

The graph of cotangent

The graph of cotangent is called cotangentoid. It can be seen in the figure below:



Cotangent is an odd unbounded periodic function with a period π . Its domain of definition and range of values are: $D(f) = \mathbb{R} \setminus \{k\pi : k \in \mathbb{Z}\}$, $H(f) = \mathbb{R}$.

Values of trigonometric functions

In the following table you can find values of trigonometric functions of some angles.

Function	$30^\circ \left(\frac{\pi}{6}\right)$	$45^\circ \left(\frac{\pi}{4}\right)$	$60^\circ \left(\frac{\pi}{3}\right)$	$90^\circ \left(\frac{\pi}{2}\right)$	$120^\circ \left(\frac{2\pi}{3}\right)$	$135^\circ \left(\frac{3\pi}{4}\right)$	$150^\circ \left(\frac{5\pi}{6}\right)$	$180^\circ \left(\pi\right)$
$\sin(\alpha)$	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\cos(\alpha)$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{3}}{2}$	-1
$\operatorname{tg}(\alpha)$	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef.	$-\sqrt{3}$	-1	$-\frac{\sqrt{3}}{3}$	0
$\operatorname{cotg}(\alpha)$	$\sqrt{3}$	1	$\frac{\sqrt{3}}{3}$	0	$-\frac{\sqrt{3}}{3}$	-1	$-\sqrt{3}$	undef.

Relations between trigonometric function

There exist many relations and formulas between trigonometric functions which can be used during the process of solution of trigonometric equations and inequations.

$$\sin^2 x + \cos^2 x = 1$$

$$\operatorname{tg} x = \frac{\sin x}{\cos x}$$

$$\sin \left(x + \frac{\pi}{2} \right) = \cos x$$

Relations for the sum of arguments

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y$$

$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

Relations for the sum of values of trigonometric functions

$$\sin x + \sin y = 2 \sin \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\sin x - \sin y = 2 \cos \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$$

$$\cos x + \cos y = 2 \cos \left(\frac{x+y}{2} \right) \cos \left(\frac{x-y}{2} \right)$$

$$\cos x - \cos y = -2 \sin \left(\frac{x+y}{2} \right) \sin \left(\frac{x-y}{2} \right)$$

Relations for a double of the argument

$$\sin 2x = 2 \sin x \cos x$$

$$\cos 2x = \cos^2 x - \sin^2 x$$

$$\operatorname{tg} 2x = \frac{2 \operatorname{tg} x}{1 - \operatorname{tg}^2 x}$$

Exercise

Exercise 1. Find the domain of definition of $f(x) = \frac{1}{1-\cos x}$.

Solution. In order to find the domain of definition, it is sufficient to find all $x \in \mathbb{R}$ such that $1 - \cos x \neq 0$ because the expression $1 - \cos x$ is in the denominator of a fraction. We solve the equation

$$1 - \cos x = 0,$$

which is equivalent to

$$\cos x = 1.$$

The value of $\cos x$ is 1 in even multiples of π , which means that the solution of the last equation is $x \in \{2k\pi : k \in \mathbb{Z}\}$. As a result, $D(f) = \mathbb{R} \setminus \{2k\pi : k \in \mathbb{Z}\}$.

Exercise

Exercise 2. Find the domain of definition of $f(x) = \sqrt{1 - \sin x}$.

Solution. In order to find the domain of definition, it is sufficient to find all $x \in \mathbb{R}$ such that $1 - \sin x \geq 0$ because the expression $1 - \sin x$ is under the square root. We solve the inequation

$$1 - \sin x \geq 0,$$

which is equivalent to

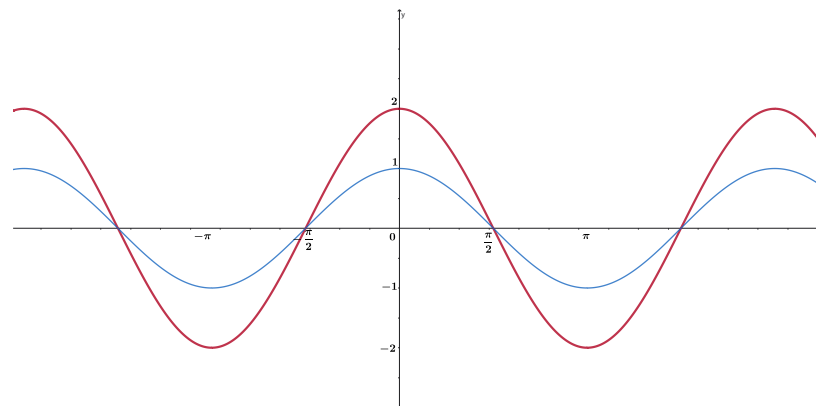
$$1 \geq \sin x.$$

Since the values of $\sin x$ are numbers from the interval $[-1, 1]$, all real numbers are the solution of the inequation. As a result, $D(f) = \mathbb{R}$.

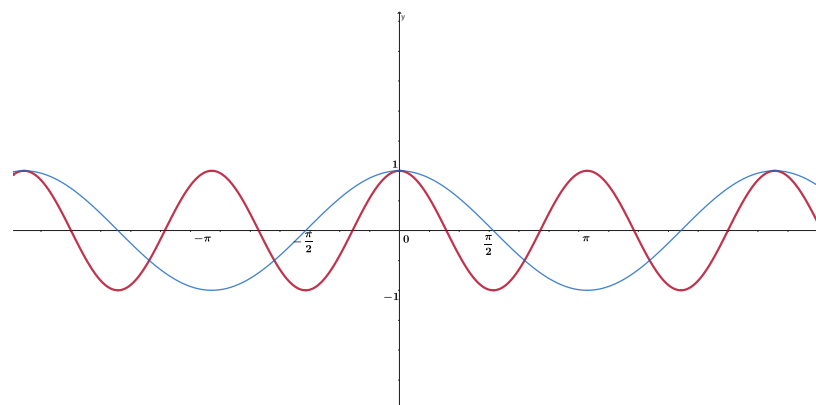
Exercise

Exercise 3. Sketch the graphs of functions $f(x) = 2 \cdot \cos x$ and $g(x) = \cos(2x)$.

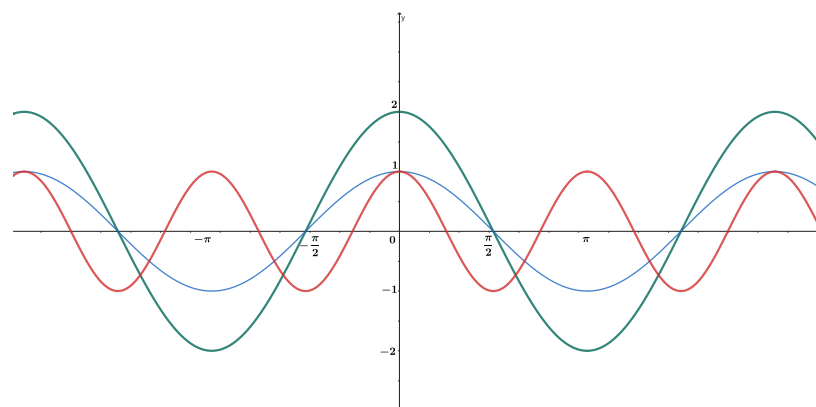
Solution. First, we sketch the graph of function f . We have to realize that $2 \cdot \cos x$ means that all function values double, which means that the graphs of $f(x) = 2 \cdot \cos x$ and $g(x) = \cos(2x)$ will intersect at points where $\cos x = 0$. This means that $\cos x = 2 \cdot \cos x$ if and only if $\cos x = 2 \cdot \cos x = 0$. Such intersections are all $x \in \left\{ \frac{2k+1}{2} \cdot \pi : k \in \mathbb{Z} \right\}$. In the following figure, the graph of the function f is given in red while the graph of $y = \cos x$ is blue.



When sketching the graph of $g(x) = \cos(2x)$, we must realize that its argument is double the argument of $\cos x$, which means that at e.g. $x = \pi$ the function value of g will be the same as at $x = 2\pi$ for $y = \cos x$. This means that the period of the function g will be one half of the period of $y = \cos x$. In the following figure, the graph the function g is given in red while the graph of $y = \cos x$ is blue.



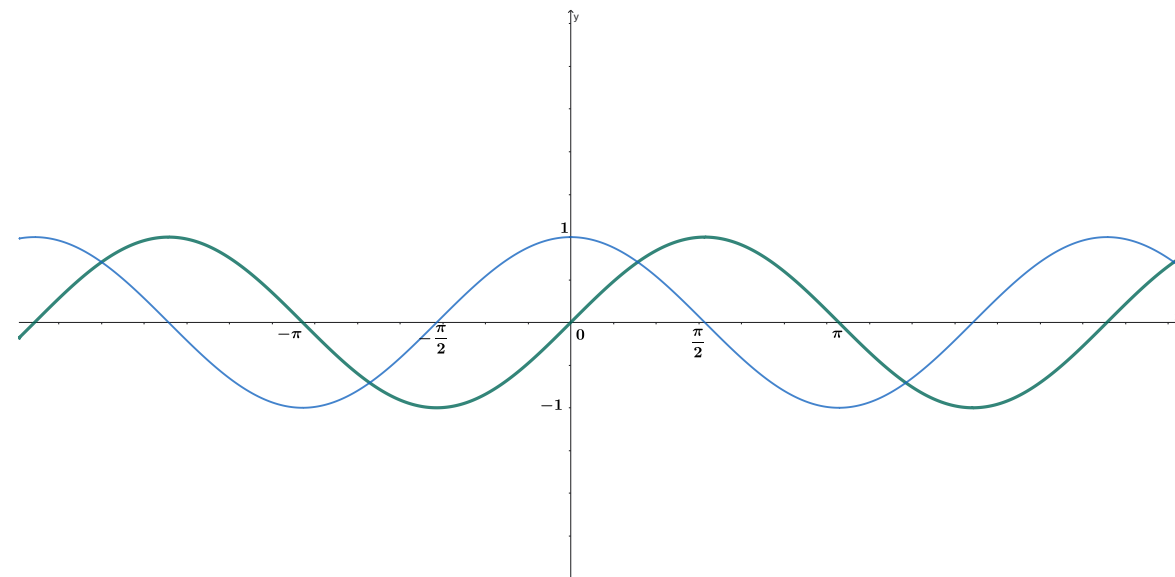
Finally, we include graphs of f, g and $y = \cos x$ in one figure:



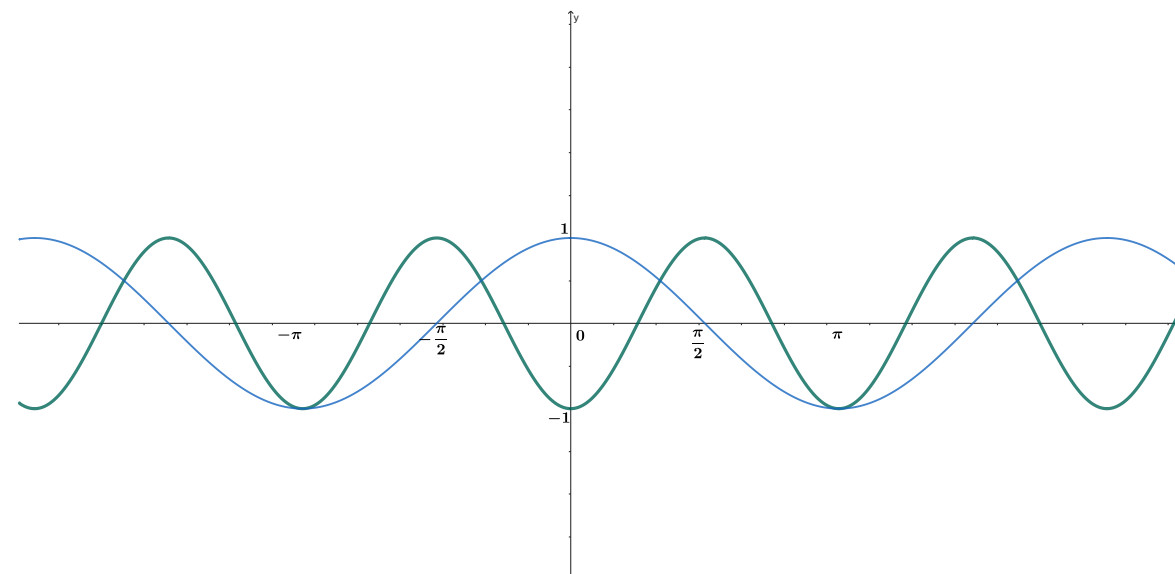
Exercise

Exercise 4. Sketch the graphs of $f(x) = \cos\left(x - \frac{\pi}{2}\right)$ and $g(x) = \cos(2x - \pi)$.

Solution. When sketching the graph of the function f , we must first find out when $x - \frac{\pi}{2} = 0$. This is a simple task, it is for $x = \frac{\pi}{2}$. The value of the function at this point is 1, same as the value of $y = \cos x$ at $x = 0$. Therefore, the graph of the function f is the same as the graph of $y = \cos x$, only it is shifted the right by $\frac{\pi}{2}$. This graph (in green) and the graph of $y = \cos x$ are given in the figure below.



When sketching the graph of the function $g(x) = \cos(2x - \pi)$, we first transform the argument $2x - \pi$ into $2\left(x - \frac{\pi}{2}\right)$. Now we can easily see that the argument is 0 for $x = \frac{\pi}{2}$. The value of the function g at this point is 1 but unlike the function f the size of its period will be one half which follows from the fact that the coefficient at x is 2. This graph (in green) and the graph of $y = \cos x$ are given in the figure below.



Exercise

Exercise 5. Transform the expression $\frac{\sin x}{1+\cos x} + \frac{1+\cos x}{\sin x}$.

Solution. First we find out all permissible $x \in \mathbb{R}$. Obviously, there is

$$1 + \cos x \neq 0 \quad \wedge \quad \sin x \neq 0 \Rightarrow x \neq (2k+1)\pi \quad \wedge \quad x \neq k\pi: k \in \mathbb{Z} \Rightarrow x \neq k\pi: k \in \mathbb{Z}.$$

We add the fractions

$$\frac{\sin x}{1 + \cos x} + \frac{1 + \cos x}{\sin x} = \frac{\sin^2 x + (1 + \cos^2 x)^2}{(1 + \cos x) \cdot \sin x}$$

and transform the sum

$$\frac{\sin^2 x + (1 + \cos x)^2}{(1 + \cos x) \cdot \sin x} = \frac{\sin^2 x + 1 + \cos^2 x + 2 \cos x}{(1 + \cos x) \cdot \sin x} = \frac{2 + 2 \cos x}{(1 + \cos x) \cdot \sin x} = \frac{2(1 + \cos x)}{(1 + \cos x) \cdot \sin x} = \frac{2}{\sin x}.$$

We have applied the formula $\sin^2 x + \cos^2 x = 1$ and common mathematical transformations such as powering or taking out of the brackets.

Exercise

Exercise 6. Find all real solutions of the equation $2 \cdot \sin^2 x - 3 \cdot \sin x + 1 = 0$.

Solution. The key to finding the solution of the equation

$$2 \cdot \sin^2 x - 3 \cdot \sin x + 1 = 0$$

is using the substitution $z = \sin x$ which turns the equation into a quadratic equation:

$$2z^2 - 3z + 1 = 0.$$

Its discriminant is:

$$D = (-3)^2 - 4 \cdot 2 \cdot 1 = 9 - 8 = 1.$$

Then

$$z_{1,2} = \frac{3 \pm \sqrt{D}}{4} = \frac{3 \pm 1}{4},$$

which means that $z_1 = 1$ and $z_2 = \frac{1}{2}$. Since there is $z = \sin x$, there must be $z \in [-1, 1]$. This is true for both z_1 and z_2 . Therefore, we solve two equations:

$$\sin x = 1 \iff x \in \left\{ \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\},$$

and

$$\sin x = \frac{1}{2} \iff x \in \left\{ \frac{\pi}{6} + 2k\pi : k \in \mathbb{Z} \right\}.$$

The solution of the original equation is

$$\left\{ \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\} \cup \left\{ \frac{\pi}{6} + 2k\pi : k \in \mathbb{Z} \right\}.$$

Exercise

Exercise 7. Find all real solutions of the equation $\operatorname{tg}(2x - 3) = 1$.

Solution. When looking for the solution of

$$\operatorname{tg}(2x - 3) = 1,$$

it is important to realize that the value of $y = \operatorname{tg} x$ is 1 at $x = \frac{\pi}{4}$, and also that the period of this function is $d = \pi$. Then we can transform the original equation into

$$2x - 3 = \frac{\pi}{4} + k \cdot \pi, \quad \text{where } k \in \mathbb{Z}.$$

After some transformations we get

$$x = \frac{\pi}{8} + \frac{3}{2} + \frac{\pi}{2} \cdot k, \quad \text{where } k \in \mathbb{Z}.$$

We transform the expression on the right-hand side and get that the solution of the original equation is $\left\{ \frac{1}{2} \left(\frac{4k+1}{4} \cdot \pi + 3 \right) : k \in \mathbb{Z} \right\}$.

Exercise

Exercise 8. In the real domain solve the inequation $\sin 2x \geq \frac{\sqrt{2}}{2}$.

Solution. When solving the inequation

$$\sin 2x \geq \frac{\sqrt{2}}{2},$$

it is important to consider that

$$\sin x = \frac{\sqrt{2}}{2} \iff x = \frac{\pi}{4} + 2k\pi \text{ nebo } x = \frac{3\pi}{4} + 2k\pi, \text{ where } k \in \mathbb{Z}.$$

Then, in the union of intervals $\bigcup_{k \in \mathbb{Z}} \langle \frac{\pi}{4} + 2k\pi, \frac{3\pi}{4} + 2k\pi \rangle$, there is $\sin x \geq \frac{\sqrt{2}}{2}$.

In our inequation there is not $\sin x$ but $\sin 2x$. Therefore, we need to find out at which points the value of $\sin 2x$ is $\frac{\sqrt{2}}{2}$. We apply what we already know and get:

$$\sin 2x = \frac{\sqrt{2}}{2} \iff \left(2x = \frac{\pi}{4} + 2k\pi \vee 2x = \frac{3\pi}{4} + 2k\pi : k \in \mathbb{Z} \right).$$

Then there is

$$x = \frac{\pi}{8} + k\pi \vee x = \frac{3\pi}{8} + k\pi : k \in \mathbb{Z}.$$

As a result, the union of intervals $\bigcup_{k \in \mathbb{Z}} \langle \frac{\pi}{8} + k\pi, \frac{3\pi}{8} + k\pi \rangle$ is the solution of our inequation.

1 TEST

1. The domain of definition of $f(x) = \frac{1}{1+\sin x}$ is

- (a) $\mathbb{R} \setminus \left\{ \frac{3}{2} \cdot \pi + 2k\pi : k \in \mathbb{Z} \right\}$
- (b) $\left\{ \frac{3}{2} \cdot \pi + 2k\pi : k \in \mathbb{Z} \right\}$
- (c) $\left\{ \frac{1}{2} \cdot \pi + 2k\pi : k \in \mathbb{Z} \right\}$
- (d) $\mathbb{R} \setminus \left\{ \frac{3}{2} \cdot \pi + k\pi : k \in \mathbb{Z} \right\}$

2. The domain of definition of $f(x) = \sqrt{2 - \sin x}$ is

- (a) $[0, 2]$
- (b) $\emptyset$
- (c) $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$
- (d) $\mathbb{R}$

3. The domain of definition of $f(x) = \frac{1}{1-\operatorname{tg} x}$ is

- (a) $\emptyset$
- (b) $\left\{ \frac{\pi}{4} + 2k\pi : k \in \mathbb{Z} \right\}$
- (c) $\left\{ \frac{\pi}{4} + k\pi : k \in \mathbb{Z} \right\}$
- (d) $\mathbb{R} \setminus \left\{ \frac{\pi}{4} + k\pi : k \in \mathbb{Z} \right\}$

4. The domain of definition of $f(x) = \frac{1}{\sqrt{1-\sin x}}$ is

- (a) $\emptyset$
- (b) $\mathbb{R} \setminus \left\{ \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$
- (c) $\mathbb{R}$
- (d) $\left\{ \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$

5. The solution of $\sin(2x) = 2$ is

- (a) $\mathbb{R}$
- (b) $\emptyset$
- (c) $\left\{ \frac{\pi}{2} + 2k\pi : k \in \mathbb{Z} \right\}$
- (d) $\left\{ \frac{\pi}{2} - k\pi : k \in \mathbb{Z} \right\}$

6. The solution of $1 - \sin(x) = 2$ is

- (a) $\left\{ \frac{3}{2}\pi + 2k\pi : k \in \mathbb{Z} \right\}$
- (b) $\left\{ \frac{3}{2}\pi + k\pi : k \in \mathbb{Z} \right\}$

(c) $\left\{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\}$

(d) $\left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$

7. The solution of $\sin^2 x - \sin x - 2 = 0$ is

(a) $\left\{\frac{3}{2}\pi + 2k\pi : k \in \mathbb{Z}\right\}$

(b) $\left\{\frac{3}{2}\pi + k\pi : k \in \mathbb{Z}\right\}$

(c) $\left\{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\}$

(d) $\left\{\frac{\pi}{2} + k\pi : k \in \mathbb{Z}\right\}$

8. The solution of the inequation $\cos x \leq \frac{1}{2}$ is

(a) $x = \frac{\pi}{3}$

(b) $\left\{\left[\frac{\pi}{3} + 2k\pi, \frac{5\pi}{3} + 2k\pi\right] : k \in \mathbb{Z}\right\}$

(c) $\mathbb{R}$

(d) $x = \frac{\pi}{6}$

9. The solution of the inequation $\frac{\sin x - 1}{1 - \cos x} \geq 0$ is

(a) $\mathbb{R}$

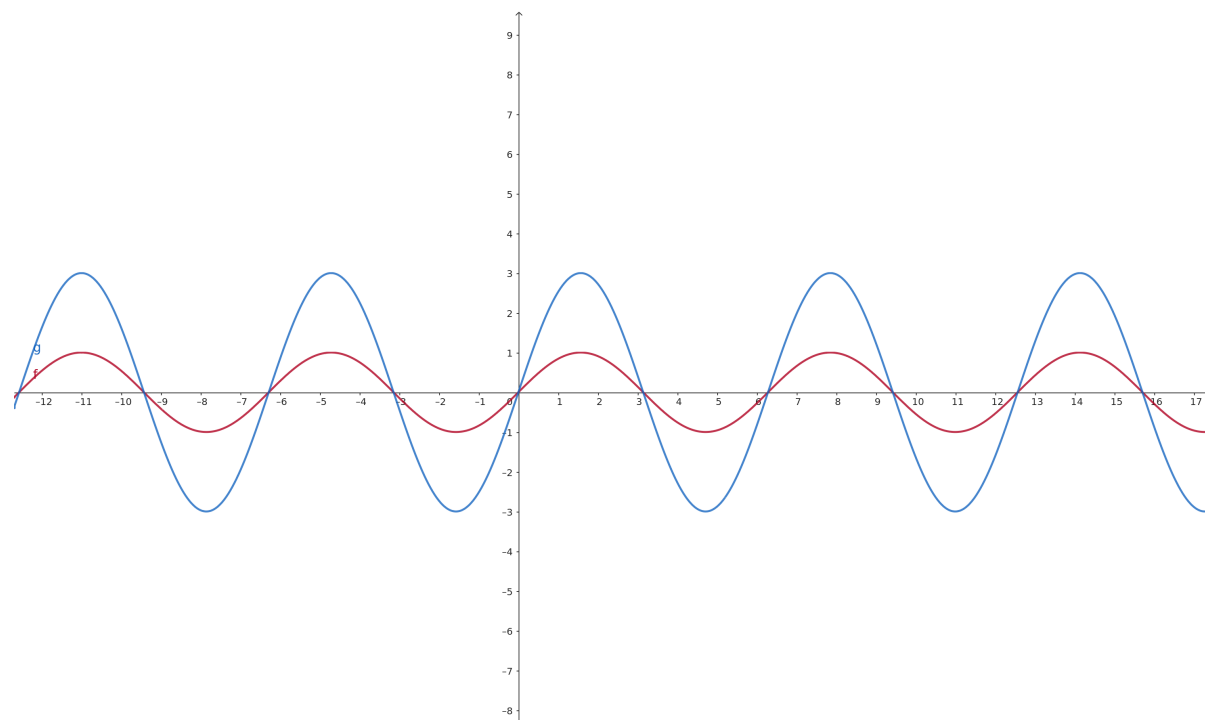
(b) $\{2kp : k \in \mathbb{Z}\}$

(c) $\left\{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\} \cup \{2kp : k \in \mathbb{Z}\}$

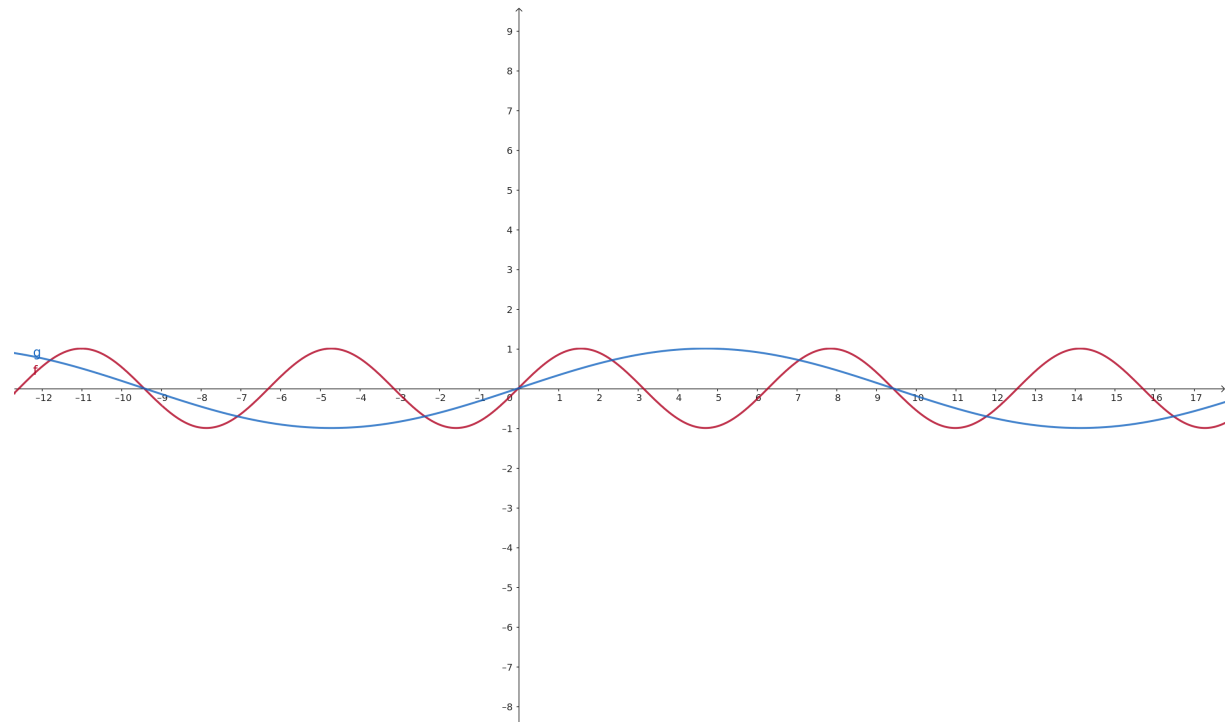
(d) $\left\{\frac{\pi}{2} + 2k\pi : k \in \mathbb{Z}\right\}$

10. Sketch the graphs of $f(x) = \sin 3x$, $g(x) = \sin x$ (the function f is red, the function g is blue).

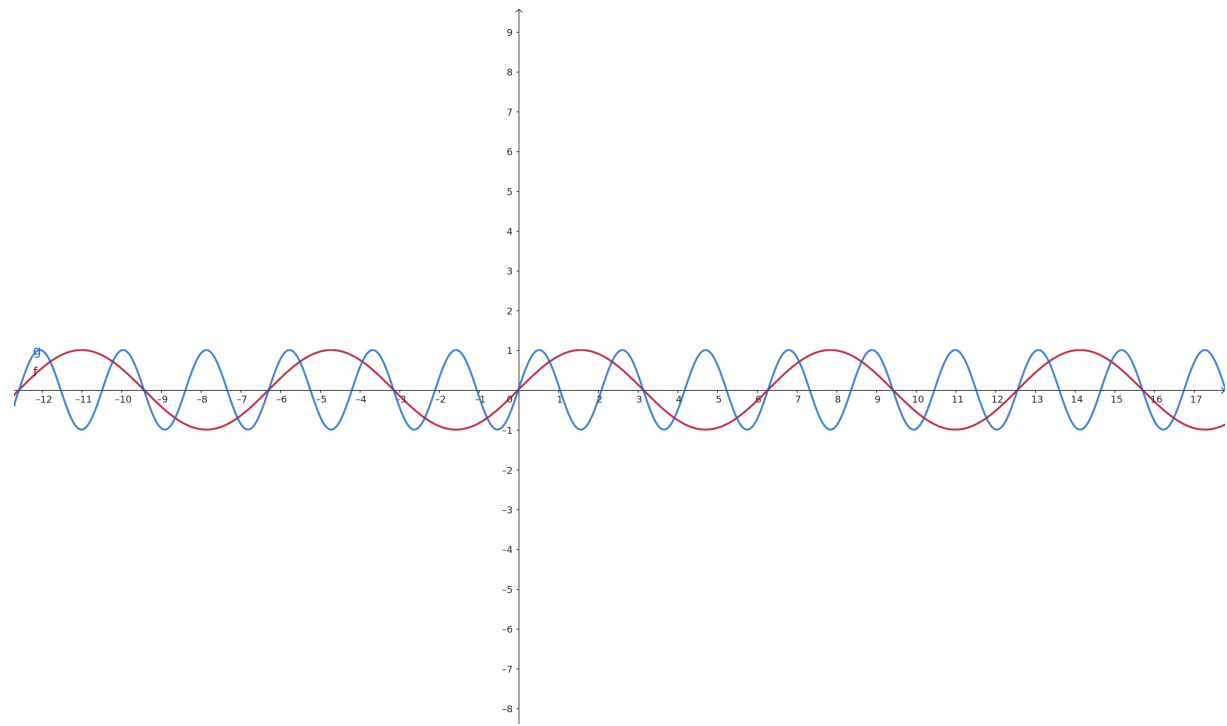
(a)



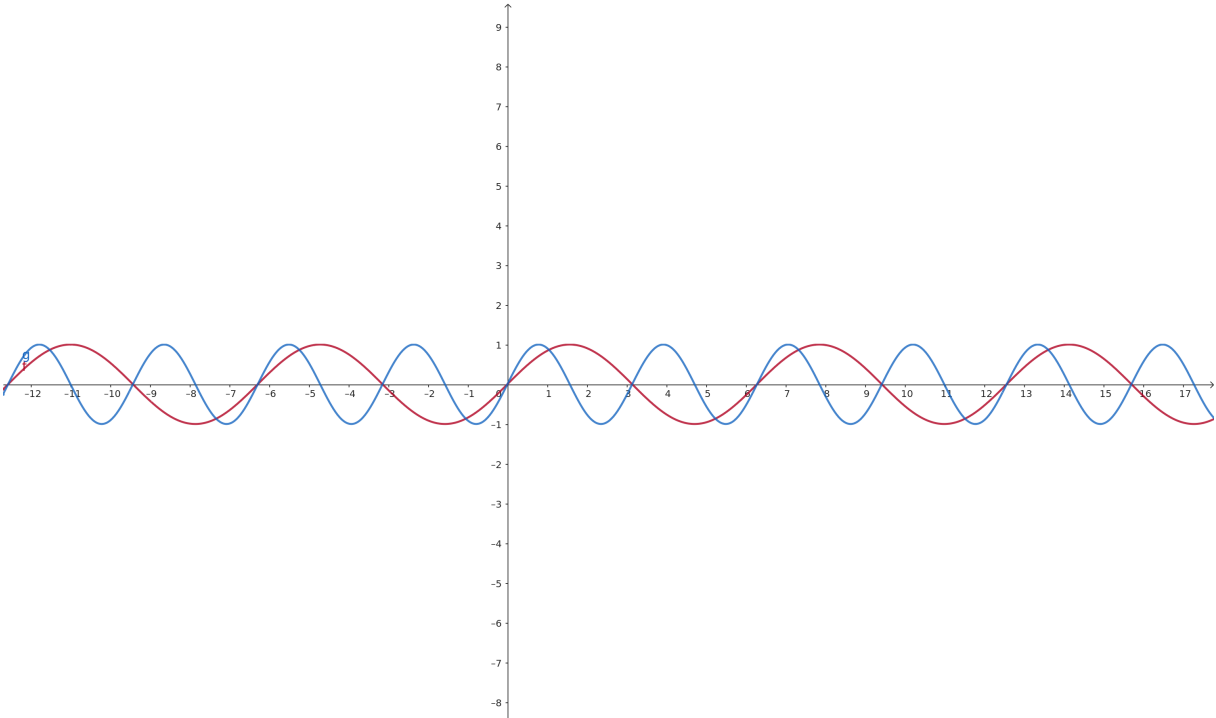
(b)



(c)



(d)



8. VECTORS, LINES AND PLANES

Analytic geometry



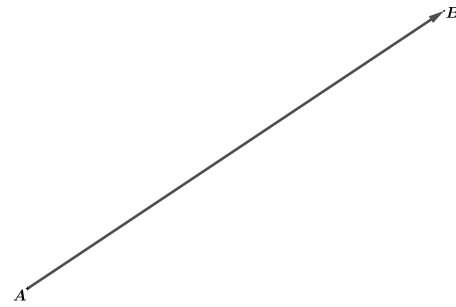
Analytic geometry is a branch of mathematics pioneered by a French philosopher and mathematician René Descartes. It was him who introduced the idea of a coordinate system – a step that made the algebraic description of geometrical objects possible. Using coordinate systems, geometrical problems can be solved by means of algebraic and analytical tools.

We will be interested in the simplest objects that can be described analytically. These include points, vectors, lines and planes.

Directed line segments

There exists exactly one line segment linking two mutually different points A and B . Such a line segment is denoted as AB , where the order of points is not relevant. The line segment AB is the same the line segment BA .

However, in physics we need to visualize not only the size of force but also its direction. Therefore, we speak about directed line segments, where the starting point and the endpoint are distinguished. As a result, in the directed line segments AB , the point A is its starting point while B is its endpoint. An example of an directed line segment is given below:

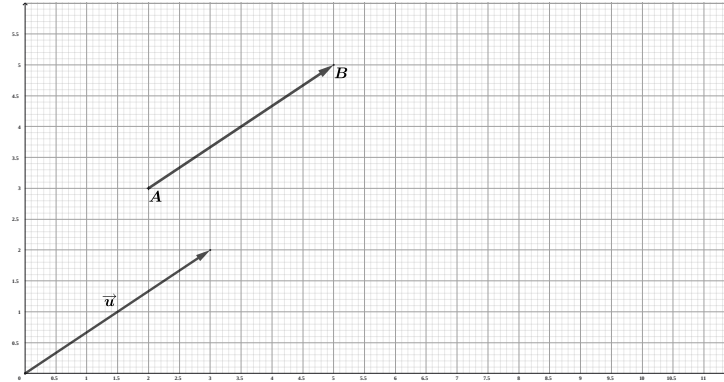


Vectors

We will use directed line segments when defining the concept of vectors. By a nonzero vector we mean a set of directed line segments which have the same nonzero size and the same direction. Line segments of zero length, i.e. line segments between two identical points, are called zero vectors.

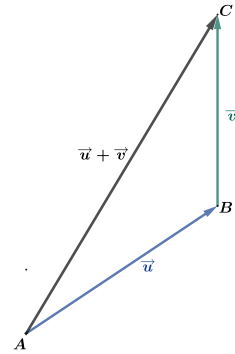
For a vector in a plane defined by an directed line segment AB , where $A = [a_1, a_2]$, $B = [b_1, b_2]$ the numbers $u_1 = b_1 - a_1$, $u_2 = b_2 - a_2$ are called coordinates of vector $\vec{u}$. We denote this as $\vec{u} = [u_1, u_2]$.

For a vector in space defined by an directed line segment AB , where $A = [a_1, a_2, a_3]$, $B = [b_1, b_2, b_3]$ the numbers $u_1 = b_1 - a_1$, $u_2 = b_2 - a_2$, $u_3 = b_3 - a_3$ are called coordinates of vector $\vec{u}$. We denote this as $\vec{u} = [u_1, u_2, u_3]$. The way of finding coordinates of the vector $\vec{u} = AB$ is symbolically denoted as $\vec{u} = B - A$. The figure below shows two different representations of the vector $\vec{u} = [2, 2]$.



The sum of vectors

The sum of vectors $\vec{u} = B - A$ and $\vec{v} = C - B$ is a vector $C - A$. We denote this as $\vec{u} + \vec{v} = C - A$. The construction of a vector sum is given in the figure:



The sum of vectors is both commutative and associative:

$$\begin{aligned}\vec{u} + \vec{v} &= \vec{v} + \vec{u}, \\ (\vec{u} + \vec{v}) + \vec{w} &= \vec{v} + (\vec{u} + \vec{w}).\end{aligned}$$

Multiplying a vector by a real number

Vectors can be multiplied by real numbers. The result of this multiplication is again a vector. Geometrically, this is represented by lengthening or shortening combined with – when vectors are multiplied by negative numbers – the change of vector orientation.

Every real multiple of a zero vector is always a zero vector. The real k multiple of a nonzero vector $\vec{u} = B - A$ is a vector $\vec{v} = C - A$, where C is such a point that

$$|AC| = |k| \cdot |AB|.$$

For $k \geq 0$ the point C is on the half-line AB , for $k < 0$, the point C is on the half-line opposite to the half-line AB . Multiplication of a vector $\vec{u}$ by a real number k is denoted as $\vec{v} = k\vec{u}$.

For each vector $\vec{u} = [u_1, u_2]$ in a plane and each real number k there is $k\vec{u} = [ku_1, ku_2]$. For each vector $\vec{u} = [u_1, u_2, u_3]$ in space and each real number k there is $k\vec{u} = [ku_1, ku_2, ku_3]$.

Suppose vectors $\vec{u}, \vec{v}, \vec{w}$, all in a plane. The vector $\vec{z} = a\vec{u} + b\vec{v} + c\vec{w}$ where $a, b, c \in \mathbb{R}$, is called linear combination of vectors $\vec{u}, \vec{v}, \vec{w}$. Linear combination of vectors in space is defined analogically.

Dot product (or inner / scalar product)

Very often we need to find the size of vectors. This is defined in a very natural way as the length of the directed line segment. The size of a vector $\vec{u}$ is denoted as $|\vec{u}|$. If there is $|\vec{u}| = 1$, we call the vector $\vec{u}$ normed (or unit) vector. From the Pythagorean theorem we get that the size of a planar vector $\vec{u} = [u_1, u_2]$ is

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2}.$$

The size of a vector $\vec{u} = [u_1, u_2, u_3]$ in space is

$$|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}.$$

The size of a zero vector $\vec{o}$ is $|\vec{o}| = 0$.

We already know how to add or subtract vectors, how to multiply them by real numbers. We also know how to find their size. Now we introduce another operation – the dot product. Notice that the dot product is often called inner product or scalar product.

By a dot product of two vectors $\vec{u} = [u_1, u_2]$, $\vec{v} = [v_1, v_2]$ in plane we mean a real number $u_1v_1 + u_2v_2$.

By a dot product of two vectors $\vec{u} = [u_1, u_2, u_3]$, $\vec{v} = [v_1, v_2, v_3]$ in space we mean a real number $u_1v_1 + u_2v_2 + u_3v_3$.

The dot product of two vectors $\vec{u}$, $\vec{v}$ is denoted as $\vec{u}\vec{v}$ or $\vec{u} \cdot \vec{v}$. Instead of $\vec{u} \cdot \vec{u}$ we write $\vec{u}^2$. In a plane there is

$$\vec{u}^2 = u_1u_1 + u_2u_2 = u_1^2 + u_2^2$$

while in space there is

$$\vec{u}^2 = u_1u_1 + u_2u_2 + u_3u_3 = u_1^2 + u_2^2 + u_3^2.$$

Therefore, there is

$$\vec{u}^2 = |\vec{u}|^2.$$

We conclude this section with one of the most important applications of the dot product. For two nonzero vectors $\vec{u}$, $\vec{v}$ represented as AB , AC we call the size of the convex angle $\sphericalangle BAC$ angle of vectors $\vec{u}$, $\vec{v}$. In case that lines AB and AC are perpendicular, we say that also vectors $\vec{u}$, $\vec{v}$ are perpendicular. In case that at least one of the two vectors is a zero vector, their angle is not defined. Now, for two nonzero vectors $\vec{u}$, $\vec{v}$ in plane or space and their angle φ there is:

$$\vec{u}\vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos \varphi \text{ where } \varphi \in [0, \pi].$$

Therefore, the angle of two nonzero vectors $\vec{u}$, $\vec{v}$ is

$$\cos \varphi = \frac{\vec{u}\vec{v}}{|\vec{u}| \cdot |\vec{v}|} \text{ where } \varphi \in [0, \pi].$$

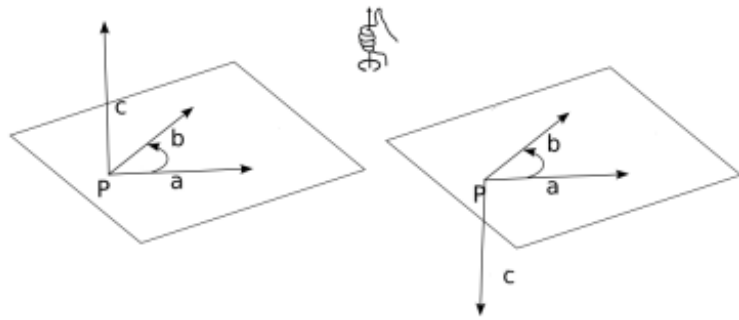
We will apply this formula when calculating the angle of two lines.

For an arbitrary pair of vectors $\vec{u}$, $\vec{v}$ in a plane or space there is $\vec{u}\vec{v} = 0$ if and only if at least one of them is a zero vector, or if both vectors are nonzero ones and they are perpendicular.

The cross product and the scalar triple product

Cross product is another vector specific operation. Unlike in dot product, the result of which is a real number, the result of a cross product of two vectors is a vector. The cross product is defined for vectors in space only. Before we introduce it, we need to discuss two new concepts.

Suppose we have three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ in space and suppose that we place them in such a way that their starting points coincide. Each triplet of vectors in space such that the vectors are not in the same plane is a basis of the space. We will denote a basis given by vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ as $[\vec{a}, \vec{b}, \vec{c}]$. Obviously, all vectors must be nonzero ones.



Now we explain what we mean by a clockwise and a counterclockwise basis. After we place all the three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ in such a way that their starting points coincide, we place our right hand on a plane generated by vectors $\vec{a}$, $\vec{b}$ in such a way that our crooked fingers show the direction for the vector $\vec{a}$ to the vector $\vec{b}$. Now we focus on our thumb. If our raised thumb points in the same direction as the vector $\vec{c}$, we say that the basis is clockwise. If our raised thumb points in the opposite direction as the vector $\vec{c}$, we say that the basis is counterclockwise. For details see the figure.

The cross product of two vectors, which are on the same line, is a zero vector. The cross product of two vectors $\vec{u}$, $\vec{v}$, which are not on the same line, is a vector $\vec{w}$ with the following properties:

- Vector $\vec{w}$ is perpendicular to both vectors $\vec{u}$, $\vec{v}$,
- Vectors $\vec{u}$, $\vec{v}$, $\vec{w}$ are a clockwise basis,
- $|\vec{w}| = |\vec{u}||\vec{v}| \cdot \sin \alpha$, where α is the angle of vectors $\vec{u}$, $\vec{v}$.

The cross product $\vec{w}$ of vectors $\vec{u}$, $\vec{v}$ is denoted as $\vec{w} = \vec{u} \times \vec{v}$. The last of the above properties has a nice geometrical interpretation: the area of a parallelogram defined by vectors $\vec{u}$, $\vec{v}$ equals the size of vector $\vec{w}$.

The coordinates of the cross product $\vec{w}$ of vectors $\vec{u} = [u_1, u_2, u_3]$ and $\vec{v} = [v_1, v_2, v_3]$ are:

$$\vec{w} = \vec{u} \times \vec{v} = [u_2v_3 - u_3v_2, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1].$$

The scalar triple product is a mix of the dot product and the cross product. Again, it is defined in space only. The scalar triple product of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is the number $(\vec{a} \times \vec{b}) \cdot \vec{c}$.

The scalar triple product of vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ is a real number equal to the volume of a parallelepiped defined by the three vectors represented in such a way that their starting points coincide.

Parametric expression of a line in plane

We know that a line is given by two mutually different points A, B . If we denote such a line as AB , then the vector $\vec{u} = B - A$ is called the direction vector of the line AB . Since each line consists of infinitely many points and since each multiple of its direction vector is again its direction vector, an arbitrary point X of a line AB can be written as $X = A + t\vec{u}$, $t \in \mathbb{R}$. The equation

$$X = A + t\vec{u} \quad t \in \mathbb{R}$$

is called the parametric equation of a line with the direction vector $\vec{u}$ which passes through the point A . The variable t is called the parameter.

If we are supposed to find out whether e.g. a point $P = [1, 2]$ is on a line $X = [2, 3] + t[-1, 2]$, $t \in \mathbb{R}$, we express the equation of a line in coordinates as:

$$\begin{aligned} x &= 2 - t, \\ y &= 3 + 2t, \quad t \in \mathbb{R} \end{aligned}$$

and substitute the coordinates of the point P for x and y . We get

$$\begin{aligned} 1 &= 2 - t, \\ 2 &= 3 + 2t, \quad t \in \mathbb{R} \end{aligned}$$

Now, from the first equation we get that $t = 1$ while from the second equation we get that $t = -\frac{1}{2}$. Since the two values are different, the point P is not on the line.

If, in the parametric equation of a line

$$X = A + t\vec{u}, \quad \text{where } t \in \mathbb{R} \text{ and } \vec{u} = B - A,$$

there is $t \in [0, \infty)$, then the point X moves along the half-line AB . For $t \in [0, 1]$, the point X moves along the line segment AB . Finally, for $t \in (-\infty, 0]$ the point X moves along the half-line BA .

Two lines defined by equations

$$X = P + t\vec{u} \quad t \in \mathbb{R} \text{ and } X = Q + s\vec{v} \quad s \in \mathbb{R}$$

are parallel if and only if the vector $\vec{u}$ is a multiple of the vector $\vec{v}$.

Standard form of a line in plane

Each line in plane can also be uniquely defined by a point and a nonzero vector perpendicular to the direction vector of the line. The vector perpendicular to the direction vector of a line is called normal segment for a given line. We denote it as $\vec{n}_p$.

If the coordinates of the normal segment for a line p are $\vec{n}_p = [a, b]$ and if the point P belonging to the line p is $P = [p_1, p_2]$, then there is

$$\vec{n}_p \cdot (X - P) = 0$$

for each point X of the line p . This means that the dot product of the normal segment $\vec{n}_p$ and an arbitrary directional vector of the line p is 0. After we substitute coordinates into this equality, we get

$$[a, b] \cdot [x - p_1, y - p_2] = 0 \Rightarrow ax - ap_1 + by - bp_2 = 0 \Rightarrow ax + by - (ap_1 + bp_2) = 0.$$

After we denote $c = -(ap_1 + bp_2)$, we get the relation

$$ax + by + c = 0.$$

This means that to every line p we may find such numbers a, b, c that the point $X = [x, y]$ is on that line if there is

$$ax + by + c = 0.$$

We already know that numbers a, b are the coordinates of the normal segment for the line p and that the number c is $c = -(ap_1 + bp_2)$, where p_1 and p_2 are coordinates of an arbitrary point of the line.

Two equations defined the same line if one is a multiple of another. Two lines are parallel with each other if and only if the normal segment for one line is a multiple of the normal segment for the other line.

Slope-intercept form of the equation of a line in plane

If there is $b \neq 0$ in the standard form of a line, we can express y from it. E.g. if the standard form is:

$$2x - 3y + 7 = 0,$$

we get

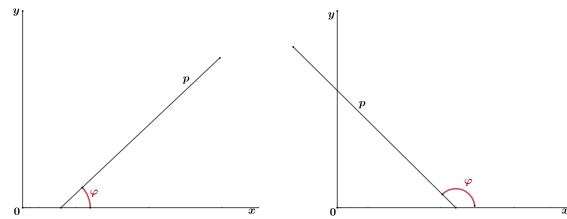
$$-3y = -2x - 7 \Rightarrow 3y = 2x + 7 \Rightarrow y = \frac{2}{3}x + \frac{7}{3}.$$

On contrary, if there is $b = 0$ in the standard form of a line p , then y is not present in it. The y -coordinate of the normal segment for p is zero, which means that it is parallel with the x -axis. As a result, the direction vector of p is parallel with the y -axis.

Therefore, all points $X = [x, y]$ coordinates of which fit the equation

$$y = kx + q,$$

belong to a line which is divergent from the y -axis. The above equation is called the slope-intercept form of a line. The number k is called the slope of the line. It describes the slope (or gradient) of the line. Its geometrical meaning is demonstrated in its angle with the positive x -half axis – see the figure:



The slope equals $\operatorname{tg}\varphi$, where φ is the angle of the line and the positive x -half axis. Recall that we have defined the angle φ as a number from the interval $[0, \pi]$.

Two parallel lines have the same angle with the positive x -half axis. Therefore, their slope is the same. The slope of a line perpendicular to a line with slope k is $k' = -\frac{1}{k}$.

Lines in space

Now we move to the study of lines in space. Beware that the standard form of a line in space does not exist! The parametric equation of a line in space can be found in a similar way as in a plane. The equation

$$X = A + t\vec{u} : t \in \mathbb{R}$$

is called the parametric equation of a line passing through the point A with the direction vector $\vec{u}$. The real number t is called the parameter.

Since there is an infinite number of choices for the point and direction vector, each line can be represented by an infinite number of parametric equations.

For example, a line p with the parametric equation $x = 2 + t, y = 3 - t, z = 1 + 2t$, where $t \in \mathbb{R}$ passes through the point $A = [2, 3, 1]$ and its direction vector is $\vec{u} = [1, -1, 2]$.

Each nonzero multiple of the vector $\vec{u}$ is its direction vector as well. Therefore, e.g. $x = 2 + 2s, y = 3 - 2s, z = 1 + 4s$ $s \in \mathbb{R}$ is one of its possible parametric equations.

If we know the parametric equation of a line, we are able to find other points on the line. For example, for $t = 1$ we get a point $B = [3, 2, 3]$, which means that also $x = 3 + r, y = 2 - r, z = 2 + 2r$ $r \in \mathbb{R}$ is a parametric equation of the same line.

Planes

Each plane is uniquely defined by two divergent lines. Lines in space are defined by parametric equations, i.e. by their direction vectors and one of their points. If we want to find a parametric equation of a plane, we make use of points A, B, C which are not collinear. If we denote $\vec{u} = B - A$, $\vec{v} = C - A$, equations

$$X = Y + s\vec{v} \quad s \in \mathbb{R} \quad \text{and} \quad Y = A + t\vec{u} \quad t \in \mathbb{R}$$

define lines with direction vectors $\vec{v}$ and $\vec{u}$, respectively. If we draw a line parallel with the line AC through each point Y of a line AB , we fill the plane defined by lines AB and AC . After we substitute the relation for Y from the second equation into the first equation, we get

$$X = A + t\vec{u} + s\vec{v} : s, t \in \mathbb{R}.$$

These equations are called parametric equations of a plane in space. Vectors $\vec{u}, \vec{v}$ are direction vectors of the plane.

However, planes are more often described using standard forms of their equations. As is the case of lines, planes are uniquely defined by means of one point and a normal segment. The normal segment for a plane ρ is perpendicular to all vectors in that plane.

The equation

$$ax + by + cz + d = 0$$

is called the standard form of the equation of a plane.

We obtain this equation in a way analogous to the equation of a line. Therefore, the coordinates of the normal segment are $[a, b, c]$ and for the coordinates of an arbitrary point $P = [p_1, p_2, p_3]$ of this plane we have

$$d = -ap_1 - bp_2 - cp_3.$$

Exercise

Exercise 1. Find the equation of the line p passing through the point $A = [2, 3]$ and parallel with the line

1. $y = 2$,
2. $x = 4$,
3. $2x + 3y = 4$,
4. $y = 3x + 2$,
5. $x = 2 - t, y = 1 + 2t, t \in \mathbb{R}$.

Solution. 1. For the x -coordinates of points of the line $y = 2$ there is $x \in \mathbb{R}$ while their y -coordinate does not change and always equals 2. Points on the line passing through the point $A = [2, 3]$ and parallel with the given line will have their y -coordinate the same as the point A while their x -coordinate can be an arbitrary real number. Therefore, the line we are looking for is $y = 3$.

2. For the y -coordinates of points of the line $x = 4$ there is $y \in \mathbb{R}$ while their x -coordinate does not change and always equals 4. Points on the line passing through the point $A = [2, 3]$ and parallel with the given line will have their x -coordinate the same as the point A while their y -coordinate can be an arbitrary real number. Therefore, the line we are looking for is $x = 2$.

3. Since the standard form is $2x + 3y = 4$, the normal segment for this line is $\vec{n} = [2, 3]$. Two parallel lines have the same normal segments, which means that the standard form of the line we are looking for is:

$$2x + 3y + c = 0,$$

where the value of c follows from substituting the point A which is on the line, i.e.:

$$2 \cdot 2 + 3 \cdot 3 + c = 0 \iff 13 + c = 0 \iff c = -13.$$

This means that the line is:

$$2x + 3y - 13 = 0.$$

4. From the slope-intercept form of the equation we get that the slope is $k = 3$. Two parallel lines have the same slope, which means that the slope-intercept form of the equation is:

$$y = 3x + q,$$

where the value of c follows from substituting the point A which is on the line, i.e.:

$$3 = 3 \cdot 2 + q \iff q = -3.$$

This means that the line is:

$$y = 3x - 3.$$

5. The parametric equation of the line $x = 2 - t, y = 1 + 2t, t \in \mathbb{R}$, suggests that the line passes through the point $B = [2, 1]$ and that its direction vector is $\vec{s} = [-1, 2]$. Two parallel lines have the same direction vector. We also know that the line we are looking for passes through the point A . This means that its parametric equation is:

$$x = 2 - t, y = 3 + 2t, t \in \mathbb{R}.$$

Exercise

Exercise 2. Find the equation of the line p passing through the point $A = [2, 3]$ and perpendicular to the line

1. $y = 2$,
2. $x = 4$,
3. $2x + 3y = 4$,
4. $y = 3x + 2$,
5. $x = 2 - t, y = 1 + 2t, t \in \mathbb{R}$.

Solution. 1. For the x -coordinates of points of the line $y = 2$ there is $x \in \mathbb{R}$ while their y -coordinate does not change and always equals 2. Points on the line passing through the point $A = [2, 3]$ and perpendicular to the given line will have their x -coordinate the same as the point A while their y -coordinate can be an arbitrary real number. Therefore, the line we are looking for is $x = 2$.

2. For the x -coordinates of points of the line $x = 4$ there is $y \in \mathbb{R}$ while their x -coordinate does not change and always equals 4. Points on the line passing through the point $A = [2, 3]$ and perpendicular to the given line will have their y -coordinate the same as the point A while their x -coordinate can be an arbitrary real number. Therefore, the line we are looking for is $y = 3$.

3. Since the standard form is $2x + 3y = 4$, the normal segment for this line is $\vec{n} = [2, 3]$. The new line is perpendicular to the original line, which means that its normal segment is its direction vector. The normal segment for the new line therefore e.g. $[3, -2]$, which means that standard form of the new line is:

$$3x - 2y + c = 0,$$

where the value of c follows from substituting the point A which is on the line, i.e.:

$$3 \cdot 2 - 2 \cdot 3 + c = 0 \iff c = 0.$$

This means that the line is:

$$3x - 2y = 0.$$

4. The slope-intercept form $y = 3x + 2$ suggests that the slope is $k = 3$. We know that the slope k' of a line perpendicular to the original line is:

$$y = -\frac{1}{3}x + q,$$

where the value of c follows from substituting the point A which is on the line, i.e.:

$$3 = -\frac{1}{3} \cdot 2 + q \iff q = \frac{11}{3}.$$

This means that the line is:

$$y = 3x + \frac{11}{3}.$$

5. The parametric equation of the line $x = 2 - t, y = 1 + 2t, t \in \mathbb{R}$ suggests the line passes through the point $B = [2, 1]$ and that its direction vector is $\vec{s} = [-1, 2]$. The direction vectors of two perpendicular lines are perpendicular. Moreover, the new line is supposed to pass through the point A . A vector perpendicular to $\vec{s} = [-1, 2]$ is e.g. $\vec{n} = [2, 1]$. Therefore, the parametric equation of the new line is

$$x = 2 + 2t, y = 3 + t, t \in \mathbb{R}.$$

Exercise

Exercise 3. Establish the mutual position of two lines p, q , where p is defined by $3x - 3y = 9$ and q is defined parametrically by $x = 2 + t, y = 1 + 2t, t \in \mathbb{R}$.

Solution. Two lines in a plane can either coincide, or they can be parallel yet not equal, or they can be divergent. The number of points in their intersection is thus infinite, zero or one, respectively. Since the points of intersection must fit into both equations, we take the parametric equation of one of the lines and substitute it into the standard form of the equation of the other line. We get:

$$3(2 + t) - 3(1 + 2t) = 9,$$

which is equivalent to

$$6 + 3t - 3 - 6t = 9 \iff 3 - 3t = 9 \iff t = -2.$$

The result $t = -2$ means that the lines have only one point in common. Its coordinates are:

$$x = 2 + (-2) = 0, y = 1 + 2 \cdot (-2) = -3 \Rightarrow A = [0, 3].$$

As a quick check we substitute the coordinates of the point A into the equation of the line p as well:

$$3 \cdot (0) - 3 \cdot (-3) = 9.$$

Since the two lines have only one point in common, they are divergent.

Exercise

Exercise 4. Establish the mutual position of a line p and a plane ρ , where p is defined parametrically as $x = 2 + t, y = 1 + 2t, z = 2 - t; t \in \mathbb{R}$ and ρ is defined as $2x - 3y + z - 1 = 0$.

Solution. A line can be part of the plane, the line and the plane can be parallel, or they can be divergent. In the first case they have infinitely many points in common, in the second case they no common points and finally, in the third case, they have one common point. The points belonging to both the line and the plane must fit into both equations. Therefore, we find them after substituting the respective equations for x, y and z from the parametric equation of the line into the equation of the plane:

$$2(2 + t) - 3(1 + 2t) + 1 \cdot (2 - t) - 1 = 0,$$

This is equivalent to

$$4 + 2t - 3 - 6t + 2 - t - 1 = 0 \iff 2 - 5t = 0 \iff t = \frac{2}{5}.$$

The result $t = \frac{2}{5}$ means that the line and the plane have one point in common. Its coordinates are:

$$x = 2 + \frac{2}{5} = \frac{12}{5}, y = 1 + 2 \cdot \frac{2}{5} = \frac{9}{5}, z = 2 - \frac{2}{5} = \frac{8}{5} \Rightarrow A = \left[\frac{12}{5}, \frac{9}{5}, \frac{8}{5} \right].$$

As a form of a quick check we substitute the coordinates of the point A into the equation of the plane. We get that:

$$2 \cdot \frac{12}{5} - 3 \cdot \frac{9}{5} + \frac{8}{5} - 1 = 0.$$

Since the line and the plane have one point in common, it means that they are divergent.

Exercise

Exercise 5. Find such a point A on the line $p : x = 5 + 2t, y = -4 - t, z = 3 + t, t \in \mathbb{R}$ which is as close to the point $B = [3, 1, 0]$ as possible. Calculate the distance of those points.

Solution. The direction vector of the line p is $[2, -1, 1]$. The point A belongs to the line, which means that there is $A = [5 + 2t, -4 - t, 3 + t]$ for some $t \in \mathbb{R}$. The point B does not belong to the line p because no such $t \in \mathbb{R}$ that there is simultaneously:

$$(3 = 5 + 2t) \quad \wedge \quad (1 = -4 - t) \quad \wedge \quad (0 = 3 + t)$$

exists. Since we want that A is a point “as close to B as possible”, the line AB must be perpendicular to the line p . Therefore, the dot product of their direction vectors must be zero. Since the direction vector of the line AB is $[-2 - 2t, 5 + t, -3 - t]$, we get that

$$[-2 - 2t, 5 + t, -3 - t] \cdot [2, -1, 1] = 0.$$

Therefore,

$$2 \cdot (-2 - 2t) + (-1) \cdot (5 + t) + 1 \cdot (-3 - t) = 0 \iff t = -2.$$

Regarding the coordinates of the point A we have that $A = [5 + (-2) \cdot 2, -4 - (-2), 3 + (-2)] = [1, -2, 1]$. The distance from the point A to the point B is

$$|AB| = \sqrt{(3 - 1)^2 + (1 - (-2))^2 + (0 - 1)^2} = \sqrt{4 + 4 + 1} = 3.$$

Exercise

Exercise 6. Find the angle of lines p, q , where $p : x = 1 + t, y = 1 + t, z = 1; t \in \mathbb{R}$ and $q : x = 2s, y = 3 + 9s, z = -1 + 6s; s \in \mathbb{R}$.

Solution. In order to find the angle of the lines we need to find the angle of their direction vectors. The direction vectors obviously are $\vec{s}_p = [1, 1, 0]$ and $\vec{s}_q = [2, 9, 6]$. Their angle is

$$\cos \varphi = \frac{\vec{s}_p \cdot \vec{s}_q}{|\vec{s}_p| \cdot |\vec{s}_q|}.$$

After substitution we get

$$\cos \varphi = \frac{[1, 1, 0] \cdot [2, 9, 6]}{\sqrt{1^2 + 1^2 + 0^2} \cdot \sqrt{2^2 + 9^2 + 6^2}} = \frac{1 \cdot 2 + 1 \cdot 9 + 0 \cdot 6}{11 \cdot \sqrt{2}} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2} \Rightarrow \varphi = \frac{\pi}{4}.$$

Therefore, the angle of the lines is $\frac{\pi}{4}$, i.e. 45° .

Exercise

Exercise 7. Establish the mutual position of planes α, β , where $\alpha: x - 2y + 4z = 3, \beta: 3x - y + 3z = 1$.

Solution. Two planes can either coincide, or be parallel with each other yet not coincide, or they can be divergent. The coefficients in the standard form of the equation of a plane are – one by one – coordinates of the vector perpendicular to the given plane. Vectors $\vec{v}_1, \vec{v}_2$, perpendicular to parallel planes are parallel, i.e. there exists $c \in \mathbb{R}$ such that $\vec{v}_1 = c \cdot \vec{v}_2$. In our case there is $\vec{v}_\alpha = [1, -2, 4], \vec{v}_\beta = [3, -1, 3]$, i.e. the two vectors obviously are not parallel with each other. As a result, neither are planes α, β parallel with each other. Therefore, they are divergent and their common points are solutions of the system:

$$\begin{aligned}x - 2y + 4z &= 3 \\ 3x - y + 3z &= 1.\end{aligned}$$

We eliminate one variable, say, y . Therefore, we multiply the second equation by -2 , add it to the first equation and get:

$$-2x - 2z = 1.$$

Our system has only two equations yet three variables. This means that we will obtain a parametric solution. We choose z as a parameter. Therefore $z = t$ and $x = -t - \frac{1}{2}$. We substitute to the first equation and express y . Thus we get:

$$\left(-t - \frac{1}{2}\right) - 2y + 4t = 3 \Rightarrow y = -\frac{3}{2}t + \frac{7}{4}.$$

The solution is $\left\{\left[-t - \frac{1}{2}, -\frac{3}{2}t + \frac{7}{4}, t\right] : t \in \mathbb{R}\right\}$, which is a parametric equation of a line.

Exercise

Exercise 8. Find the equation of the axis of the line segment AB , where $A = [2, 1]$, $B = [4, 7]$.

Solution. The axis of a line segment is perpendicular to it and passes through the center of the line segment. First, we find the center of the line segment AB . It is $S = \left[\frac{2+4}{2}, \frac{1+7}{2}\right] = [3, 4]$. Next we find the coordinates of the vector $AB = B - A = [2, 6]$. The vector, perpendicular to AB , is $[6, -2]$. Therefore, one of the possible parametric equations of the axis of the line segment is $x = 3 + 6t, y = 4 - 2t: t \in \mathbb{R}$. The standard form of the equation is $2x + 6y + c = 0$ because the vector AB is perpendicular to this axis. After we substitute the center, we get that $2 \cdot 3 + 6 \cdot 4 + c = 0$, i.e. $c = -30$. As a result, the standard form of the equation is $2x + 6y - 30 = 0$, or rather $x + 3y - 15 = 0$.

Exercise

Exercise 9. Find the value of the parameter $c \in \mathbb{R}$ so that vectors $\vec{a} = [-2, 3, c]$, $\vec{b} = [5, c, -8]$ were perpendicular.

Solution. Two vectors are perpendicular if their dot product is zero. Therefore,

$$[-2, 3, c] \cdot [5, c, -8] = 0 \iff (-2) \cdot 5 + 3c - 8c = 0 \iff -10 - 5c = 0 \iff c = -1.$$

Vectors $\vec{a} = [-2, 3, c]$, $\vec{b} = [5, c, -8]$ are perpendicular for $c = -1$.

1 TEST

1. The equation of a line passing through points $A = [1, 1]$ and $B = [2, 4]$ is
 - (a) $3x + y - 2 = 0$
 - (b) $3x - y - 2 = 0$
 - (c) $3x - y + 2 = 0$
 - (d) $-3x - y - 2 = 0$
2. The equation of a line p passing through the point $A = [2, 3]$ and parallel with the line $p : y = 7$ is
 - (a) $x = 3$
 - (b) $y = 2$
 - (c) $x = 2$
 - (d) $y = 3$
3. The equation of a line p passing through the point $A = [2, 3]$ and parallel with the line $p : 2x + 3y = 6$ is
 - (a) $2x + 3y - 13 = 0$
 - (b) $2x - 3y - 13 = 0$
 - (c) $2x + 3y + 13 = 0$
 - (d) $-2x + 3y + 13 = 0$
4. The equation of a line p passing through the point $A = [2, 3]$ and perpendicular to the line $p : y = 2x - 7$, má tvar
 - (a) $y = \frac{1}{2}x + 4$
 - (b) $y = -\frac{1}{2}x + 4$
 - (c) $y = \frac{1}{2}x - 4$
 - (d) $y = 2x - 4$
5. Lines $p : x - 2y = 7, q : 2x + y = 5$ are
 - (a) identical,
 - (b) parallel lines,
 - (c) skew lines,
 - (d) perpendicular lines.
6. The common points of lines $p : x - 2y = 7, q : 2x - y = 5$ are
 - (a) $A = \left[\frac{17}{5}, -\frac{9}{5}\right], B = [0, 0]$
 - (b) $A = \left[\frac{17}{5}, -\frac{9}{5}\right]$
 - (c) such points do not exist
 - (d) $A = \left[-\frac{17}{5}, -\frac{9}{5}\right], B = [0, 1]$
7. The angle of lines p, q , where $p : x = 1 + t, y = 1 - t, z = 2; t \in \mathbb{R}$ and $q : x = 5, y = 3 - s, z = 6 + s; s \in \mathbb{R}$ is

- (a) 45°
- (b) 90°
- (c) 60°
- (d) 15°

8. On the line $p : x = 5 - 2t, y = -4 - 2t, z = 3 - t, t \in \mathbb{R}$ find such a point A which is as close as possible to the point $B = [3, 2, 4]$.

- (a) $A = [7, -2, 4]$
- (b) $A = [1, 2, 4]$
- (c) $A = [3, -6, 2]$
- (d) $A = [-7, 2, -4]$

9. The line of the median from the point A in the triangle ABC , where $A = [0, 3], B = [3, 1], C = [1, 0]$, is:

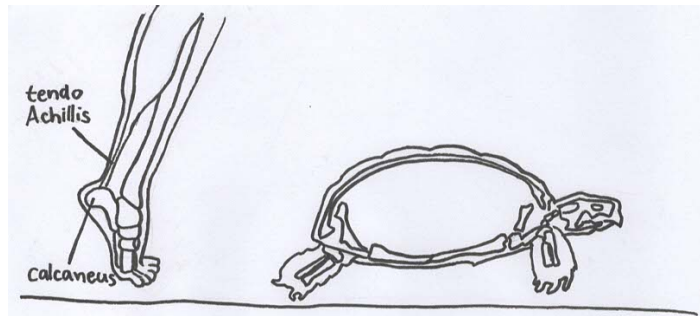
- (a) $5x - 4y - 9 = 0$
- (b) $5x + 4y - 12 = 0$
- (c) $5x + 4y + 12 = 0$
- (d) $4x + 5y - 12 = 0$

10. The value of the parameter c for which the vectors $\vec{a} = [-1, 4, c], \vec{b} = [2, c, -1]$ are perpendicular is

- (a) $c = \frac{2}{3}$
- (b) $c = \frac{3}{2}$
- (c) $c = -\frac{2}{3}$
- (d) such value $c \in \mathbb{R}$ does not exist

9. SEQUENCES AND SERIES

Sequences and series



“Achilles and a tortoise” is a well-known paradox which (provided we accept a certain way of reasoning) demonstrates that Achilles will never manage to overcome the tortoise because every time he runs to the spot where the tortoise had already been, the tortoise is a bit further. This problem can be written in the form of an infinite series of shortening lengths such as e.g.

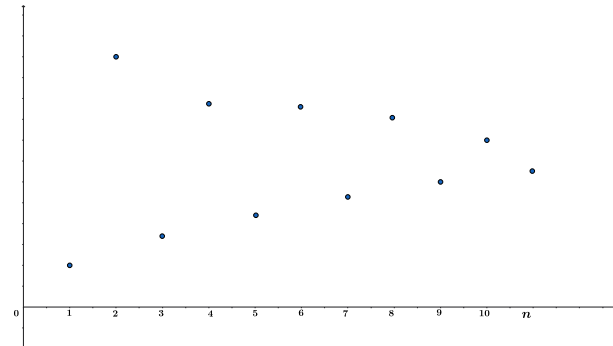
$$10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots + .$$

However, the sum of such a series is finite, which in the past contradicted the idea that the sum of an infinite series must be always infinite. In this chapter we will deal with infinite sequences and series.

Sequences

Sequences are mappings with domains of definition D being either the set of natural numbers or some of their subsets. If the domain of definition of a sequence is a finite set, the sequence is finite. If the domain of definition of a sequence is an infinite set, the sequence is infinite. Usually, we denote the elements of the domain of definition by n (or k, l, i , etc.) The function values of sequences are usually denote as a_n (or b_n, a_k , etc.). We call them terms (or members) of a sequence.

Infinite sequences may be defined by a relation for evaluating the n^{th} term of a sequence. We denote this as $(a_n)_{n=1}^{\infty}$. Alternatively, we use directly the defining formula instead of a_n and write e.g. $(2n - 6)_{n=1}^{\infty}$. Such a notation is to be understood as $a_1 = 2 \cdot 1 - 6 = -4, a_2 = 2 \cdot 2 - 6 = -2, \dots, a_n = 2 \cdot n - 6, \dots$. Sequences may also be defined by recursive formulas, which means that we know values of the first k terms and the formula for the $(k + 1)^{\text{th}}$ term is given by means of the preceding k terms. For example, the well-known Fibonacci sequence is defined recursively by $a_1 = a_2 = 1, a_{n+2} = a_n + a_{n+1}$. Finally, sequences may be defined by graphs as well:



Exercise

Exercise 1. Find the first five terms of a sequence $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^n \cdot \frac{n^2}{n+1}$.

Solution. We find the respective terms by substituting $n = 1$ to $n = 5$ into the relation $a_n = (-1)^n \cdot \frac{n^2}{n+1}$. For $n = 1$ we get

$$a_1 = (-1)^1 \cdot \frac{1^2}{1+1} = -\frac{1}{2}.$$

Analogically, there is:

$$a_2 = (-1)^2 \cdot \frac{2^2}{2+1} = \frac{4}{3},$$

$$a_3 = (-1)^3 \cdot \frac{3^2}{3+1} = -\frac{9}{4},$$

$$a_4 = (-1)^4 \cdot \frac{4^2}{4+1} = \frac{16}{5},$$

$$a_5 = (-1)^5 \cdot \frac{5^2}{5+1} = -\frac{25}{6}.$$

Some properties of sequences

Since sequences are special types of functions, properties of sequences are similar to those of functions. The same is true for their definitions. We will be especially interested in the monotonicity of sequences:

A sequence (a_n) is called increasing if there is

$$\forall n, m \in D: n < m \Rightarrow a_n < a_m.$$

A sequence (a_n) is called decreasing if there is

$$\forall n, m \in D: n < m \Rightarrow a_n > a_m.$$

A sequence (a_n) is called non-increasing if there is

$$\forall n, m \in D: n < m \Rightarrow a_n \geq a_m.$$

A sequence (a_n) is called non-decreasing if there is

$$\forall n, m \in D: n < m \Rightarrow a_n \leq a_m.$$

Remark 2. *The following theorem, included for infinite sequences yet easily modifiable for finite ones, is a useful tool for establishing monotonicity of sequences:*

A sequence $(a_n)_{n=1}^{\infty}$ is *increasing* if there is

$$\forall n \in \mathbb{N}: a_n < a_{n+1}.$$

A sequence $(a_n)_{n=1}^{\infty}$ is *decreasing* if there is

$$\forall n \in \mathbb{N}: a_n > a_{n+1}.$$

A sequence $(a_n)_{n=1}^{\infty}$ is *non-increasing* if there is

$$\forall n \in \mathbb{N}: a_n \geq a_{n+1}.$$

A sequence $(a_n)_{n=1}^{\infty}$ is *non-decreasing* if there is

$$\forall n \in \mathbb{N}: a_n \leq a_{n+1}.$$

Boundedness of sequences is again a special case of boundedness of functions.

A sequence $(a_n)_{n=1}^{\infty}$ is bounded from above if there is

$$\exists h \in \mathbb{R} \forall n \in D: a_n \leq h.$$

A sequence $(a_n)_{n=1}^{\infty}$ is bounded from below if there is

$$\exists d \in \mathbb{R} \forall n \in D: a_n \geq d.$$

A sequence $(a_n)_{n=1}^{\infty}$ is bounded if there is

$$\exists h, d \in \mathbb{R} \forall n \in D: d \leq a_n \leq h.$$

Exercise

Exercise 3. Find the first five terms of a sequence $(a_n)_{n=1}^{\infty}$, where $a_n = \frac{n-2}{n+1}$ and find out whether the sequence is increasing or decreasing and whether it is bounded.

Solution. We find the first five terms in the same way as in the previous exercise:

$$\begin{aligned} a_1 &= \frac{1-2}{1+1} = -\frac{1}{2}, \\ a_2 &= \frac{2-2}{2+1} = 0, \\ a_3 &= \frac{3-2}{3+1} = \frac{1}{4}, \\ a_4 &= \frac{4-2}{4+1} = \frac{2}{5}, \\ a_5 &= \frac{5-2}{5+1} = \frac{3}{6}. \end{aligned}$$

The values of the first five terms will direct us in finding answers to the remaining questions. The defining relation of the sequence suggests that – with the exception of the first two terms – all values of the sequence will be nonnegative. Given the first two terms we can write that there is $a_n \geq -\frac{1}{2}$ for an arbitrary $n \in \mathbb{N}$. This means that the sequence is bounded from below by $-\frac{1}{2}$. The defining relation of the sequence can be transformed into:

$$a_n = \frac{n-2}{n+1} = \frac{n+1-3}{n+1} = \frac{n+1}{n+1} - \frac{3}{n+1} = 1 - \frac{3}{n+1}.$$

Obviously, there is $a_n < 1$ for an arbitrary $n \in \mathbb{N}$. This means that the sequence is bounded from above by 1. Therefore, the sequence is bounded. Moreover, for the first five terms we have that:

$$a_1 < a_2 < a_3 < a_4 < a_5.$$

This means that the sequence cannot be decreasing. Using Remark 2 we verify whether the sequence is increasing. In other words, we find out whether, for an arbitrary $n \in \mathbb{N}$, there is $a_n < a_{n+1}$:

$$a_n < a_{n+1} \iff \frac{n-2}{n+1} < \frac{(n+1)-2}{(n+1)+1} \iff \frac{n-2}{n+1} < \frac{n-1}{n+2}.$$

We nullify and transform the last inequation:

$$\begin{aligned} \frac{n-2}{n+1} &< \frac{n-1}{n+2} \\ \frac{n-2}{n+1} - \frac{n-1}{n+2} &< 0 \\ \frac{(n+2) \cdot (n-2) - (n+1) \cdot (n-1)}{(n+1) \cdot (n+2)} &< 0 \\ \frac{n^2 - 4 - n^2 + 1}{(n+1) \cdot (n+2)} &< 0 \\ \frac{-3}{(n+1) \cdot (n+2)} &< 0. \end{aligned}$$

The last inequation is true, which means that the expression on the left-hand side is negative for all natural numbers n . Its nominator is negative and its denominator has the form of a product of natural numbers. As a result, the sequence is increasing.

Arithmetic sequence (or arithmetic progression)

Now we focus on a special type of a sequence, in which the differences between two consecutive terms are constant. Such sequences are called arithmetric sequences (or arithmetic progressions) and the constant difference is called difference of the sequence. In other words, a sequence is called arithmetic if

$$\exists d \in \mathbb{R} \forall n \in \mathbb{N}: a_{n+1} = a_n + d.$$

Thanks to the constant difference we can write the n^{th} term of each arithmetic sequence with the help of its first term and difference as

$$a_n = a_1 + d(n - 1).$$

Moreover, the sum of the first n terms of each arithmetic sequence can be written as:

$$s_n = \frac{n}{2} \cdot (a_1 + a_n).$$

Exercise

Exercise 4. We know that, in some arithmetic sequence, there is, for some $n \in \mathbb{N}$: $a_{n-1} = 35, a_{n+1} = 43$. Find a_n and the difference d .

Solution. We know that for the terms and the difference of arithmetic sequences there is:

$$\begin{aligned}a_n &= a_1 + (n - 1) \cdot d, \\a_n - a_{n-1} &= d.\end{aligned}$$

Then there is

$$\begin{aligned}a_{n-1} &= a_1 + (n - 2) \cdot d, \\a_{n+1} &= a_1 + n \cdot d.\end{aligned}$$

After we make a sum of terms a_{n-1}, a_{n+1} , we get:

$$a_{n-1} + a_{n+1} = a_1 + (n - 2) \cdot d + a_1 + n \cdot d = 2a_1 + 2nd - 2d = 2 \cdot (a_1 + (n - 1) \cdot d) = 2a_n.$$

Therefore, there is

$$a_n = \frac{a_{n-1} + a_{n+1}}{2},$$

which means that in each arithmetic sequence the terms a_n is the arithmetic average of the previous and following terms. Therefore, since there is $a_{n-1} = 35, a_{n+1} = 43$, we get that $a_n = 39$ and the difference is $d = 4$.

Geometric sequence (or geometric progression)

Geometric sequence (or geometric progression) is another special type of sequences. In geometric sequences, the ratio of two consecutive terms is constant. There is

$$\exists q \in \mathbb{R} \forall n \in \mathbb{N}: a_{n+1} = a_n \cdot q.$$

We call the number q the quotient of the geometric sequence.

Thanks to the constant quotient we can write the n^{th} term of each geometric sequence with the help of its first term and the quotient as

$$a_n = a_1 \cdot q^{n-1}.$$

Moreover, the sum of the first n terms of each geometric sequence can be written as:

$$\begin{aligned} s_n &= n \cdot a_1, \text{ pro } q = 1, \\ s_n &= a_1 \frac{q^n - 1}{q - 1}, \text{ pro } q \neq 1. \end{aligned}$$

Exercise

Exercise 5. We know that, in some geometric sequence, there is $a_1 = 3, q = -\frac{1}{2}$. Find the first five terms of the sequence and the sum of the first ten terms of the sequence.

Solution. We know that, for all terms of geometric sequences, there is

$$a_n = a_1 \cdot q^{n-1}.$$

This means that there is

$$a_2 = a_1 \cdot q = 3 \cdot \left(-\frac{1}{2}\right) = -\frac{3}{2},$$

$$a_3 = a_1 \cdot q^2 = 3 \cdot \left(-\frac{1}{2}\right)^2 = \frac{3}{4},$$

$$a_4 = a_1 \cdot q^3 = 3 \cdot \left(-\frac{1}{2}\right)^3 = -\frac{3}{8},$$

$$a_5 = a_1 \cdot q^4 = 3 \cdot \left(-\frac{1}{2}\right)^4 = \frac{3}{16}.$$

Given the fact that $q \neq 1$, we find the sum of the first 10 terms using the formula:

$$s_n = a_1 \frac{q^n - 1}{q - 1}.$$

Therefore, there is

$$s_{10} = 3 \cdot \frac{\left(-\frac{1}{2}\right)^{10} - 1}{-\frac{1}{2} - 1} = 3 \cdot \frac{\frac{1}{1024} - 1}{-\frac{3}{2}} = 3 \cdot \frac{-\frac{1023}{1024}}{-\frac{3}{2}} = \frac{1023}{512}.$$

Proper limit

The real number a is called the proper limit of the sequence $(a_n)_{n=1}^{\infty}$ if there is

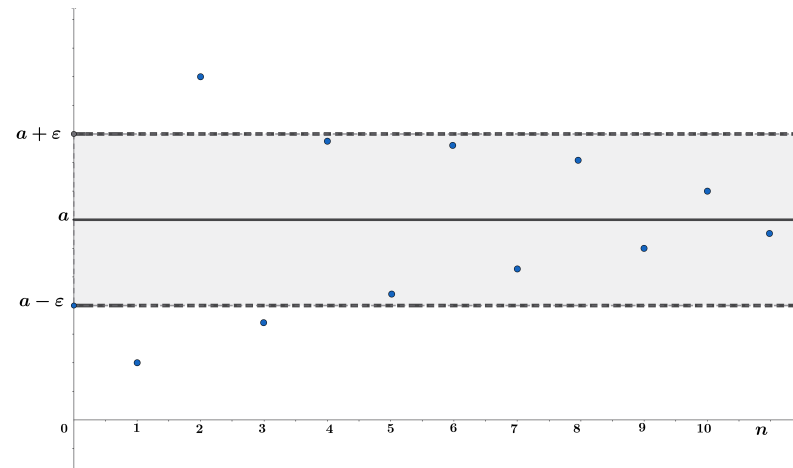
$$\forall \varepsilon > 0 \quad \exists n_0; \quad n > n_0 \implies |a_n - a| < \varepsilon.$$

We write this as

$$\lim_{n \rightarrow \infty} a_n = a.$$

Sequences with proper limits are called convergent. All other sequences are called divergent.

What is the interpretation of the definition? See the figure below.



In the figure we can see a small segment of the sequence. Imagine that all further terms of the sequence (those to the right) are within the infinite highlighted ε stripe. We usually express distance by means of the absolute value. This means that to a positive ε we can find $n_0 = 3$ so that for all $n > 3$ there was $|a - a_n| < \varepsilon$. If we think of a smaller ε , we shift n_0 to the right. If we, after an arbitrary decrease of ε , are able to find n_0 such that the distance of any term following this n_0 from a is smaller than ε , i.e. all these terms will be in the respective ε -stripe, the sequence will converge to the number a .

Obviously, each sequence can have maximum one limit. The notion of a limit is closely linked to the notion of boundedness. Indeed, each convergent sequence is bounded. Moreover, if a bounded sequence is monotonic, it is convergent.

Improper limit

Suppose that for each real number K there exists an n_0 such that for each $n > n_0$ there is $a_n > K$. In such a case we say that the sequence has an improper limit $+\infty$ and we write:

$$\lim_{n \rightarrow \infty} a_n = +\infty.$$

We understand this definition in such a way that for an arbitrarily great real number K we are able to find an n_0 such that all terms to the right of a_{n_0} are greater than K .

The improper limit $-\infty$ is defined analogically:

Suppose that for each real number K there exists an n_0 such that for each $n > n_0$ there is $a_n < K$. In such a case we say that the sequence has an improper limit $-\infty$ and write:

$$\lim_{n \rightarrow \infty} a_n = -\infty.$$

As has been already mentioned, sequences converge if they have proper limits. In the opposite case, i.e. if the limits are either improper ones, or they do not exist, the sequences diverge.

Exercise

Exercise 6. Prove that the sequence $\left(\frac{n}{n+1}\right)_{n=1}^{\infty}$ converges to 1.

Solution. We need to show that for an arbitrary real number ε we are able to find such a natural number n_0 that for each natural number $n > n_0$ there is:

$$\left|\frac{n}{n+1} - 1\right| < \varepsilon.$$

After we make the difference of the terms and remove the absolute value, we get that

$$\left|\frac{n - n - 1}{n + 1}\right| = \left|\frac{-1}{n + 1}\right| = \frac{1}{n + 1} < \varepsilon.$$

Since ε as well as $n + 1$ are positive numbers, we can rewrite the inequation as:

$$\frac{1}{\varepsilon} < n + 1,$$

which means that

$$n > \frac{1}{\varepsilon} - 1.$$

Since there is $\frac{1}{\varepsilon} - 1 \geq 1$, the n_0 we are looking for can be the closest smaller or equal natural number. If there is $\frac{1}{\varepsilon} - 1 < 1$, then $n_0 = 1$. As a result, if there is e.g. $\varepsilon = \frac{1}{100}$, then $\frac{1}{\varepsilon} - 1 = 99 \in \mathbb{N}$. Therefore, there is $n_0 = 99$.

You may ask how we found out that the sequence converges to 1 and not to some other number. After you write several terms of this sequence, you will find out that with n increasing the values of terms approach 1.

Properties of limits

Since treating every sequence the way we did in the previous exercise is rather impractical, we now include several properties of limits which will simplify our future computations:

Suppose that there is $\lim_{n \rightarrow \infty} a_n = a$, $\lim_{n \rightarrow \infty} b_n = b$. Then there is

1. $\lim_{n \rightarrow \infty} (a_n + b_n) = a + b$.
2. $\lim_{n \rightarrow \infty} a_n \cdot b_n = a \cdot b$.
3. For $b \neq 0$ there is $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{a}{b}$.
4. $\lim_{n \rightarrow \infty} |a_n| = |a|$.
5. Suppose that $\lim_{n \rightarrow \infty} a_n = 0$ and (b_n) is a bounded sequence. Then there is $\lim_{n \rightarrow \infty} a_n \cdot b_n = 0$.
6. $\lim_{n \rightarrow \infty} |a_n| = \infty \wedge a_n \neq 0 \ \forall n \in \mathbb{N} \Rightarrow \lim_{n \rightarrow \infty} \frac{1}{a_n} = 0$.
7. the fact that $\lim_{n \rightarrow \infty} \frac{1}{a_n} = 0$ and almost all a_n are positive, means that $\lim_{n \rightarrow \infty} a_n = \infty$.
8. the fact that $\lim_{n \rightarrow \infty} \frac{1}{a_n} = 0$ and almost all a_n are negative, means that $\lim_{n \rightarrow \infty} a_n = -\infty$.
9. $\lim_{n \rightarrow \infty} \frac{1}{n^r} = 0$.
10. $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$.

Exercise

Exercise 7. Find $\lim_{n \rightarrow \infty} \frac{3-5n^2-2n^4}{3-2n^5}$.

Solution. There is a polynomial both in the nominator and in the denominator. In such cases, the computations are easy. First we take n^k , where k is the greatest exponent present in both polynomials, out of the brackets both in the nominator and in the denominator. In our case we take out n^5 :

$$\lim_{n \rightarrow \infty} \frac{3-5n^2-2n^4}{3-2n^5} = \lim_{n \rightarrow \infty} \frac{n^5 \left(\frac{3}{n^5} - \frac{5}{n^3} - \frac{2}{n} \right)}{n^5 \left(\frac{3}{n^5} - 2 \right)} = \lim_{n \rightarrow \infty} \frac{\frac{3}{n^5} - \frac{5}{n^3} - \frac{2}{n}}{\frac{3}{n^5} - 2}.$$

We know that sequences $\left(\frac{1}{n^r}\right)_{n=1}^{\infty} : r > 0$ converge to 0. Moreover, from the properties of limits we know that if sequences have proper limits, then their sum (or difference, or ratio) converge to the sum (or difference, or ratio) of the limits. Therefore, there is

$$\lim_{n \rightarrow \infty} \frac{\frac{3}{n^5} - \frac{5}{n^3} - \frac{2}{n}}{\frac{3}{n^5} - 2} = \frac{0 - 0 - 0}{0 - 2} = \frac{0}{2} = 0.$$

Exercise

Exercise 8. Find $\lim_{n \rightarrow \infty} \frac{2n^2+3n-1}{3n^2-2n+1}$.

Solution. We work in a similar fashion as in the previous exercise:

$$\lim_{n \rightarrow \infty} \frac{2n^2 + 3n - 1}{3n^2 - 2n + 1} = \lim_{n \rightarrow \infty} \frac{n^2 \left(+\frac{3}{n} - \frac{1}{n^2} \right)}{n^2 \left(-\frac{2}{n} + \frac{1}{n^2} \right)} = \lim_{n \rightarrow \infty} \frac{2 + \frac{3}{n} - \frac{1}{n^2}}{3 - \frac{2}{n} + \frac{1}{n^2}} = \frac{2 + 0 - 0}{3 - 0 + 0} = \frac{2}{3}.$$

Exercise

Exercise 9. Find $\lim_{n \rightarrow \infty} \frac{2n^3+1}{3n^2+n}$.

Solution. Once again, our reasoning is remains the same:

$$\lim_{n \rightarrow \infty} \frac{2n^3+1}{3n^2+n} = \lim_{n \rightarrow \infty} \frac{n^3 \left(2 + \frac{1}{n^3}\right)}{n^3 \left(\frac{3}{n} + \frac{1}{n^2}\right)} = \lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n^3}}{\frac{3}{n} + \frac{1}{n^2}}.$$

Notice the inverted value of the last expression. There is

$$\lim_{n \rightarrow \infty} \frac{1}{\frac{2 + \frac{1}{n^3}}{\frac{3}{n} + \frac{1}{n^2}}} = \lim_{n \rightarrow \infty} \frac{\frac{3}{n} + \frac{1}{n^2}}{2 + \frac{1}{n^3}} = \lim_{n \rightarrow \infty} \frac{0 + 0}{2 + 0} = 0.$$

Since all terms of the sequence are positive and given the properties of limits, we can conclude that the original sequence diverges to $+\infty$. Therefore, there is

$$\lim_{n \rightarrow \infty} \frac{2 + \frac{1}{n^3}}{\frac{3}{n} + \frac{1}{n^2}} = +\infty.$$

Exercise

Exercise 10. Find $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^5$.

Solution. After we transform the expression in the brackets, we find its fifth power:

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^5 = \lim_{n \rightarrow \infty} \left(\frac{n \cdot 1}{n \left(1 + \frac{1}{n}\right)}\right)^5 = \lim_{n \rightarrow \infty} \left(\frac{n}{n}\right)^5 \cdot \left(\frac{1}{1 + \frac{1}{n}}\right)^5 = (1)^5 \cdot \left(\frac{1}{1+0}\right)^5 = 1.$$

Exercise

Exercise 11. Find $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+5}$.

Solution. This task is more complicated than the previous ones because the exponent is $(n + 5)$. Therefore, we adjust the defining formula of the sequence so that we could use the fact that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$:

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{n+5} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \cdot \left(1 + \frac{1}{n}\right)^5 = e \cdot (1 + 0)^5 = e.$$

Compare this procedure with the procedure in the previous exercise.

Exercise

Exercise 12. Find $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{3n}$.

Solution. We will solve this exercise in the same way as in the previous exercise. Be careful when powering the expression.

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{3n} = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^3.$$

Since there is $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$, we get that:

$$\lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{n}\right)^n\right]^3 = e^3.$$

Exercise

Exercise 13. Find $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{5n}\right)^n$.

Solution. We will solve this exercise in the same way as in the previous exercise. Once again we will make use of the fact that $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{5n}\right)^{5n} = e$. Therefore, there is

$$\lim_{n \rightarrow \infty} \left(1 + \frac{1}{5n}\right)^n = \lim_{n \rightarrow \infty} \left[\left(1 + \frac{1}{5n}\right)^{5n} \right]^{\frac{1}{5}} = e^{\frac{1}{5}}.$$

Exercise

Exercise 14. Find $\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n$.

Solution. Once again we make use of the fact that there is $\lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$. We adjust the defining formula of the sequence in the following way:

$$\left(\frac{n}{n+1}\right)^n = \frac{1}{\left(\frac{n}{n+1}\right)^{-n}} = \frac{1}{\left(\frac{n+1}{n}\right)^n} = \left(\frac{n+1}{n}\right)^{-n}.$$

We will apply this fact and we get:

$$\lim_{n \rightarrow \infty} \left(\frac{n}{n+1}\right)^n = \lim_{n \rightarrow \infty} \left(\frac{n+1}{n}\right)^{-n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{-n} = \lim_{n \rightarrow \infty} \frac{1}{\left(1 + \frac{1}{n}\right)^n} = \frac{1}{e}.$$

Infinite series

Series is a concept linked to sequences because we get a series when we make the sum of terms of sequences. If the sequence is a finite one, we get a finite series. For infinite sequences we get infinite series. To be more precise, for a sequence $(a_n)_{n=1}^k$ we call the expression

$$a_1 + a_2 + a_3 + \cdots + a_k = \sum_{i=1}^k a_i$$

a finite series.

For a sequence $(a_n)_{n=1}^\infty$ we call the expression

$$a_1 + a_2 + a_3 + \cdots + a_k + \cdots = \sum_{i=1}^\infty a_i$$

an infinite series. The terms of the sequence we make the sum of are called terms of the series.

Since each series is defined as a sum, we will be interested especially in whether there exists the sum of the series and it does, what its value is.

If there exists a real number which is the sum of the series, we call the series convergent. In the opposite case we call the series divergent. We are already familiar with the notions of convergence and divergence. In the context of series it is linked to the notion of the limit of partial sums.

Suppose we have a sequence $(a_n)_{n=1}^\infty$. We make the sequence of partial sums where $s_k = a_1 + a_2 + a_3 + \cdots + a_k$. This means that

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ &\dots \\ s_k &= a_1 + a_2 + a_3 + \cdots + a_k, \\ &\dots \end{aligned}$$

The series is convergent if and only if the sequence of partial sums is convergent. If that is the case, the limit of the series of partial sums equals the sum of the series:

$$\sum_{n=1}^\infty a_i = \lim_{n \rightarrow \infty} s_n.$$

Infinite geometric series are special cases of infinite series. We obtain them after we make the sum of the terms of geometric sequences. There exists a very simple convergence criterion for convergence of geometric series. The sums of convergent geometric series can be computed using a simple formula.

Every infinite geometric series with the first term a_1 and quotient q such that $|q| < 1$ converges and its sum is

$$\sum_{n=1}^\infty a_i = \lim_{n \rightarrow \infty} s_n = \frac{a_1}{1 - q}.$$

For $|q| \geq 1$ the series diverges.

Exercise

Exercise 15. Show that the infinite series $\sum_{n=1}^{\infty} 10^{-n}$ is convergent and find its sum.

Solution. We write the series $\sum_{n=1}^{\infty} 10^{-n}$ as

$$10^{-1} + 10^{-2} + 10^{-3} + \cdots + 10^{-n} + \cdots .$$

This is an infinite geometric series with quotient $q = 10^{-1}$ and the first term $a_1 = 10^{-1}$. Since there is $-1 < 10^{-1} < 1$, this series is convergent and its sum is

$$\sum_{n=1}^{\infty} 10^{-n} = \frac{10^{-1}}{1 - 10^{-1}} = \frac{0,1}{0,9} = \frac{1}{9}.$$

Exercise

Exercise 16. Show that the infinite series $\frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \cdots$ is convergent and find its sum.

Solution. First, notice that the given terms may be written as $\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots$, which means that they make a series:

$$\frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots \frac{1}{2^{2n+1}} + \cdots \sum_{n=0}^{\infty} \frac{1}{2^{2n+1}}.$$

This is the infinite geometric series with quotient $q = \frac{1}{4}$ and the first term $a_1 = \frac{1}{2}$. Since there is $-1 < \frac{1}{4} < 1$, this series is convergent and its sum is

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n+1}} = \frac{\frac{1}{2}}{1 - \frac{1}{4}} = \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}.$$

Exercise

Exercise 17. Write $0,1\overline{45}$ as a fraction with both the nominator and the denominator being integers.

Solution. The number $0,1\overline{45}$ may be written as:

$$0,1\overline{45} = \frac{1}{10} + \frac{45}{10^3} + \frac{45}{10^5} + \frac{45}{10^7} + \cdots + \frac{45}{10^{2n-1}} + \cdots = 1 \cdot 10^{-1} + 45 \cdot 10^{-3} + 45 \cdot 10^{-5} + 45 \cdot 10^{-7} + \cdots + 45 \cdot 10^{-2n-1} + \cdots .$$

The sum

$$45 \cdot 10^{-3} + 45 \cdot 10^{-5} + 45 \cdot 10^{-7} + \cdots + 45 \cdot 10^{-2n-1} + \cdots$$

is the infinite geometric series with quotient $q = 10^{-2}$ and the first term $a_1 = 45 \cdot 10^{-3}$. Since there is $-1 < 10^{-2} < 1$, the series is convergent and its sum is

$$\sum_{n=1}^{\infty} 45 \cdot 10^{-2n-1} = \frac{45 \cdot 10^{-3}}{1 - 10^{-2}}.$$

This can be rewritten as

$$\sum_{n=1}^{\infty} 45 \cdot 10^{-2n-1} = \frac{0,045}{0,99} = \frac{45}{990} = \frac{9}{198} = \frac{1}{22}.$$

Thus

$$0,1\overline{45} = 1 \cdot 10^{-1} + \frac{1}{22} = \frac{1}{10} + \frac{1}{22} = \frac{16}{110} = \frac{8}{55},$$

which means that there is $0,1\overline{45} = \frac{8}{55}$.

Exercise

Exercise 18. Show that the infinite series $1 - 1 + 1 - 1 + \cdots + (-1)^{n+1}$ is divergent.

Solution. The series $1 - 1 + 1 - 1 + \cdots + (-1)^{n+1}$ can be written as

$$\sum_{n=1}^{\infty} (-1)^{n+1}.$$

This is the infinite geometric series with quotient $q = -1$ and the first term $a_1 = 1$. Since there is $|q| \geq 1$, the series is divergent. This task can be solved using the sequence of partial sums as well.

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 - 1 = 0, \\ s_3 &= 1 - 1 + 1 = 1, \\ s_4 &= 1 - 1 + 1 - 1 = 0, \\ &\dots \\ s_{2k} &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots - 1 = 0, \\ s_{2k+1} &= 1 - 1 + 1 - 1 + 1 - 1 + \cdots - 1 + 1 = 1, \\ &\dots \end{aligned}$$

Obviously, there is $s_{2k} = 0$ and $s_{2k+1} = 1$, i.e. the sequence of partial sums is

$$1, 0, 1, 0, \dots$$

This sequence obviously has no limit, which means that the series is divergent.

1 TEST

1. The first five terms of the sequence $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^{n+1} \cdot \frac{n}{n+2}$, is:

(a) $-\frac{1}{3}, \frac{1}{2}, -\frac{3}{5}, \frac{2}{3}, -\frac{5}{7}$.

(b) $\frac{1}{3}, -\frac{1}{2}, \frac{3}{5}, -\frac{2}{3}, \frac{5}{7}$.

(c) $\frac{1}{3}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{5}{7}$.

(d) $-\frac{1}{2}, \frac{2}{3}, -\frac{3}{4}, \frac{4}{5}, -\frac{5}{6}$.

2. The sequence $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^{n+1} \cdot n$, is:

(a) increasing,

(b) decreasing,

(c) neither increasing nor decreasing,

(d) bounded.

3. Choose a sequence which is decreasing:

(a) $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^{n+1}$,

(b) $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^{n+1} \frac{1}{n}$,

(c) $(a_n)_{n=1}^{\infty}$, where $a_n = \frac{1}{n}$,

(d) $(a_n)_{n=1}^{\infty}$, where $a_n = \frac{2^n}{3}$.

4. The sequence $(a_n)_{n=1}^{\infty}$, where $a_n = 2n - 3$, is

(a) an increasing arithmetic one,

(b) an increasing geometric one,

(c) neither arithmetic nor geometric,

(d) a decreasing arithmetic one.

5. The sequence $(a_n)_{n=1}^{\infty}$, where $a_n = \frac{2n^3 - 5n^2 - 1}{3n^3 - 4n + 38}$ converges to:

(a) 0,

(b) 1,

(c) $\frac{2}{3}$,

(d) $\frac{3}{2}$.

6. The sequence $(a_n)_{n=1}^{\infty}$, where $a_n = \left(1 + \frac{1}{n}\right)^{7n}$ converges to:

(a) $7e$,

(b) e^7 ,

(c) $\frac{7}{e}$,

(d) e .

7. The sequence $(a_n)_{n=1}^{\infty}$, where $a_n = (-1)^n \cdot 3 \cdot \frac{1}{2^n}$ is

- (a) arithmetic with $d = -3$,
- (b) geometric with $q = \frac{1}{2}$,
- (c) geometric with $q = \frac{3}{2}$,
- (d) geometric with $q = -\frac{1}{2}$.

8. Write $0.\overline{4}$ as a fraction with both the nominator and the denominator being integers. The fraction is:

- (a) $\frac{4}{9}$,
- (b) $\frac{3}{7}$,
- (c) $\frac{5}{11}$,
- (d) $\frac{9}{4}$.

9. The infinite series $\sum_{n=1}^{\infty} (-1)^n \cdot b^n$ converges for $b \in \mathbb{R}$, where:

- (a) $b = 1$,
- (b) $b = -1$,
- (c) $b = \frac{2}{5}$,
- (d) $b = \frac{5}{2}$.

10. The sum of the infinite series $\sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{5}$ is:

- (a) $b = 1$,
- (b) $b = -1$,
- (c) $b = -\frac{1}{6}$,
- (d) $b = \frac{1}{6}$.

10. COMPLEX NUMBERS

Complex numbers



The history of complex numbers begins in the 15th century and is linked to mathematicians of the University of Bologna. At that time, when looking for solutions of cubic equations, European mathematicians first faced radicals of negative numbers. The general solution of cubic equations is linked to names such as Scipion del Ferro, Niccolo Tartaglia and Gerolamo Cardano. In the second half of the 16th century, a Bolognese mathematician Rafaello Bombelli, who studied solutions of quadratic equations with negative determinants, enlarged our knowledge of complex numbers. Abraham de Moivre linked complex numbers to trigonometry. In the 18th century, Leonhard Euler introduced the notation for the complex unit and Carl Friedrich Gauss established a way to visualize complex numbers. Naturally, the above names are a selection of the most important researchers only since it was a number of many others who gave the study of complex domain its present shape.

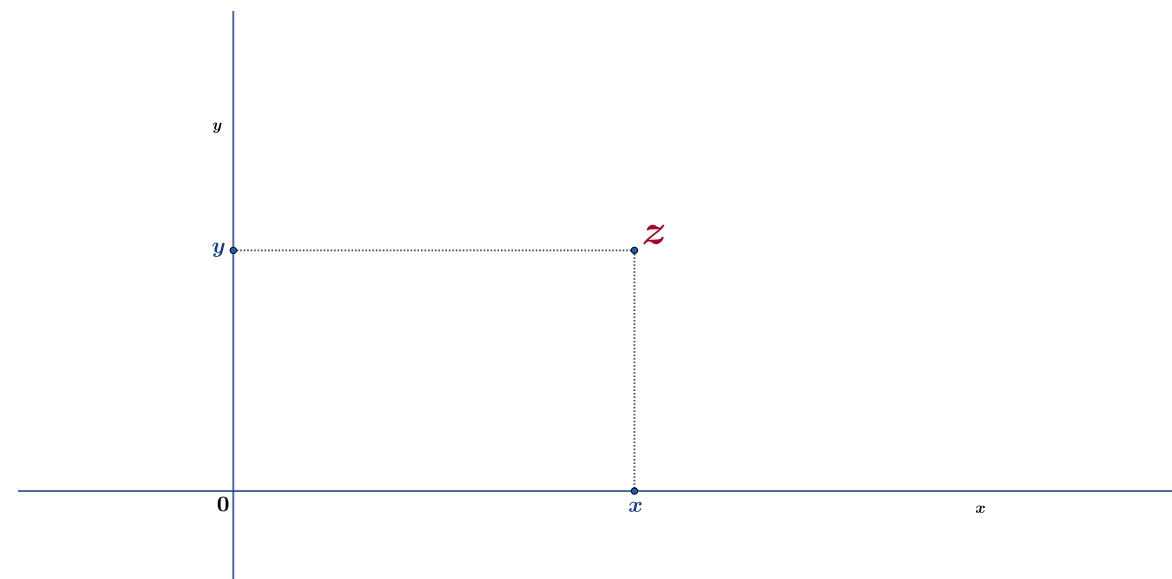
Complex numbers

Suppose our task is to solve the equation $x^2 + 3x + 3 = 0$. The first step we usually do is calculating the discriminant:

$$D = b^2 - 4 \cdot a \cdot c = 9 - 4 \cdot 1 \cdot 3 = -3.$$

Since for this equation we obtain $D < 0$, there are no real solutions. In order to be able to find solutions of equations such that $D < 0$, we need to introduce a new number domain.

We define a complex number z as an ordered pair $[x, y]$, where $x, y \in \mathbb{R}$. We call x the real part of z while y is called the imaginary part of z . The set of all complex numbers is denoted by $\mathbb{C}$. Complex numbers are usually visualized as points in the planar Cartesian coordinate system. Such a plane is called Gaussian or Argand plane; its x -axis is called the real axis while its y -axis is called the imaginary axis. Real parts of complex numbers correspond to the x -coordinates of the points in the Gaussian plane while imaginary parts of complex numbers correspond to the y -coordinates.



The notation $\operatorname{Re} z$, or $\operatorname{Re}(z)$, stands for the real part of z while the notation $\operatorname{Im} z$, or $\operatorname{Im}(z)$, stands for the imaginary part of z . As a result, for $z = [x, y]$ there is $\operatorname{Re} z = x$ and $\operatorname{Im} z = y$.

Two complex numbers $z_1 = [x_1, y_1]$ and $z_2 = [x_2, y_2]$ are equal if both their real and imaginary parts are equal, i.e. if the ordered pairs $[x_1, y_1], [x_2, y_2]$ are equal:

$$z_1 = z_2 \iff (\operatorname{Re} z_1 = \operatorname{Re} z_2 \wedge \operatorname{Im} z_1 = \operatorname{Im} z_2) \iff (x_1 = x_2 \wedge y_1 = y_2).$$

Graphically, two complex numbers are equal if their images in the Gaussian plane coincide.

Algebraic form of complex numbers

Usually we denote complex numbers $z = [x, y]$ in the algebraic form $z = x + iy$, where i is the imaginary unit defined by

$$i^2 = -1.$$

Given this, the second power of $-i$ is:

$$(-i)^2 = (-i) \cdot (-i) = (-1)^2 \cdot i^2 = -1.$$

As a result, also the second power of $-i$ equals -1 .

The absolute value of a complex number z is the distance of its image from the origin of the Gaussian plane, i.e. from the point $0 + 0i$. We denote the absolute value of z by $|z|$. The absolute value of a complex number z can be computed using the Pythagorean theorem:

$$|z| = \sqrt{x^2 + y^2}.$$

Complex numbers, whose absolute values are 1, are called complex units.

For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ we have that:

- The sum of complex numbers z_1 and z_2 is a complex number:

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2).$$

- The difference of complex numbers z_1 and z_2 is a complex number:

$$z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2).$$

- The product of complex numbers z_1 and z_2 is a complex number:

$$z_1 \cdot z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2).$$

The multiplication of complex numbers in the algebraic form corresponds to the multiplication of binomials.

Before we introduce the notion of a quotient of complex numbers, we need to introduce a new concept. We say that complex numbers z and $\bar{z}$ are complex conjugates if there is $z = x + yi$ and $\bar{z} = x - yi$. In other words, complex conjugates have the same real parts yet their imaginary parts are such that their absolute value is the same but the sign is different. In the Gaussian plane, complex conjugates are symmetric with respect to the x -axis. For the product of complex conjugates we have that:

$$z \cdot \bar{z} = (x + iy) \cdot (x - iy) = x^2 + y^2.$$

The quotient of complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2 \neq 0$ is a complex number

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} \cdot \frac{\bar{z}_2}{\bar{z}_2} = \frac{x_1y_1 + x_2y_2}{y_1^2 + y_2^2} + i \cdot \frac{x_2y_1 - x_1y_2}{y_1^2 + y_2^2}.$$

Later on we include an exercise in which this formula is derived.

Finally, we include the n^{th} power of a complex number. Let $z = x + iy$ be a complex number and let $n \in \mathbb{N}$. Then the n^{th} power of the complex number z is the complex number:

$$z^n = (x + iy)^n = \underbrace{(x + iy) \cdot (x + iy) \cdots (x + iy)}_n.$$

Remark 1. Since the value of electric current is denoted also by i , we denote the imaginary unit by j in electrical engineering. However, in this chapter we keep denoting the imaginary unit by i .

Exercise

Exercise 2. Find the absolute value of the complex number $z = \sqrt{3} - i\frac{\sqrt{2}}{2}$.

Solution. We apply the formula for the absolute value and get

$$|z| = \sqrt{(\sqrt{3})^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{3 + \frac{1}{2}} = \sqrt{\frac{7}{2}}.$$

Exercise

Exercise 3. Find the sum, difference and product of complex numbers $z_1 = \frac{1}{3} - \frac{3}{4}i$, $z_2 = -\frac{5}{3} + \frac{5}{2}i$.

Solution. The results are:

- For the sum we need to find the sum of the real and imaginary parts, respectively:

$$z_1 + z_2 = \left(\frac{1}{3} - \frac{3}{4}i\right) + \left(-\frac{5}{3} + \frac{5}{2}i\right) = \left(\frac{1}{3} - \frac{5}{3}\right) + i\left(-\frac{3}{4} + \frac{5}{2}\right) = -\frac{4}{3} + \frac{7}{4}i.$$

- For the difference we need to find the difference of the real and imaginary parts, respectively:

$$z_1 - z_2 = \left(\frac{1}{3} - \frac{3}{4}i\right) - \left(-\frac{5}{3} + \frac{5}{2}i\right) = \left(\frac{1}{3} - \left(-\frac{5}{3}\right)\right) + i\left(-\frac{3}{4} - \frac{5}{2}\right) = 2 - \frac{13}{4}i.$$

- In order to find the product, we treat the complex numbers as binomials:

$$z_1 \cdot z_2 = \left(\frac{1}{3} - \frac{3}{4}i\right) \cdot \left(-\frac{5}{3} + \frac{5}{2}i\right) = \left(\frac{1}{3}\right) \cdot \left(-\frac{5}{3}\right) + \left(\frac{1}{3}\right) \cdot \left(\frac{5}{2}i\right) + \left(-\frac{3}{4}i\right) \cdot \left(-\frac{5}{3}\right) + \left(-\frac{3}{4}i\right) \cdot \left(\frac{5}{2}i\right),$$

which simplifies to

$$\left(-\frac{5}{9}\right) + \left(\frac{5}{6}i\right) + \left(\frac{5}{4}i\right) + (-i) \cdot i \cdot \left(\frac{15}{8}\right) = \left(-\frac{5}{9}\right) + \left(\frac{15}{8}\right) + i \cdot \left(\frac{5}{6} + \frac{5}{4}\right) = \frac{95}{72} + \frac{25}{12}i.$$

Exercise

Exercise 4. Find the complex conjugate $\bar{z}$ of $z = 2 - 3i$. Then find the sum, difference and product of numbers z and $\bar{z}$.

Solution. The complex conjugate of $z = 2 - 3i$ is the complex number $\bar{z} = 2 - (-3i) = 2 + 3i$.

Then the sum of z and $\bar{z}$ is

$$z + \bar{z} = 2 - 3i + 2 + 3i = 2 + 2 = 4.$$

Their difference is:

$$z - \bar{z} = (2 - 3i) - (2 + 3i) = 2 - 3i - 2 - 3i = -6i.$$

Finally, their product is:

$$z \cdot \bar{z} = (2 - 3i) \cdot (2 + 3i) = 2 \cdot 2 + 2 \cdot i \cdot 3 + (-i) \cdot 3 \cdot 2 + (-i) \cdot 3 \cdot i \cdot 3 = 4 + 6i - 6i + 9 = 13.$$

Exercise

Exercise 5. Find the quotient of complex numbers $z_1 = 3 - 2i$ and $z_2 = 1 + 2i$.

Solution. We need to find the quotient:

$$\frac{z_1}{z_2} = \frac{3 - 2i}{1 + 2i}.$$

We need to adjust the denominator in such a way that we get a real number. We apply the result of the previous exercise where we could see that the product of complex conjugates is a real number. Therefore, we multiply the fraction with 1 expressed in such a way that both the nominator and the denominator are complex conjugates of the number in the denominator of the original fraction. In our case, we need to consider the complex conjugate of z_2 , which is $\bar{z} = 1 - 2i$. We get:

$$\frac{3 - 2i}{1 + 2i} = \frac{3 - 2i}{1 + 2i} \cdot \frac{1 - 2i}{1 - 2i} = \frac{(3 - 2i) \cdot (1 - 2i)}{1 + 4} = \frac{3 - 8i + i^2 \cdot 4}{5} = \frac{-1 - 8i}{5} = -\frac{1}{5} - \frac{8}{5}i.$$

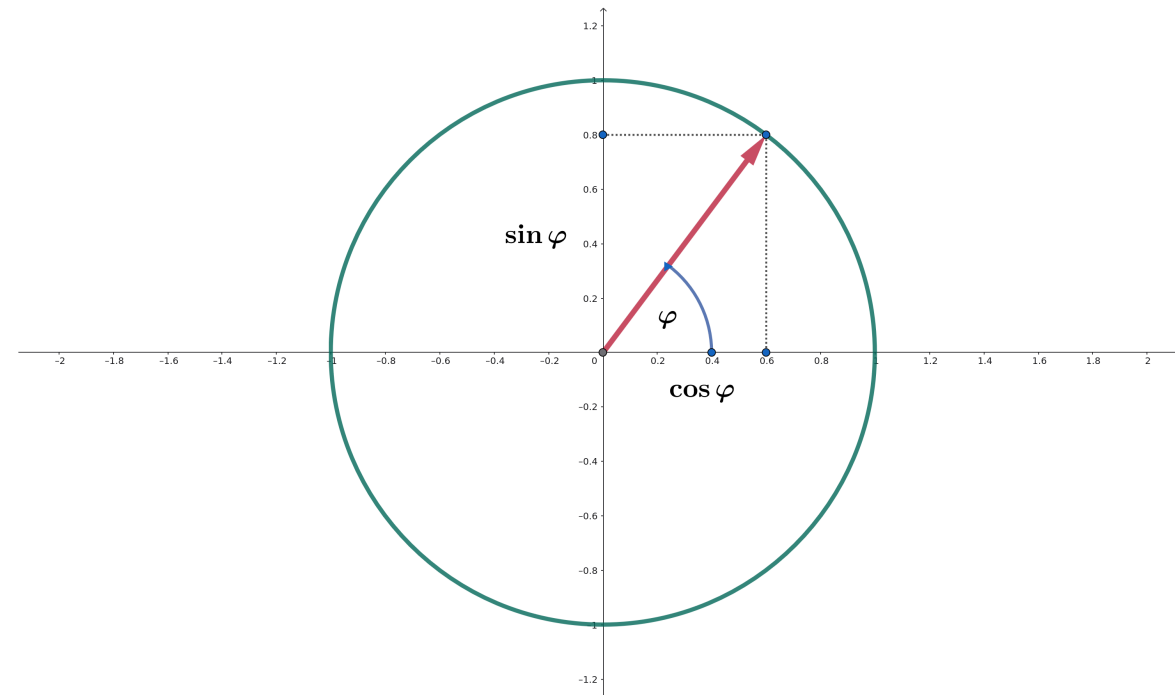
Trigonometric form of complex units

The image of a nonzero complex number in the Gaussian plane can be determined also in a way different from the algebraic form. This includes the distance from the origin of the Gaussian plane and a size of the oriented angle φ between the positive x -semi-axis and a half-line originating in the origin of the complex plane and terminating at the image of the complex number.

Let us first discuss such complex numbers that their images are on the unit circle centered at the origin of Gaussian plane. Such points are images of complex units. Their x -coordinates are $\cos \varphi$ while their y -coordinates are $\sin \varphi$. Thus, complex units, i.e. numbers $z \in \mathbb{C}$ such that $|z| = 1$, can be written as:

$$z = \cos \varphi + i \sin \varphi.$$

This notation is called the trigonometric form of complex units. See the figure below.

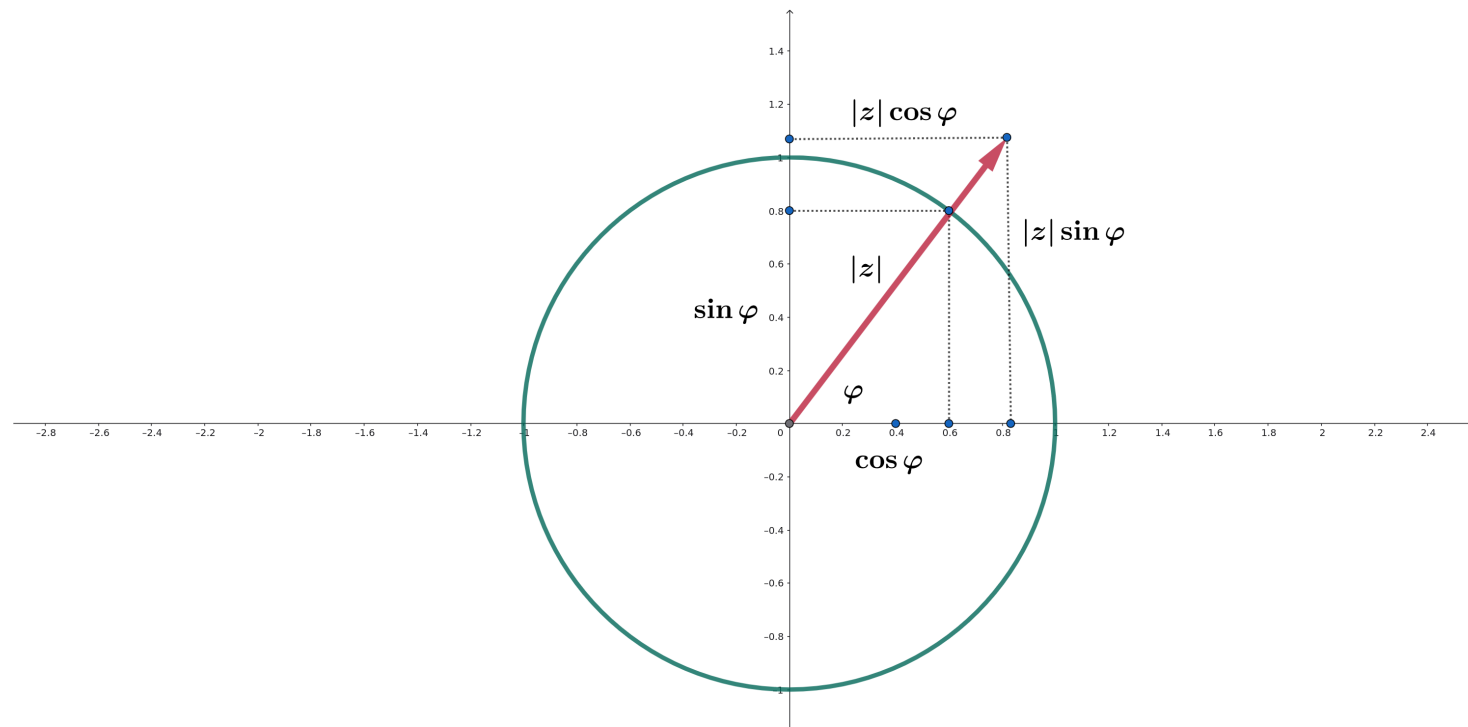


Trigonometric form of complex numbers

For an arbitrary complex number z , its trigonometric form is

$$z = |z| \cdot (\cos \varphi + i \sin \varphi)$$

as is given in the figure:



Since trigonometric functions are periodic, the argument φ of complex numbers is not unique. Each complex number with an argument φ has arguments $\varphi + 2k\pi$, where $k \in \mathbb{Z}$, as well. However, we usually suppose that $\varphi \in [0, 2\pi)$.

Two nonzero complex numbers $a = |a|(\cos \alpha + i \sin \alpha)$ and $b = |b|(\cos \beta + i \sin \beta)$ are equal when their absolute values are equal and when their arguments differ by $2k\pi$, where $k \in \mathbb{Z}$.

If we want to express a complex number $z = a + bi$ in the trigonometric form, we need to find its absolute value $|z|$ and its argument φ . In order to find the argument, we make use of the previous figure and get relations:

$$\cos \varphi = \frac{a}{|z|}, \quad \sin \varphi = \frac{b}{|z|}.$$

These relations uniquely define the basic argument $\varphi \in [0, 2\pi)$.

Conversely, in order to express the complex number, given in the trigonometric form, in the algebraic form, we evaluate $\sin \varphi$ and $\cos \varphi$.

Exercise

Exercise 6. Express in the trigonometric form with the argument from $[0, 2\pi)$ the complex number $z = \frac{\sqrt{3}}{2} - \frac{1}{2}i$

Solution. The number z is given in the algebraic form. First we find the absolute value:

$$|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(-\frac{1}{2}\right)^2} = 1.$$

For the argument of z there holds:

$$\cos \alpha = \frac{\operatorname{Re} z}{|z|} = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \wedge \alpha \in \langle 0, 2\pi) \Rightarrow \alpha \in \left\{ \frac{\pi}{6}, \frac{11\pi}{6} \right\},$$

$$\sin \alpha = \frac{\operatorname{Im} z}{|z|} = \frac{-\frac{1}{2}}{1} = -\frac{1}{2} \wedge \alpha \in \langle 0, 2\pi) \Rightarrow \alpha \in \left\{ \frac{7\pi}{6}, \frac{11\pi}{6} \right\}.$$

As a result, the argument of z is $\alpha = \frac{11\pi}{6}$, which means that its trigonometric form is

$$z = |z| \cdot (\cos \alpha + i \sin \alpha) = 1 \cdot \left(\cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6} \right) = \cos \frac{11\pi}{6} + i \sin \frac{11\pi}{6}.$$

Exercise

Exercise 7. Express in the algebraic form $z = 2 \cdot \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$.

Solution. First we find the values of sine and cosine for the angle $\alpha = \frac{3\pi}{4}$:

$$\sin \frac{3\pi}{4} = \frac{\sqrt{2}}{2},$$

$$\cos \frac{3\pi}{4} = -\frac{\sqrt{2}}{2}.$$

We substitute these values in the trigonometric form and simplify the result:

$$z = 2 \cdot \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) = 2 \cdot \left(-\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = -\sqrt{2} + i\sqrt{2} = -\sqrt{2} \cdot (1 - i).$$

Operations with complex numbers in the trigonometric form

The trigonometric form of complex numbers is convenient when we multiply or divide complex numbers. Using the relations between trigonometric functions one can derive that the product of two nonzero numbers in the trigonometric form $a = |a|(\cos \alpha + i \sin \alpha)$ and $b = |b|(\cos \beta + i \sin \beta)$ is

$$a \cdot b = |a|(\cos \alpha + i \sin \alpha) \cdot |b|(\cos \beta + i \sin \beta) = |a| \cdot |b| \cdot (\cos(\alpha + \beta) + i \sin(\alpha + \beta))$$

while their quotient is

$$\frac{a}{b} = \frac{|a|(\cos \alpha + i \sin \alpha)}{|b|(\cos \beta + i \sin \beta)} = \frac{|a|}{|b|} \cdot (\cos(\alpha - \beta) + i \sin(\alpha - \beta)).$$

The relations for multiplication enable us to define the n^{th} power, $n \in \mathbb{N}$, of a complex number in the trigonometric form:

$$\begin{aligned} a^n &= (|a|(\cos \alpha + i \sin \alpha))^n = \underbrace{(|a|(\cos \alpha + i \sin \alpha)) \cdot (|a|(\cos \alpha + i \sin \alpha)) \cdots (|a|(\cos \alpha + i \sin \alpha))}_n = \\ &= |a|^n \cdot \left(\cos \underbrace{(\alpha + \alpha + \cdots + \alpha)}_n + i \sin \underbrace{(\alpha + \alpha + \cdots + \alpha)}_n \right) = |a|^n \cdot (\cos(n\alpha) + i \sin(n\alpha)). \end{aligned}$$

Remark 8. For all above operations, the argument, i.e. also numbers $\alpha + \beta$, $\alpha - \beta$ and $n\alpha$ is an angle from the interval $[0, 2\pi)$.

Exercise

Exercise 9. Find the product and quotient of complex numbers $a = 7 \cdot \left(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}\right)$ and $b = 2 \cdot \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)$.

Solution. We apply the formulas for the product and quotient of complex numbers in the trigonometric form and get:

$$a \cdot b = 7 \cdot \left(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}\right) \cdot 2 \cdot \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right) = 14 \cdot \left(\cos \left(\frac{4\pi}{5} + \frac{\pi}{3}\right) + i \sin \left(\frac{4\pi}{5} + \frac{\pi}{3}\right)\right) = 14 \cdot \left(\cos \frac{17\pi}{15} + i \sin \frac{17\pi}{15}\right),$$

$$\frac{a}{b} = \frac{7 \cdot \left(\cos \frac{4\pi}{5} + i \sin \frac{4\pi}{5}\right)}{2 \cdot \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3}\right)} = \frac{7}{2} \cdot \left(\cos \left(\frac{4\pi}{5} - \frac{\pi}{3}\right) + i \sin \left(\frac{4\pi}{5} - \frac{\pi}{3}\right)\right) = \frac{7}{2} \cdot \left(\cos \frac{7\pi}{15} + i \sin \frac{7\pi}{15}\right).$$

Exercise

Exercise 10. Find the product and quotient of complex numbers $a = 2 \cdot \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)$ and $b = \frac{1}{3} \cdot \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)$.

Solution. We solve the task using the same techniques as in the previous exercise:

$$a \cdot b = 2 \cdot \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right) \cdot \frac{1}{3} \cdot \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right) = \frac{2}{3} \cdot \left(\cos \left(\frac{3\pi}{2} + \frac{5\pi}{6} \right) + i \sin \left(\frac{3\pi}{2} + \frac{5\pi}{6} \right) \right) = \frac{3}{2} \cdot \left(\cos \frac{14\pi}{6} + i \sin \frac{14\pi}{6} \right) = \frac{3}{2} \cdot \left(\cos \frac{7\pi}{3} + i \sin \frac{7\pi}{3} \right).$$

Obviously, there is $\frac{7\pi}{3} \notin [0, 2\pi)$, which means that we need to find the corresponding angle in the interval $[0, 2\pi)$. Therefore, we write the angle $\frac{7\pi}{3}$ as the sum $\alpha + 2k\pi$, where $k \in \mathbb{Z}$:

$$\frac{7\pi}{3} = \frac{(6+1)\pi}{3} = \frac{6\pi}{3} + \frac{\pi}{3} = \frac{\pi}{3} + 2\pi.$$

We get

$$a \cdot b = \frac{3}{2} \cdot \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right).$$

We find the quotient $\frac{a}{b}$ in an analogous way:

$$\frac{a}{b} = \frac{2 \cdot \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)}{\frac{1}{3} \cdot \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)} = 6 \cdot \left(\cos \left(\frac{3\pi}{2} - \frac{5\pi}{6} \right) + i \sin \left(\frac{3\pi}{2} - \frac{5\pi}{6} \right) \right) = 6 \cdot \left(\cos \frac{4\pi}{6} + i \sin \frac{4\pi}{6} \right) = 6 \cdot \left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \right).$$

Were we asked to find the quotient $\frac{b}{a}$, we would proceed in an analogous way:

$$\frac{b}{a} = \frac{\frac{1}{3} \cdot \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right)}{2 \cdot \left(\cos \frac{3\pi}{2} + i \sin \frac{3\pi}{2} \right)} = \frac{1}{6} \cdot \left(\cos \left(\frac{5\pi}{6} - \frac{3\pi}{2} \right) + i \sin \left(\frac{5\pi}{6} - \frac{3\pi}{2} \right) \right) = \frac{1}{6} \cdot \left(\cos \left(-\frac{4\pi}{6} \right) + i \sin \left(-\frac{4\pi}{6} \right) \right) = \frac{1}{6} \cdot \left(\cos \left(-\frac{2\pi}{3} \right) + i \sin \left(-\frac{2\pi}{3} \right) \right).$$

As was the case of the product, nor the angle $-\frac{2\pi}{3}$ is in the interval $[0, 2\pi)$. Therefore, we have to find the corresponding angle from the interval $[0, 2\pi)$. In this case we write the angle $-\frac{2\pi}{3}$ as the sum $\alpha + 2k\pi$, where $k \in \mathbb{Z}$. Since there is $-\frac{2\pi}{3} \in (-2\pi, 0)$, it is sufficient if we add 2π to it. In this way we get an angle from $[0, 2\pi)$. There is:

$$\alpha = -\frac{2\pi}{3} + 2\pi,$$

i.e. $\alpha = \frac{4\pi}{3}$. As a result,

$$\frac{b}{a} = \frac{1}{6} \cdot \left(\cos \frac{4\pi}{3} + i \sin \frac{4\pi}{3} \right).$$

Exponential (Euler) form of complex numbers

We conclude with yet another form of complex numbers, called exponential (or Euler one).

If we denote $e^{i\alpha} = (\cos \alpha + i \sin \alpha)$, we can express the complex number $a = |a| (\cos \alpha + i \sin \alpha)$ as:

$$a = |a| \cdot e^{i\alpha},$$

where $|a|$ is its absolute value and α is its argument.

When transforming the complex number in the exponential form into the algebraic form, it is sufficient to calculate the values of sine and cosine of the argument α .

The process of transforming the nonzero complex number expressed in the algebraic form into the exponential form is the same as the process of transformation into the trigonometric form. Once again, we need to find the absolute value and argument of the given complex number.

The exponential form of complex numbers is (as is the case of the trigonometric form) convenient when we multiply and divide complex numbers.

The product of two nonzero complex numbers $a = |a| \cdot e^{i\alpha}$ and $b = |b| \cdot e^{i\beta}$ is

$$a \cdot b = |a| \cdot e^{i\alpha} \cdot |b| \cdot e^{i\beta} = |a| \cdot |b| \cdot e^{i(\alpha+\beta)}$$

while their quotient is

$$\frac{a}{b} = \frac{|a|}{|b|} \cdot e^{i(\alpha-\beta)}.$$

Exercise

Exercise 11. Express the following numbers in the exponential (Euler) form:

1. $z_1 = 2 \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right),$

2. $z_2 = \frac{1}{2} + i\frac{\sqrt{3}}{2}.$

Solution. The results are:

1. The number z_1 is given in the trigonometric form. Since there is:

$$z_1 = 2 \cdot \left(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6} \right),$$

we have that $|z_1| = 2$ and the argument is $\frac{5\pi}{6}$. As a result, the expression of z_1 in the trigonometric form is

$$z_1 = 2 \cdot e^{i\frac{5\pi}{6}}.$$

2. The number z_2 is given in the algebraic form. First we find its absolute value:

$$|z_2| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1.$$

For the argument z_2 we have

$$\cos \alpha = \frac{\frac{1}{2}}{1} = \frac{1}{2} \Rightarrow \alpha \in \left\{ \frac{\pi}{3}, \frac{5\pi}{3} \right\},$$

$$\sin \alpha = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \Rightarrow \alpha \in \left\{ \frac{\pi}{3}, \frac{2\pi}{3} \right\}.$$

Therefore, the argument of this number is $\alpha = \frac{\pi}{3}$. Its exponential form is thus

$$z_2 = e^{i\frac{\pi}{3}}.$$

Exercise

Exercise 12. Write the complex number $z = 3 \cdot e^{i\frac{3\pi}{4}}$ in the algebraic form.

Solution. First of all, we must consider that there is $z = a + ib$, where $a = |z| \cdot \cos \frac{3\pi}{4}$ and $b = |z| \cdot \sin \frac{3\pi}{4}$. Therefore, there is obviously $|z| = 3$. Thus

$$a = 3 \cdot \cos \frac{3\pi}{4} = 3 \cdot \left(-\frac{\sqrt{2}}{2} \right) = -\frac{3\sqrt{2}}{2}$$

and

$$b = 3 \cdot \sin \frac{3\pi}{4} = 3 \cdot \left(\frac{\sqrt{2}}{2} \right) = \frac{3\sqrt{2}}{2}.$$

As a result, there is $z = -\frac{3\sqrt{2}}{2} + i\frac{3\sqrt{2}}{2}$.

Exercise

Exercise 13. Find the product and quotient of complex numbers $a = 7 \cdot e^{i\frac{4\pi}{5}}$ and $b = 2 \cdot e^{i\frac{\pi}{3}}$.

Solution. We apply the formulas for calculating the product and quotient:

$$a \cdot b = 7 \cdot e^{i\frac{4\pi}{5}} \cdot 2 \cdot e^{i\frac{\pi}{3}} = 7 \cdot 2 \cdot e^{i(\frac{4\pi}{5} + \frac{\pi}{3})} = 14 \cdot e^{i\frac{17\pi}{15}}.$$

$$\frac{a}{b} = \frac{7 \cdot e^{i\frac{4\pi}{5}}}{2 \cdot e^{i\frac{\pi}{3}}} = \frac{7}{2} \cdot e^{i(\frac{4\pi}{5} - \frac{\pi}{3})} = \frac{7}{2} \cdot e^{i\frac{7\pi}{15}}.$$

Exercise

Exercise 14. Find z^6 , where $z = \frac{\sqrt{3}}{2} + i\frac{1}{2}$.

Solution. The number z is given in the algebraic form. Finding its power will be easier if it was in the trigonometric or exponential form. First, we need to find the absolute value of z :

$$|z| = \sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 1.$$

For the argument of z we have

$$\cos \alpha = \frac{\frac{\sqrt{3}}{2}}{1} = \frac{\sqrt{3}}{2} \wedge \alpha \in \langle 0, 2\pi \rangle \Rightarrow \alpha \in \left\{ \frac{\pi}{6}, \frac{11\pi}{6} \right\}$$

and at the same time

$$\sin \alpha = \frac{\frac{1}{2}}{1} = \frac{1}{2} \wedge \alpha \in \langle 0, 2\pi \rangle \Rightarrow \alpha \in \left\{ \frac{\pi}{6}, \frac{5\pi}{6} \right\},$$

which means that the argument of z is $\alpha = \frac{\pi}{6}$. Therefore, its trigonometric form is

$$z = |z| \cdot (\cos \alpha + i \sin \alpha) = 1 \cdot \left(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right) = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}.$$

The formula for computing the n^{th} power of our complex number z in the trigonometric form is

$$z^n = 1^n \cdot \left(\cos \left(n \cdot \frac{\pi}{6} \right) + i \sin \left(n \cdot \frac{\pi}{6} \right) \right).$$

For $n = 6$ we get:

$$z^6 = 1^6 \cdot \left(\cos \left(6 \cdot \frac{\pi}{6} \right) + i \sin \left(6 \cdot \frac{\pi}{6} \right) \right) = (\cos \pi + i \sin \pi) = -1 + i0 = -1.$$

For the exponential form of z there is:

$$z = e^{i\frac{\pi}{6}}.$$

Its 6^{th} power is thus:

$$z^6 = \left(e^{i\frac{\pi}{6}} \right)^6 = e^{i\pi}.$$

Exercise

Exercise 15. Find the solution of the equation $x^2 + 3x + 3 = 0$.

Solution. The equation is a quadratic one with real coefficients. However, after we substitute the coefficients into the discriminant formula

$$D = b^2 - 4ac,$$

we get:

$$D = 3^2 - 4 \cdot 1 \cdot 3 = 9 - 12 = -3 < 0.$$

After we apply the formula for the solutions, i.e.

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a},$$

we get that:

$$x_{1,2} = \frac{-3 \pm \sqrt{-3}}{2},$$

i.e.

$$x_{1,2} = \frac{-3 \pm \sqrt{3}\sqrt{-1}}{2}.$$

And now we can apply the fact that $\sqrt{-1} = i$ which will result into:

$$x_1 = \frac{-3 + i\sqrt{3}}{2},$$

and

$$x_2 = \frac{-3 - i\sqrt{3}}{2}.$$

Thus we can write that

$$x^2 + 3x + 3 = \left(x - \frac{-3 + i\sqrt{3}}{2}\right) \cdot \left(x - \frac{-3 - i\sqrt{3}}{2}\right),$$

which expands to

$$x^2 - x \cdot \frac{-3 - i\sqrt{3}}{2} - x \cdot \frac{-3 + i\sqrt{3}}{2} + \frac{-3 + i\sqrt{3}}{2} \cdot \frac{-3 - i\sqrt{3}}{2} = x^2 + \frac{3x + ix\sqrt{3} + 3x - ix\sqrt{3}}{2} + \frac{9 + i3\sqrt{3} - i3\sqrt{3} + 3}{4} = x^2 + 3x + 3.$$

Notice that x_1 and x_2 are complex conjugates. This is the case always when the quadratic equation has real coefficients and negative discriminant.

Exercise

Even though the formula for solutions of the quadratic equation is usually applied for equations with real coefficients, it can be used also for the general case of equations with complex coefficients.

Exercise 16. Find the solution of the equation $x^2 - 3ix - 2 = 0$.

Solution. The equation is quadratic yet not all of its coefficients are real. As a result, its solutions will not be complex conjugates. As in real domain we denote: $a = 1$, $b = -3i$, $c = -2$. Applying the formula

$$D = b^2 - 4ac$$

we get that the discriminant is:

$$D = (-3i)^2 - 4 \cdot 1 \cdot (-2) = 9 \cdot i^2 + 8 = -9 + 8 = -1.$$

Now we substitute the value into the formula for obtaining the solutions, i.e. into:

$$x_{1,2} = \frac{-b \pm \sqrt{D}}{2a}.$$

We get that the solutions of the equation are:

$$x_{1,2} = \frac{3i \pm \sqrt{-1}}{2},$$

i.e.

$$x_1 = \frac{3i + i}{2} = \frac{4i}{2} = 2i$$

and

$$x_2 = \frac{3i - i}{2} = \frac{2i}{2} = i.$$

Thus we can write that

$$x^2 - 3ix - 2 = (x - 2i)(x - i),$$

which is easy to verify after we multiply the brackets.

1 TEST

1. The absolute value of $x = 3 - 4i$ is:
 - (a) 5,
 - (b) 25,
 - (c) $\sqrt{7}$,
 - (d) -5 .
2. For the imaginary unit i the value of $(-i)^3$ is:
 - (a) 1,
 - (b) -1 ,
 - (c) i ,
 - (d) $-i$.
3. For the imaginary unit i the value of $(-i)^{39}$ is:
 - (a) 1,
 - (b) -1 ,
 - (c) i ,
 - (d) $-i$.
4. The sum of complex numbers $x = 3 - 2i$ and $y = 2 + 3i$ is:
 - (a) 0,
 - (b) $5 + i$,
 - (c) $1 - 5i$,
 - (d) $12 - 5i$.
5. The trigonometric form of $x = 1 + i$ is:
 - (a) $\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}$,
 - (b) $\sqrt{2} \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$,
 - (c) $\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$,
 - (d) $\sqrt{2} \left(\cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)$.
6. The exponential form of $x = -1 + i\sqrt{3}$ is:
 - (a) $e^{i\frac{2\pi}{3}}$,
 - (b) $2e^{i\frac{2\pi}{3}}$,
 - (c) $2e^{i\frac{\pi}{3}}$,
 - (d) $2e^{i\frac{\pi}{6}}$.

7. For $x = 1 - 2i, y = 3 + 7i$, the complex number $\frac{x}{y}$ is:

- (a) $\frac{11}{58} + i\frac{13}{58}$,
- (b) $-\frac{11}{58} - i\frac{13}{58}$,
- (c) $-\frac{11}{58} + i\frac{13}{58}$,
- (d) $\frac{11}{58} - i\frac{13}{58}$.

8. Choose all complex numbers absolute value of which is 2:

- (a) $2\sqrt{2} - i\sqrt{2}$,
- (b) $-\sqrt{3} - i$,
- (c) $2i$,
- (d) $\sqrt{2}$.

9. For $x = 5e^{i\frac{4\pi}{3}}, y = 3e^{i\frac{3\pi}{2}}$ the complex number $x \cdot y$ is:

- (a) $15 \cdot e^{i\frac{5\pi}{6}}$,
- (b) $15 \cdot e^{i2\pi}$,
- (c) $15 \cdot e^{i\frac{11\pi}{6}}$,
- (d) 15.

10. For $x = 1, y = e^{i\frac{\pi}{2}}$ the complex number $\frac{x}{y}$ is:

- (a) e^i ,
- (b) 4,
- (c) $e^{i\frac{\pi}{2}}$,
- (d) $e^{i\frac{3\pi}{2}}$.